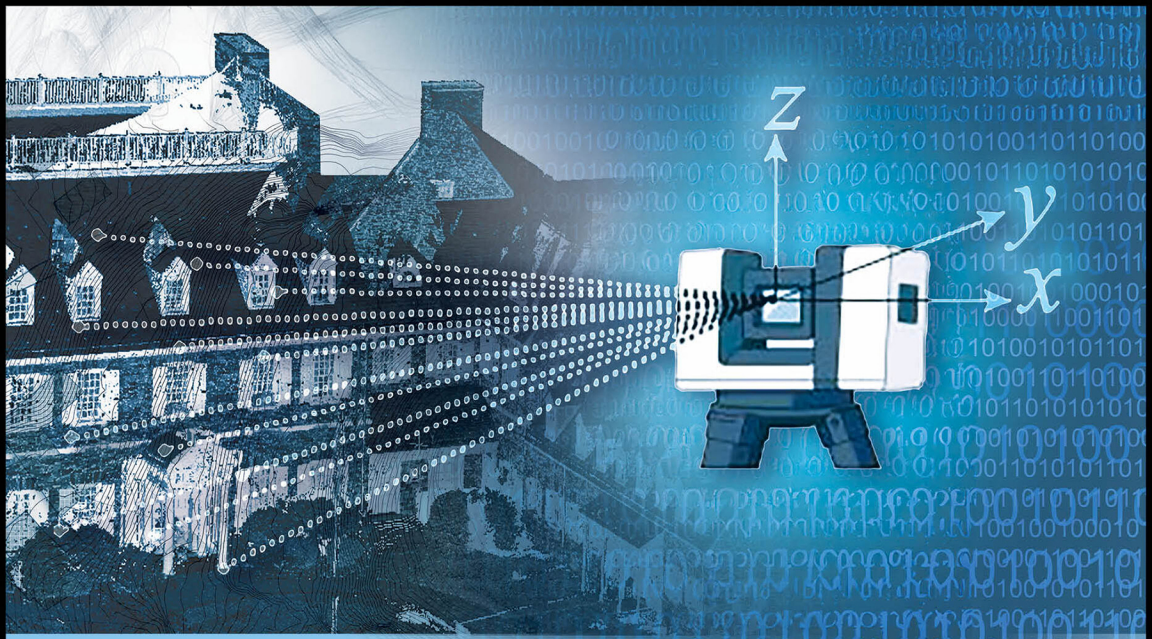


SEVENTH EDITION

ADJUSTMENT COMPUTATIONS

SPATIAL DATA ANALYSIS

CHARLES D. GHILANI • DIMITRIOS BOLKAS
MICHAEL J. OLSEN

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ADJUSTMENT COMPUTATIONS

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Spatial Data Analysis

Seventh Edition

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PREFACE

No observation is exact. As a corollary, every observation contains error. These statements are fundamental and accepted universally. Therefore, it follows logically that surveyors, who are measurement specialists, should have a thorough understanding of errors. They must be familiar with the different types of errors, their sources, and their expected magnitudes. Armed with this knowledge, they will be able to (1) adopt procedures to reduce error sizes when obtaining measurements and (2) account rigorously for the presence of errors as they analyze and adjust their data. This book is devoted to providing surveyors and users of spatial data with a better understanding of these topics.

The least-squares method to adjust spatial data is well-established to analyze and adjust survey data. This should not be surprising because the method is the most rigorous adjustment procedure available. It is soundly based on the mathematical theory of probability; it allows for appropriate weighting of all observations in accordance with their estimated precisions; and it enables complete statistical analyses to be performed following adjustments so that the estimated precisions of adjusted quantities can be determined. Procedures for employing the method of least squares and then statistically analyzing the results are major topics covered in this book.

In years past, the least-squares method was seldom used for adjusting surveying data because the time required to set up and solve the necessary equations was too great for hand methods. Readily available computing capabilities have eliminated this disadvantage. Besides advances in computer technology, some other recent developments have also led to the increased use of least squares. Prominent among these are the global navigation satellite systems (GNSS) such

as the global positioning system (GPS) and geographic information systems (GISs). These systems rely heavily on rigorous error propagation methods, adjustment of data, and statistical analysis of the results. But perhaps the most compelling of all reasons for the continually increasing interest in least-squares adjustments is that modern accuracy standards depend on quantities obtained from least-squares adjustments. Thus, surveyors of the future will not be able to test their observations for compliance with these standards unless they adjust their data using least squares. For example, the American Society of Photogrammetry and Remote Sensing (ASPRS) Positional Accuracy Standards for Digital Geospatial Data, the joint American Land Title Association/National Society of Professional Surveyors (ALTA/NSPS) standards for boundary surveys, and the forthcoming Transportation Research Board (TRB) survey guide for transportation applications are based on statistical results as is. Clearly modern surveyors must be able to apply the method of least squares to adjust their observed data, and they must also be able to perform a statistical evaluation of the results after the adjustment.

The advent of newer technologies such as laser scanning and Uncrewed Aircraft Systems (UASs) has marked a new era for geospatial professions. Today, professionals can map natural surfaces and model objects and structures with unprecedented detail using hundreds of millions to billions of points. However, detailed point clouds are not free from error, and professionals still need to demonstrate compliance with accuracy requirements and survey standards. This can be achieved through error propagation, allowing professionals to understand the quality of point clouds, which can considerably vary spatially, making error propagation processes more important than ever. Furthermore, geospatial professionals, having to manage massive datasets of points, are faced with new problems such as three-dimensional modeling of man-made objects and natural surfaces. This can be achieved through robust methods such as least-squares fitting.

The seventh edition addresses the evolving knowledge requirements with two new chapters. Chapter 26 covers basic laser scanner measurement principles, major error sources, and error propagation of uncertainty to point cloud coordinates. Chapter 27 covers the least-squares method to fit planes, spheres, and cylinders to point clouds. Additionally, this chapter covers polynomial fitting and interpolation for digital elevation model (DEM) estimation. Chapter 19 of the new edition provides updated information related to the new ASPRS positional accuracy standards. Chapter 24 provides updated information about transformations in the modernized National Spatial Reference System (NSRS). Lastly, Chapter 24 covers the use of total station pseudo-observations to adjust data for a combined GNSS and total station survey.

As with previous editions, the seventh edition provides many helpful instructional videos on the subject matter covered in the book. They also demonstrate the solution of example problems using spreadsheets and software that accompanies the book. For instructors who adopt this text in their classes, an *Instructor's Manual to Accompany Adjustment Computations* is available from the publisher's

website. This manual includes detailed solutions to all the problems in the book, along with suggested course outlines and exams, which can be used in their courses. To obtain access to the manual, please contact your regional Wiley representative.

Users of the book can visit the Wiley website for the book to download software packages STATS, ADJUST, and MATRIX, which will run on any PC-compatible computer in the Windows environment. The first package, called STATS, performs basic statistical analyses. For any given set of observed data, it will compute the mean, median, mode, and standard deviation, and develop and plot the histogram and normal distribution curve. It will also compute critical values for the t, τ, χ^2, and F distributions. This software can also compute confidence intervals for the population mean, variance, and ratio of two variances, and perform statistical tests for the population mean, variance, and ratio of two variances.

The second package, called ADJUST, contains a collection of programs to perform specific least-squares adjustments covered in the book. For users in the United States, this software allows you to make geodetic corrections to observed surveying data using the National Geodetic Survey's geoid model and deflection of the vertical files. When performing least-squares adjustments, ADJUST allows the user to select either data snooping or the tau criterion for post-adjustment blunder detection. The user can also specify whether the angular data is in the sexagesimal system of degrees, minutes, and seconds or the grad system. The program contains a variety of coordinate transformations and performs least-squares fitting of points to a line, parabola, or circle. Additionally, the horizontal and vertical least-squares problems compute the radial error of a point. It contains editors for the State Plane Coordinate System (SPCS) of 1983 and 2022. The editor for SPCS 2022 allows users to enter the projection parameters for SPCS 2022 or any other map projection using the Lambert conformal conic, transverse Mercator, or oblique Mercator map projections. Additionally, the software can reduce individual survey observations or files of observations to the map projection surface.

A feature in MATRIX is its ability to perform unweighted and weighted least-squares adjustments with a single command. Using this program, systems of simultaneous linear equations can be solved quickly and conveniently, and the basic algorithm for performing least-squares adjustments can be solved in a stepwise fashion. For those who wish to develop their own software, the book provides several helpful computer algorithms in the languages of BASIC, C, FORTRAN, and PASCAL. Additionally, the Mathcad worksheets on the companion website demonstrate the use of functions in developing modular programs in least-squares adjustments. These worksheets are presented as ebooks, which can be installed under Mathcad's Handbook subdirectory. For those who do not own Mathcad, these worksheets are converted to html files, which can be viewed in any web browser. These files demonstrate the programming of example problems in the book as well as providing additional background information for items such as map projections, geodetic reductions of survey observations, and so on. For students, in the process of learning to program, these worksheets demonstrate modular programming methods. For the examples and problems

of Chapters 26 and 27, the authors provide MatLab scripts (m files). For those not owning MatLab, they can open these scripts with any text editor such as notepad.

The chapters of this book are arranged in the order found most convenient in teaching college courses on adjustment computations. The content in this book can be covered in two or three typical undergraduate, college-level courses. It could also be taught in a single graduate course where students should already have sufficient statistical and mathematical background to move through the book efficiently. It is believed that this order also best facilitates practicing surveyors, graduate students, and other geospatial professionals who use the book for self-study. In earlier chapters, we define terms and introduce students to the fundamentals of errors and methods for analyzing them. The next several chapters are devoted to the subject of error propagation in the diverse types of traditional surveying measurements. Then the chapters which follow describe observation weighting and introduce the least-squares method to adjust observations. Applications of least squares in adjusting basic types of surveys are then presented in separate chapters, including level nets, trilateration, triangulation, traverses and horizontal networks, GNSS networks, and conventional three-dimensional surveys. The subject of error ellipses and error ellipsoids is covered in a separate chapter. Procedures for applying least squares in curve fitting and in computing coordinate transformations are also presented. The more advanced topics of blunder detection and the method of general least-squares adjustments are covered next. The last chapters cover the topics of laser scanner measurements and error propagation in laser scanning, point cloud three-dimensional fitting, and computer optimization.

As with previous editions, matrix methods, which are so well adapted to adjustment computations, continue to be used in this edition. For those students who have never studied matrices, or those who wish to review this topic, an introduction to matrix methods is given in Appendices A and B. Those students who have already studied matrices can conveniently skip this subject or use it for review. Least-squares adjustments often require the formation and solution of nonlinear equations. Procedures for linearizing nonlinear equations by Taylor's theorem are presented in Appendix C, given their importance in adjustment computations. Appendix D contains several statistical tables, including the standard normal error distribution, the χ^2 distribution, t distribution, and a set of F distribution tables. These tables are described at appropriate locations in the text, and their use is demonstrated with example problems. These tables are useful to verify that the correct functions are being utilized in spreadsheets or other software.

Basic courses in surveying, statistics, and calculus are necessary prerequisites to understanding some of the theoretical coverage and equation derivations given herein. Nevertheless, those who do not have these courses as background but who wish to learn how to apply least squares in adjusting surveying observations can follow the discussions on data analysis.

Besides being appropriate for use as a textbook in college classes, this book will be of value to anyone working with geospatial information including surveyors, engineers, and GIS professionals. With the inclusion of video lessons, it is possible for practitioners to learn the subject matter at their leisure. The authors hope that through the publication of this book, the least-squares adjustment method and the rigorous statistical analyses of surveying data will become more commonplace, as they should.

ACKNOWLEDGMENTS

Through the years, many people have contributed to the development of this book. As noted in the preface, the book has been used in continuing education classes taught to practicing surveyors as well as in classes taken by students at the University of California–Berkeley, the University of Wisconsin–Madison, Oregon State University, and the Pennsylvania State University–Wilkes-Barre. The students in these classes have provided data for some of the example problems and have supplied numerous helpful suggestions for improvements throughout the book. The authors gratefully acknowledge their contributions.

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ABOUT THE COMPANION WEBSITE

This book is accompanied by a companion website:



www.wiley.com/go/Ghilani/adjustmentcomputations7e

The website includes:

- Software
- Others

CHAPTER 1

INTRODUCTION

1.1 INTRODUCTION



We currently live in what is often termed *the information age*. Aided by new and emerging technologies, data are being collected at unprecedented rates in all walks of life. For example, in the field of surveying, total station instruments, global navigation satellite systems (GNSS) equipment, digital metric cameras, laser-scanning systems, lidar, mobile mapping systems, and satellite imaging systems are only some of the new instruments that are now available for rapid generation of vast quantities of observational data.

Geographic Information Systems (GISs) have evolved concurrently with the development of these new data acquisition instruments. GISs are now used extensively for management, planning, and design. They are being applied worldwide at all levels of government, in business and industry, by public utilities, and in private engineering and surveying offices. Implementation of a GIS depends on large quantities of data from a variety of sources, many of them consisting of observations made with the new instruments, such as those noted above, and others collected by instruments no longer used in practice.

However, before data can be utilized whether for surveying and mapping projects, for engineering design, or for use in a GIS, they must be processed. One of the most important aspects of this is to account for the fact that *no measurements are exact. That is, they always contain errors*.

The steps involved in accounting for the existence of errors in observations consist of (1) performing statistical analyses of the observations to assess the magnitudes of their errors, studying their distributions, and determining whether they are within acceptable tolerances and if the observations are acceptable and (2) adjusting them so they conform to exact geometric conditions or other required constraints. Procedures for performing these two steps in processing measured data are the principal subjects of this text.

1.2 DIRECT AND INDIRECT MEASUREMENTS

Measurements are defined as observations made to determine unknown quantities. They may be classified as either direct or indirect. *Direct measurements* are made by applying an instrument directly to the unknown quantity and observing its value, usually by reading it directly from graduated scales on the device. Determining the distance between two points by making a direct measurement using a graduated tape or measuring an angle by making a direct observation from the graduated circle of a total station instrument are examples of direct measurements.

Indirect measurements are obtained when it is not possible or practical to make direct measurements. In such cases, the quantity desired is determined from its mathematical relationship to direct measurements. For example, surveyors may observe angles and lengths of lines between points directly and use these observations to compute station coordinates. From these coordinate values, other distances and angles that were not observed directly may be derived indirectly by computation. During this procedure, the errors that were present in the original direct observations are *propagated* (distributed) by the computational process into the indirect values. Thus, the indirect measurements (computed station coordinates, distances, directions, and angles) contain errors that are functions of the original errors. This distribution of errors is known as *error propagation*. The analysis of how errors propagate is also a principal topic of this text.

1.3 MEASUREMENT ERROR SOURCES

It can be stated unconditionally that (1) *no measurement is exact*, (2) *every measurement contains errors*, (3) *the true value of a measurement is never known*, *and thus* (4) *the exact size of the error present is always unknown*. These facts can be illustrated by the following. If an angle is measured with a scale divided into degrees, its value can be read only to perhaps the nearest tenth of a

degree. However, if a better scale graduated in minutes were available and read under magnification, the same angle might be estimated to tenths of a minute. With a scale graduated in seconds, a reading to the nearest tenth of a second might be possible. From the foregoing, it should be clear that no matter how well the observation is taken, a better one may be possible. Obviously, in this example, observational accuracy depends on the division size of the scale. But accuracy depends on many other factors, including the overall reliability and refinement of the equipment used, environmental conditions that exist when the observations are taken, and human limitations (e.g., the ability to estimate fractions of a scale division). As more precise equipment is developed, environmental conditions improve, and observer ability increases, observations will approach their true values more closely, but they can never be exact.

An *error* is the difference between a measured value for any quantity and its true value, or

$$\varepsilon = y - \mu \tag{1.1}$$

where ε is the error in an observation, y the measured value, and μ its true value.

As discussed earlier, errors stem from three sources, which are classified as instrumental, natural, and personal. These are described as follows:

1. *Instrumental errors*. These errors are caused by imperfections in instrument construction or adjustment. For example, the divisions on a theodolite or total station instrument may not be spaced uniformly. These error sources are present whether the equipment is read manually or digitally.
2. *Natural errors*. These errors are caused by changing conditions in the surrounding environment. These include variations in atmospheric pressure, temperature, wind, gravitational fields, and magnetic fields.
3. *Personal errors*. These errors arise due to limitations in human senses, such as the ability to read a micrometer or to center a level bubble. The sizes of these errors are affected by personal ability to see and by manual dexterity. These factors may be influenced further by temperature, insects, and other physical conditions that cause humans to behave in a less precise manner than they would under ideal conditions.

1.4 DEFINITIONS

From the discussion thus far, it can be stated with absolute certainty that all measured values contain errors, whether due to lack of refinement in readings, instabilities in environmental conditions, instrumental imperfections, or human limitations. Some of these errors result from physical conditions that cause them to occur in a systematic way, whereas others occur with apparent randomness. Accordingly, errors are classified as either systematic or random. But before

defining systematic and random errors, it is helpful to define mistakes. These three terms are defined as follows:

1. *Mistakes*. These are caused by confusion or by an observer's carelessness. They are not classified as errors and must be removed from any set of observations. Examples of mistakes include (a) forgetting to set the proper parts-per-million (ppm) scaling correction on an electronic distance meter (EDM) instrument or failure to read the correct air temperature, (b) incorrectly reading graduated scales, and (c) recording erroneous values (i.e., writing down 27.55 for 25.75). Mistakes are also known as *blunders* or *gross errors*.
2. *Systematic errors*. These errors, which are sometimes called biases, follow some physical law and thus can be predicted. Some systematic errors are removed by following correct observational procedures (e.g., balancing backsight and foresight distances in differential leveling to compensate for earth curvature and refraction). Others are removed by deriving corrections based on the physical conditions that were responsible for their creation (e.g., applying a computed correction for earth curvature and refraction on a trigonometric leveling observation). Additional examples of systematic errors are (a) temperature not being standard while taping, (b) an indexing error of the vertical circle of a total station instrument, and (c) use of a level rod that is not standard length. Corrections for systematic errors can be computed and applied to observations to eliminate their effects.
3. *Random errors*. These are the errors that remain after all mistakes and systematic errors have been removed from the observed values. In general, they are the result of human and instrument imperfections. They are generally small and are as likely to be negative as positive. They usually do not follow any physical law and therefore must be dealt with according to the mathematical laws of probability. Examples of random errors are (a) imperfect centering over a point during distance measurement with an EDM instrument, (b) bubble not centered at the instant a level rod is read, and (c) small errors in reading graduated scales. It is impossible to avoid random errors in measurements entirely. Although they are often called accidental errors, their occurrence should not be considered an accident.

1.5 PRECISION VERSUS ACCURACY



Due to errors, repeated measurement of the same quantity will often yield different values. A *discrepancy* is defined as the algebraic difference between two observations of the same quantity. When small discrepancies exist between repeated observations, it is generally believed that only small errors exist. Thus, the tendency is to give higher credibility to such data and to call the observations *precise*. However, precise values are not necessarily accurate values.

To help understand the difference between precision and accuracy, the following definitions are given:

1. *Precision*. Precision is the degree of consistency between observations and is based on the sizes of the discrepancies in a data set. The degree of precision attainable is dependent on the stability of the environment during the time of measurement, the quality of the equipment used to make the observations, and the observer's skill with the equipment and observational procedures.
2. *Accuracy*. Accuracy is the measure of the absolute nearness of an observed quantity to its true value. Since the true value of a quantity can never be determined, accuracy is always an unknown.

The difference between precision and accuracy can be demonstrated using distance observations. Assume that the distance between two points is paced, taped, and measured electronically and that each procedure is repeated five times. The resulting observations, recorded in international feet (SI), are as follows:

Observation	Pacing (p)	Taping (t)	EDM (e)
1	571	567.17	567.133
2	563	567.08	567.124
3	566	567.12	567.129
4	588	567.38	567.165
5	557	567.01	567.114

The arithmetic means for these sets of data are 569, 567.15, and 567.133, respectively. A line plot illustrating relative values of the electronically measured distances denoted by e, and the taped distances denoted by t, is shown in Figure 1.1. Notice that although the means of the EDM data and of the taped observations are relatively close, the EDM set has smaller discrepancies. This indicates that the EDM instrument produced a higher precision. However, this higher precision does not necessarily prove that the mean of the EDM data is implicitly more accurate than the mean of the taped values. In fact, the opposite may be true if, for example, the reflector constant was entered incorrectly, causing a large systematic error to be

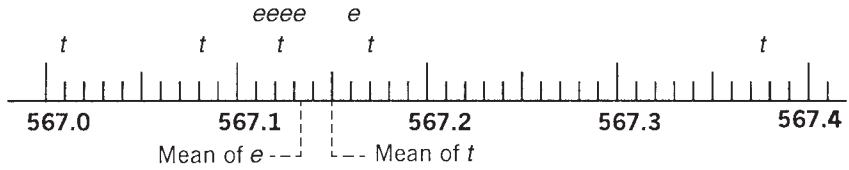


FIGURE 1.1 Line plot of distance quantities.

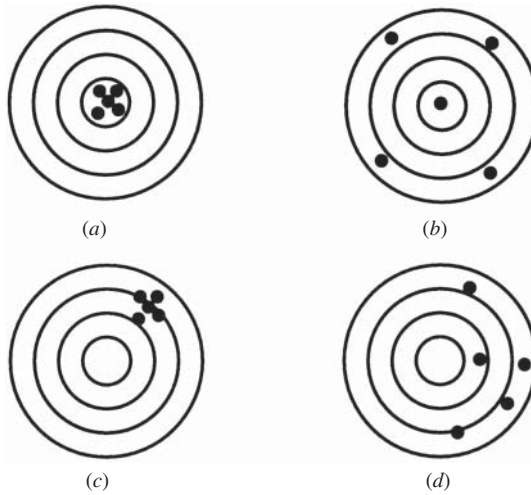


FIGURE 1.2 Examples of precision versus accuracy. (a) Accurate and precise, (b) Accurate when averaged but not precise, (c) precise but not accurate, and (d) neither accurate nor precise.

present in all the EDM readings. Because of the larger discrepancies, it is unlikely that the mean of the paced distances is as accurate as either of the other two values. But its mean could be more accurate if large systematic errors were present in both the taped and electronically measured distances.

Another illustration explaining differences between precision and accuracy involves target shooting, as depicted in Figure 1.2. As shown, four situations can occur. If accuracy is considered as closeness of shots to the center of a target at which a marksman shoots and precision as the closeness of the shots to each other, then (1) the data may be both precise and accurate, as shown in Figure 1.2(*a*); (2) the data may produce an accurate mean but not be precise, as shown in Figure 1.2(*b*); (3) the data may be precise but not accurate, as shown in Figure 1.2(*c*); or (4) the data may be neither precise nor accurate, as shown in Figure 1.2(*d*).

Figure 1.2(*a*) is the desired result when observing quantities. The other cases can be attributed to the following situations. The results shown in Figure 1.2(*b*) occur when there is little refinement in the observational process. Someone skilled at pacing may achieve these results. Figure 1.2(*c*) generally occurs when systematic errors are present in the observational process. This can occur, for example, in taping if corrections are not made for tape length and temperature, or with electronic distance measurements when using the wrong combined instrument–reflector constant. Figure 1.2(*d*) shows results obtained when the observations are not corrected for systematic errors and are taken carelessly by the observer (or the observer is unskilled at the measurement process).

In general, when making measurements, data such as those shown in Figure 1.2(*b*) and 1.2(*d*) are undesirable. Rather, results like those shown in

Figure 1.2(*a*) are preferred. However, in making measurements, the results of Figure 1.2(*c*) can be just as acceptable if proper steps are taken to correct for the presence of systematic errors. (This correction would be equivalent to the marksman realigning the sights after taking the shots.) To make these corrections, (1) the specific types of systematic errors that have occurred in the observations must be known, and (2) the procedures used in correcting them must be understood.

1.6 REDUNDANT OBSERVATIONS IN SURVEYING AND THEIR ADJUSTMENT

As noted earlier, errors exist in all observations. In surveying, the presence of errors is obvious in many situations, where the observations must meet certain conditions. In level loops that begin and close on the same benchmark, for example, the elevation difference for the loop must equal zero. However, in practice, this is hardly ever the case due to the presence of random errors. (For this discussion, it is assumed that all mistakes have been eliminated from the observations and appropriate corrections have been applied to remove all systematic errors.) Other conditions that disclose errors in surveying observations are that (1) the three measured angles in a plane triangle must total 180°, (2) the sum of the angles measured around the horizon at any point must equal 360°, and (3) the algebraic sum of the latitudes (and departures) must equal zero for closed traverses that begin and end on the same station. Many other conditions could be cited; however, in any of them, the observations rarely, if ever, meet the required conditions due to the presence of random errors.

The examples mentioned earlier not only demonstrate that errors are present in surveying observations but also illustrate the importance of *redundant observations*, which are those measurements that are more than the minimum number needed to determine the unknowns. Two measurements of the length of a line, for example, yield one redundant observation. The first observation would be sufficient to determine the unknown length, and the second is redundant. However, this second observation is very valuable. First, by examining the discrepancy between the two values, an assessment of the size of the error in the observations can be made. If a large discrepancy exists, a blunder or large error is likely to have occurred. In that case, observations of the line would be repeated until at least two values having an acceptably small discrepancy were obtained. Second, the redundant observation permits an adjustment to be made to obtain a final value for the unknown line length, and that final adjusted value will be more precise statistically than either of the individual observations. In this case, if the two observations were of equal precision, the adjusted value would be the simple mean.

Each of the specific conditions cited in the first paragraph of this section involves one redundant observation. For example, there is one redundant observation when the three angles of a planar triangle are observed. This is true because with two observed angles, say A and B, the third could be computed based on fundamental

geometric principles as $C = 180° - A - B$, and thus observation of C is unnecessary. However, measuring angle C enables an assessment of the errors in the angles to be made, and it also makes an adjustment possible to obtain final angles with statistically improved precision. Assuming that the angles were of equal precision, the adjustment would enforce the 180° sum for the three angles by distributing the total discrepancy in equal parts to each angle.

Although the examples cited here are indeed simple, they help to define redundant observations and to illustrate their importance. In large surveying networks, the number of redundant observations can become extremely large, and the adjustment process is somewhat more involved than it was for the simple examples given here.

Prudent surveyors always make redundant observations in their work, for the two important reasons indicated earlier: (1) to enable assessing errors and making decisions regarding acceptance or rejection of the observations and (2) to make possible an adjustment, whereby final values with higher precisions are determined for the unknowns.

1.7 ADVANTAGES OF LEAST SQUARES ADJUSTMENT

As indicated previously, in surveying, it is recommended that redundant observations always be made and that adjustments of the observations always be performed. These adjustments account for the presence of errors in the observations and increase the precision of the final values computed for the unknowns. When an adjustment is completed, all observations are corrected so that they are consistent throughout the survey network (i.e., the same values for the unknowns are determined no matter which corrected observation(s) are used to compute them).

Many methods have been derived for performing adjustments in surveying; however, the method of least squares should be used because it has significant advantages over all other rule-of-thumb procedures. The advantages of least squares over other methods can be summarized with the following four general statements: (1) it is the most rigorous of adjustments; (2) it can be applied with greater ease than other adjustments; (3) it enables rigorous post-adjustment analyses to be made; and (4) it can be used to perform pre-survey planning. These advantages are discussed further below.

The least squares method is rigorously based on the theory of mathematical probability, whereas, in general, other methods do not have this rigorous base. As described later in the book, in a least squares adjustment, the following condition of mathematical probability is enforced: *The sum of the squares of the errors times their respective weights is minimized*. By enforcing this condition in any adjustment, the set of errors that is computed has the highest probability of occurrence. Another aspect of least squares adjustment that adds to its rigor is that it permits all observations, regardless of their number or type, to be entered into the adjustment and used simultaneously in the computations. Thus, an adjustment can combine

distances, horizontal angles, azimuths, zenith or vertical angles, height differences, coordinates, and even GNSS observations. One important additional asset of the least squares method is that it enables "relative weights" to be applied to the observations in accordance with their expected relative reliabilities. These reliabilities are based on expected precisions. Thus, if distances were observed in the same survey by pacing, taping, and using an EDM instrument, they could all be combined in an adjustment by assigning appropriate relative weights.

Years ago, because of the comparatively heavy computational effort involved in least squares, non-rigorous or so-called rule-of-thumb adjustments were often applied. However, now that computers have eliminated the computational expense, the reverse is true, and least squares adjustments are performed more easily than these rule-of-thumb techniques. Least squares adjustments are less complicated because the same fundamental principles are followed regardless of the type of survey or the kind of observations. Also, the same basic procedures are used regardless of the geometric figures involved (e.g., triangles, closed polygons, quadrilaterals, or other more complicated networks). Rules of thumb, on the other hand, are not the same for all types of surveys (e.g., level nets use one rule and traverses use another), and the rules vary for different geometric shapes of each. Furthermore, the rule of thumb applied for a particular survey by one surveyor may be different from that applied by another surveyor. A favorable characteristic of least squares adjustments is that there is only one rigorous approach to the procedure, and thus, no matter who performs the adjustment for any survey, the same results are obtained.

The least squares method has the advantage that after an adjustment has been finished, a complete statistical analysis can be performed on the results. Based on the sizes and distribution of the errors, various tests can be conducted to determine if a survey meets acceptable tolerances or whether the observations must be repeated. If blunders exist in the data, these can be detected and eliminated. The least squares method enables the precisions for the adjusted quantities to be determined easily, and these precisions can be expressed in terms of error ellipses for a clear and lucid depiction. Procedures for accomplishing these tasks are described in subsequent chapters.

Besides its advantages in adjusting survey data, the least squares method can be used to plan surveys. In this application, prior to conducting a needed survey, simulated surveys can be run in a trial-and-error procedure. For any project, an initial trial geometric figure for the survey is selected. Based on the figure, trial observations are either computed or scaled. Relative weights are assigned to the observations in accordance with the precisions that are estimated by considering the different combinations of equipment and field procedures. A least squares adjustment of this initial network is then performed, and the results are analyzed. If goals have not been met, the geometry of the figure and the observation precisions are varied, and the adjustment is performed again. In this process, different types of observations can be used, and observations can be added or deleted. These different combinations of geometric figures and observations are varied until one is achieved that produces either optimum or satisfactory results. The survey

crew can then proceed to the field, confident that if the project is conducted with care according to the optimum design, satisfactory results will be obtained. This technique of applying the least squares method in survey planning is discussed further in later chapters.

1.8 OVERVIEW OF THE BOOK

In the remainder of the book, the interrelationship between observational errors and their adjustment is explored. In Chapters 2 through 5, methods used to determine the reliability of observations are described. In these chapters, the way that errors of multiple observations tend to be distributed is illustrated, and techniques used to compare the quality of different sets of observed values are examined. In Chapters 6 through 9 and in Chapter 13, methods used to model error propagation in observed and computed quantities are discussed. In particular, error sources present in traditional and modern surveying techniques are examined, and the ways these errors propagate throughout the observational and computational processes are explained. For example, Chapter 26 describes how to perform uncertainty estimation with terrestrial laser scanners. The fundamental *principles of least squares* to adjust observations in accordance with random error theory are presented in Chapter 11 with applications to level nets (Chapter 12), trilateration (Chapter 14), triangulation (Chapter 15), traverses (Chapter 16), GNSS networks (Chapter 17), and coordinate system transformations (Chapter 18). Strategies to improve adjustments, such as weighting (Chapter 10) and constraint equations (Chapter 20), and computer optimizations (Chapter 28), are shown. Several chapters are dedicated to the analysis of the quality of the results of an adjustment. Chapter 19 demonstrates how errors can be propagated to provide error ellipses. In Chapters 21 and 25, methods used to locate mistakes in sets of observations after an adjustment are explored. Chapter 22 presents the generalized least squares method with applications to geodetic control networks (Chapter 23), combined GNSS and terrestrial observations (Chapter 24), and geometric shape fitting (Chapter 27) of point cloud data. Several appendices provide fundamental statistical and linear algebra mathematical concepts with supporting computer code and statistical tables.

PROBLEMS

Note: Partial answers to problems marked with an asterisk are given in Appendix G.

1.1 Describe the steps involved in accounting for the existence of errors in observations.

1.2 List two examples of direct measurements.

1.3 List two examples of indirect measurements.

1.4 List four unconditional statements about measurements.

1.5 Define the sources of errors and provide an example of each.

1.6 State whether the following are mistakes, systematic errors, or random errors in the context of how they will affect the measurement.

(a) Earth curvature and refraction.

(b) Imperfect centering of an instrument over a point.

***(c)** Use of an uncalibrated thermometer resulting in the entry of incorrect temperature values for an EDM.

(d) Recording a length of 135.48 as 135.44.

(e) Recording an elevation difference of 95.041 m as 95.

1.7 Discuss the difference between a mistake and an error.

1.8 Is forgetting to change the temperature and pressure correction values for an EDM a mistake or a systematic error?

1.9 Give an example of

(a) a random instrumental error.

(b) a random natural error.

(c) a random personal error.

1.10 Define the term discrepancy.

1.11 Identify each of the following errors as either systematic or random.

(a) Bubble not precisely centered at the instance of reading a rod.

(b) Level compensator out of adjustment and not calibrated.

(c) A vertical circle indexing error on a total station.

***(d)** Use of a tape that is not of standard length.

(e) Reading the graduated horizontal circle on a total station.

1.12 Define discrepancy.

1.13 Identify each of the following errors according to their source (natural, instrumental, personal):

***(a)** Level rod length.

(b) EDM–reflector constant.

(c) Air pressure in an EDM observation.

(d) Precision in reading a graduation on a level rod.

(e) Earth curvature in leveling observations.

(f) An inclined line of sight in an automatic level.

***1.14** The calibrated length of a particular line is 100.003 m. A length of 99.999 m is obtained using an EDM. What is the error in the observation?

1.15 In Problem 1.14 if the observed length is 100.006 m, what is the error in the observation?

1.16 Why do surveyors measure angles using both faces of a total station (i.e., direct and reversed)?

1.17 List two examples of redundant observations in surveying.

1.18 A line is observed as 201.34 ft, 201.34 ft, 201.35 ft, and 201.38 ft. How many redundant observations were observed?

1.19 Why do prudent surveyors always obtain redundant measurements?

1.20 What mathematical condition is enforced in a least squares adjustment?

1.21 List at least three advantages that the least squares method has over rule-of-thumb adjustment procedures.

CHAPTER 2

OBSERVATIONS AND THEIR ANALYSIS

2.1 INTRODUCTION

Sets of data can be represented and analyzed using either graphical or numerical methods. Simple graphical analyses to depict trends commonly appear in newspapers or on television. A plot of the daily variation of the closing Dow Jones industrial average over the past year is an example. A bar chart showing daily high temperatures over the past month is another. While graphical forms are helpful for communication of information, when data are presented in numerical form, they can be subjected to numerical analysis. Instead of using the bar chart, the high temperatures could be tabulated and their mean computed over several time windows of interest. In surveying, observational data can also be represented and analyzed either graphically or numerically. In this chapter, some rudimentary methods for doing so are discussed.

2.2 SAMPLE VERSUS POPULATION



Due to time and financial constraints in statistical analyses, generally, only a small sample of data is collected from a much larger, possibly infinite population.

For example, political parties may wish to know the percentage of voters who support their candidate. It would be prohibitively expensive to query the entire voting population to obtain the desired information. Instead, polling agencies select a subset of voters from the voting population. This is an example of *population sampling*.

As another example, suppose that an employer wishes to determine the relative measuring capabilities of two prospective new employees. The candidates could theoretically spend days or even weeks demonstrating their abilities. Obviously, this would not be very practical, so instead, the employer could have each person record a sample of readings, and from the readings, predict the person's abilities. For instance, the employer could have each candidate read a micrometer 30 times. The 30 readings would represent a *sample* of the entire *population* of possible readings. In fact, in surveying, every time that distances, angles, or elevation differences are measured, samples are being collected from an infinite population of measurements.

From the preceding discussion, the following definitions can be made:

1. *Population*. A population consists of all possible measurements that can be made on a particular item or procedure. Often, a population has an infinite number of data elements.
2. *Sample*. A sample is a subset of data selected from the population.

2.3 RANGE AND MEDIAN

Suppose that a one-second (1″) micrometer theodolite is used to read a direction 50 times. The second's portions of the readings are shown in Table 2.1. These readings constitute what is called a *dataset*. How can these data be organized to make them more meaningful? How can one answer the question: Are the data representative of readings that should reasonably be expected with this instrument and a competent operator? What statistical tools can be used to represent and analyze this dataset?

TABLE 2.1 Fifty Readings

22.7	25.4	24.0	20.5	22.5
22.3	24.2	24.8	23.5	22.9
25.5	24.7	23.2	22.0	23.8
23.8	24.4	23.7	24.1	22.6
22.9	23.4	25.9	23.1	21.8
22.2	23.3	24.6	24.1	23.2
21.9	24.3	23.8	23.1	25.2
26.1	21.2	23.0	25.9	22.8
22.6	25.3	25.0	22.8	23.6
21.7	23.9	22.3	25.3	20.1

TABLE 2.2 Data in Ascending Order

20.1	20.5	21.2	21.7	21.8
21.9	22.0	22.2	22.3	22.3
22.5	22.6	22.6	22.7	22.8
22.8	22.9	22.9	23.0	23.1
23.1	23.2	23.2	23.3	23.4
23.5	23.6	23.7	23.8	23.8
23.8	23.9	24.0	24.1	24.1
24.2	24.3	24.4	24.6	24.7
24.8	25.0	25.2	25.3	25.3
25.4	25.5	25.9	25.9	26.1

One quick numerical method used to analyze data is to compute its *range*, also called *dispersion*. Range is the difference between the highest and lowest values. It provides an indication of the precision of the data. From Table 2.1, the lowest value is 20.1 and the highest is 26.1. Thus, the range is 26.1−20.1, or 6.0. The range for this dataset can be compared with the ranges of other sets, but this comparison has little value when the two sets differ in size. For instance, would a set of 100 data points with a range of 8.5 be better than the set in Table 2.1? Clearly, other methods of statistically analyzing datasets would be useful.

To assist in analyzing data, it is often helpful to list the values in order of increasing size. This was done with the data from Table 2.1 to produce the results shown in Table 2.2. By looking at this ordered set, it is possible to determine quickly the data's middle value or *midpoint*. In this example, it lies between the values of 23.4 and 23.5. The midpoint value is known as the *median*. Since there are an even number of values in this example, the median is given by the average of the two values closest to (which straddle) the midpoint. That is, the median is assigned the average of the 25th and 26th entries in the ordered set of 50 values, and thus for the dataset of Table 2.2, the median is the average of 23.4 and 23.5 or 23.45.

2.4 GRAPHICAL REPRESENTATION OF DATA



Although an ordered numerical tabulation of data allows for some data distribution analysis, it can be improved with a *frequency histogram*, usually simply called a *histogram*. Histograms are bar graphs that show frequency distributions indicating how often certain values appear in the dataset. To create a histogram, the data are divided into *classes*. These are subregions of data that usually have a uniform range in values, or *class width*. Although there are no universally applicable rules for the selection of class width, generally 5–20 classes are used. As a rule of thumb, a dataset of 30 values may have only five or six classes, whereas a dataset of 100 values may have as many as 15–20 classes. In general, the smaller the dataset, the lower the number of classes used.

The histogram *class width* (range of data represented by each histogram bar) is determined by dividing the total range by the number of classes to be used. For example, consider the data in Table 2.2. If they were divided into seven classes, the *class width* would be the range divided by the number of classes, or 6.0/7 = 0.857, or 0.86. The first *class interval* is found by adding the class width to the lowest data value. For the data in Table 2.2, the first class interval is from 20.1 to (20.1 + 0.86) or 20.96. This class interval includes all data from 20.1 up to, but not including, 20.96. The next class interval is from 20.96 up to (20.96 + 0.86) or 21.82. Remaining class intervals are found by adding the class width to the upper boundary value of the preceding class. The class intervals for the data in Table 2.2 are listed in column (1) of Table 2.3.

After creating class intervals, the number of data values in each interval is tallied. This is called the *class frequency*. Obviously, having data ordered consecutively as shown in Table 2.2 aids greatly in this counting process. Column (2) of Table 2.3 shows the class frequency for each class interval of the data in Table 2.2.

Often it is also useful to calculate the *class relative frequency* for each interval. This is found by dividing the class frequency by the total number of observations. For the data in Table 2.2, the class relative frequency for the first class interval is 2/50 = 0.04. Similarly, the class relative frequency for the fourth interval (from 22.67 to 23.53) is 13/50 = 0.26. The class relative frequencies for the data in Table 2.2 are given in column (3) of Table 2.3. *Notice that the sum of all class relative frequencies is always 1*. The class relative frequency enables easy determination of percentages, which are called *percentage points*. For instance, the class interval from 21.82 to 22.67 contains 16% (0.16 × 100%) of the sample observations.

A *histogram* is a bar graph plotted with either class frequencies or relative class frequencies on the ordinate, versus values of the class interval bounds on the abscissa. Using the data from Table 2.3, the histogram shown in Figure 2.1 was

TABLE 2.3 Frequency Table

(1) Class Interval	(2) Class Frequency	(3) Class Relative Frequency
20.10–20.96	2	2/50 = 0.04
20.96–21.82	3	3/50 = 0.06
21.82–22.67	8	8/50 = 0.16
22.67–23.53	13	13/50 = 0.26
23.53–24.38	11	11/50 = 0.22
24.38–25.24	6	6/50 = 0.12
25.24–26.10	7	7/50 = 0.14
	$\sum = 50$	$\sum = 50/50 = 1$

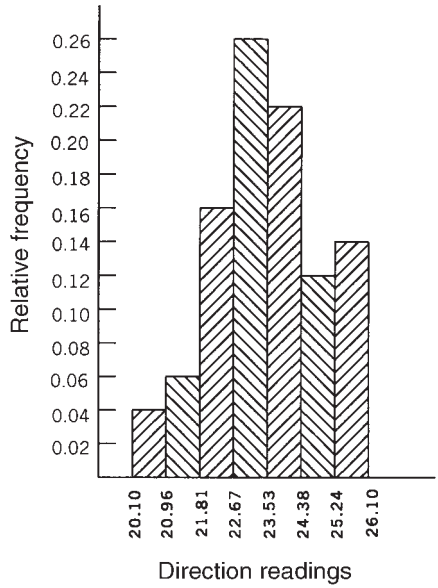


FIGURE 2.1 Frequency histogram.

constructed. Notice that in this figure, relative frequencies have been plotted as ordinates.

Histograms drawn with the same ordinate and abscissa scales can be used to compare two different datasets. If one dataset is more precise than the other, it will have comparatively tall bars in the center of the histogram, with relatively short bars near its edges. Conversely, the less precise dataset will yield a wider range of abscissa values, with shorter bars at the center.

A summary of items seen easily on a histogram includes:

- Whether the data are symmetrical about a central value.
- The range or dispersion in the measured values.
- The frequency of occurrence of the measured values.
- The steepness of the histogram, which is an indication of measurement precision.

Figure 2.2 shows several possible histogram shapes. Figure 2.2(*a*) depicts a histogram that is symmetric about its central value with a single peak in the middle. Figure 2.2(*b*) is also symmetric about the center but has a steeper slope than Figure 2.2(*a*), with a higher peak for its central value. Assuming the ordinate and abscissa scales to be equal, the data used to plot Figure 2.2(*b*) are more precise than

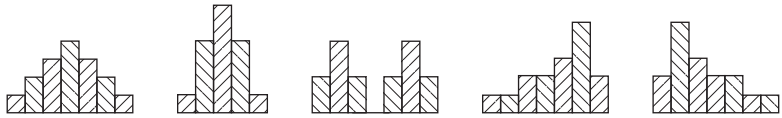


FIGURE 2.2 Common histogram shapes with (a) symmetry about its central value with single peak, (b) symmetry similar to a but with a steeper slope indicating higher precision, (c) a bimodal distribution with two peaks, (d) a distribution skewed to the right, and (e) a distribution skewed to the left.

those used for Figure 2.2(*a*). Symmetric histogram shapes are common in surveying practice, as well as in many other fields. In fact, they are so common that the shapes are said to be examples of a *normal* distribution. In Chapter 3, the reasons why these shapes are so common are discussed.

Figure 2.2(*c*) has two peaks and is said to be a *bimodal* histogram. In the histogram of Figure 2.2(*d*), there is a single peak with a long tail to the left. This results from a *skewed* dataset, and in particular, these data are said to be *skewed to the right*. The data of histogram Figure 2.2(*e*) is *skewed to the left*.

In surveying, the varying histogram shapes just described result from variations in personnel, physical conditions, and equipment – for example, repeated observations of a long distance made with an EDM instrument and by taping. The EDM procedure would probably produce data having a very narrow range; thus, the resulting histogram would be narrow and steep with a tall central bar like that in Figure 2.2(*b*). The histogram of the same distance measured by tape and plotted at the same scales would probably be wider with neither sides as steep nor the central value as great, as shown in Figure 2.2(*a*). Since observations in surveying practice tend to be normally distributed, bimodal or skewed histograms from measured data are not expected. The appearance of such a histogram should lead to an investigation for the cause of this shape. For instance, if a dataset from an EDM calibration plots as a bimodal histogram, it could raise questions about whether the instrument or reflector was moved during the measuring process, or if atmospheric conditions changed dramatically during the session. Similarly, a skewed histogram in EDM work may indicate the appearance of a weather front that stabilized over time. The existence of multipath errors in GNSS observations could also produce these types of histogram plots.

2.5 NUMERICAL METHODS OF DESCRIBING DATA

Numerical descriptors are values computed from a dataset that are used to interpret the data's precision or quality. Numerical descriptors fall into three categories: (1) *measures of central tendency*, (2) *measures of data variation*, and (3) *measures of relative standing*. These categories are all called *statistics*. *Simply described, a statistic is a numerical descriptor computed from sample data*.

2.6 MEASURES OF CENTRAL TENDENCY

Measures of central tendency are computed statistical quantities that give an indication of the value within a dataset that tends to exist at the center. The arithmetic mean, median, and mode are three such measures. They are described as follows:

1. *Arithmetic mean:* For a set of n observations, $y_1, y_2, \ldots, y_n$, the arithmetic mean is the average of the observations. Its value, $\bar{y}$, is computed from the following equation:

$$\bar{y} = \frac{\sum_{i=1}^{n} y_i}{n} \tag{2.1}$$

   Typically, the symbol $\bar{y}$ is used to represent the sample's arithmetic mean, and the symbol μ is used to represent the population mean. Otherwise, the same equation applies. Using Equation (2.1), the mean of the observations in Table 2.2 is 23.5.
2. *Median:* As mentioned previously, this is the midpoint of a sample set when arranged in ascending or descending order. One-half of the data are above the median and one-half are below it. When there are an odd number of quantities, only one such value satisfies this condition. For a dataset with an even number of quantities, the average of the two observations that straddle the midpoint is used to represent the median. Due to the relatively small number of observations in conventional surveying, it is seldom used. However, it can be very useful for filtering dense point cloud data sets, particularly given that it is less sensitive to outliers compared with the mean.
3. *Mode:* Within a sample of data, the mode is the most frequently occurring value. It is seldom used in surveying because of the relatively small number of values observed in a typical set of observations. Additionally, in small sample sets, several different values may occur with the same frequency, and hence, the mode can be meaningless as a measure of central tendency. The mode for the data in Table 2.2 is 23.8. It is possible for a set of data to have more than one mode. A common example is a dataset with two modes, which is said to be *bimodal*.

2.7 ADDITIONAL DEFINITIONS

Nine other terms, pertinent to the study of observations and their analysis, are listed and defined here:

1. *True Value*, μ*:* A quantity's theoretically correct or exact value. As noted in Section 1.3, the true value can never be determined.

2. *Error*, ε*:* The error is the difference between any individual observed quantity and its true value. The true value is simply the population's arithmetic mean if all repeated observations have equal precision. Since the true value of an observed quantity is indeterminate, errors are also indeterminate and are therefore only theoretical quantities. As given in Equation (1.1), and repeated for convenience here, errors are expressed as

$$\varepsilon_i = y_i - \mu \tag{2.2}$$

where y_i is the individual observation associated with ε_i, and μ is the true value for that quantity.

3. *Most Probable Value*, $\bar{y}$*:* The most probable value is that value for a measured quantity, which, based upon the observations, has the highest probability of occurrence. It is derived from a sample set of data rather than the population and is simply the mean if the repeated observations have the same precision.
4. *Residual*, v*:* A residual is the difference between any individual measured quantity and the most probable value for that quantity. Residuals are the values that are used in adjustment computations since the most probable values can be determined. The term *error* is frequently used when residual is meant, and although they are very similar and behave in the same manner, there is a theoretical distinction. The mathematical expression for a residual is

$$v_i = \bar{y} - y_i \tag{2.3}$$

where v_i is the residual in the ith observation, y_i, and $\bar{y}$ is the most probable value for the unknown.

5. *Degrees of Freedom:* Also called redundancies, the degrees of freedom are the number of observations that are more than the number necessary to solve for the unknowns. In other words, the number of degrees of freedom equals the number of *redundant* observations (see Section 1.6). As an example, if a distance between two points is measured three times, one observation would determine the unknown distance and the other two are redundant. These redundant observations reveal the discrepancies and inconsistencies in observed values. This, in turn, makes possible the practice of adjustment computations for obtaining the most probable values based on the measured quantities.
6. *Variance*, σ^2*:* This is a value by which the precision for a set of data is given. *Population variance* applies to a dataset consisting of an entire *population*. It is the mean of the squares of the *errors* and is given by:

$$\sigma^2 = \frac{\sum_{i=1}^{n} \varepsilon_i^2}{n} \tag{2.4}$$

Sample variance applies to a sample set of data. It is an unbiased estimate for the population variance given in Equation (2.4) and is calculated as:

$$S^2 = \frac{\sum_{i=1}^{n} v_i^2}{n-1} \tag{2.5}$$

Note that Equations (2.4) and (2.5) are identical except that ϵ has been changed to v, and n has been changed to $(n-1)$ in Equation (2.5). The validity of these modifications is demonstrated in Section 2.10.

It is important to note that the simple algebraic average of all errors in a dataset cannot be used as a meaningful precision indicator. This is because random errors are as likely to be positive as negative, and thus the algebraic average will equal zero. This fact is shown for a population of data in the following simple proof. Summing Equation (2.2) for n samples gives

$$\sum_{i=1}^{n} \epsilon_i = \sum_{i=1}^{n} (y_i - \mu) = \sum_{i=1}^{n} y_i - n\mu \tag{a}$$

Then substituting Equation (2.1) into Equation (*a*), yields:

$$\sum_{i=1}^{n} \epsilon_i = \sum_{i=1}^{n} y_i - n\left(\frac{\sum_{i=1}^{n} y_i}{n}\right) = \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} y_i = 0 \tag{b}$$

Similarly, it can be shown that the mean of all residuals of a sample dataset equals zero.

7. *Standard Error*, σ*:* This is the square root of the population variance. From Equation (2.4) and this definition, the following equation is written for the standard error:

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n} \epsilon_i^2}{n}} \tag{2.6}$$

where n is the number of observations, and $\sum_{i=1}^{n} \epsilon_i^2$ is the sum of the squares of the errors. Note that the population variance, σ^2, and standard error, σ, are indeterminate because true values, and hence errors, are indeterminate.

As will be explained in Section 3.5, 68.3% of all observations in a population dataset lie within $\pm\sigma$ of the true value, μ. Thus, the larger the standard error, the more dispersed are the values in the dataset and the less precise is the measurement.

8. *Standard Deviation*, *S:* This is the square root of the *sample variance*. It is calculated using the following expressions:

$$S = \sqrt{\frac{\sum_{i=1}^{n} v_i^2}{n-1}} \tag{2.7}$$

where S is the standard deviation, $n-1$ the degrees of freedom, and $\sum_{i=1}^{n} v_i^2$ the sum of the residuals squared. A standard deviation is an estimate for the standard error of the population. Since the standard error cannot be determined, the standard deviation is a practical expression for the precision of a *sample* set of data. Residuals are used rather than errors because they can be calculated from the most probable values, whereas errors cannot be determined. As discussed in Section 3.5 for a sample set of data, 68.3% of the observations will theoretically lie between the most probable value plus and minus the standard deviation, S. The meaning of this statement will be clarified in an example that follows.

9. *Standard Deviation of the Mean:* Because all observed values contain errors, the mean, which is computed from a sample set of measured values, will also contain an error. The standard deviation of the mean is computed from the sample standard deviation according to the following equation:

$$S_{\bar{y}} = \pm\frac{S}{\sqrt{n}} \tag{2.8}$$

Notice that as n approaches infinity, $S_{\bar{y}}$ approaches 0. This illustrates that as the size of the sample set approaches the total population, the computed mean $\bar{y}$ will approach the true mean μ. This equation is derived in Section 6.2.3.

2.8 ALTERNATIVE FORMULAS FOR DETERMINING VARIANCE

Calculation of the standard deviation in the previous section requires two passes through the data set. The first pass computes the mean and the second computes the residuals before the standard deviation can be computed. In computationally intensive applications, it can be advantageous to compute the standard deviation in a single pass. This section presents two such methods. From the definition of residuals, Equation (2.5) is rewritten as

$$S^2 = \frac{\sum_{i=1}^{n} (\bar{y} - y_i)^2}{n-1} \tag{2.9}$$

Expanding Equation (2.9) yields

$$S^2 = \frac{1}{n-1}\left[(\bar{y} - y_1)^2 + (\bar{y} - y_2)^2 + \cdots + (\bar{y} - y_n)^2\right] \qquad (c)$$

Substituting Equation (2.1) for $\bar{y}$ into Equation (c), and dropping the bounds for the summation,

$$S^2 = \frac{1}{n-1}\left[\left(\frac{\sum y_i}{n} - y_1\right)^2 + \left(\frac{\sum y_i}{n} - y_2\right)^2 + \cdots + \left(\frac{\sum y_i}{n} - y_n\right)^2\right] \qquad (d)$$

Expanding Equation (d),

$$S^2 = \frac{1}{n-1}\left[\begin{array}{c}\left(\frac{\sum y_i}{n}\right)^2 - 2y_1\frac{\sum y_i}{n} + y_1^2 + \left(\frac{\sum y_i}{n}\right)^2 - 2y_2\frac{\sum y_i}{n} \\ + y_2^2 + \cdots + \left(\frac{\sum y_i}{n}\right)^2 - 2y_n\frac{\sum y_i}{n} + y_n^2\end{array}\right] \qquad (e)$$

Rearranging Equation (e) and recognizing that $\left(\frac{\sum y_i}{n}\right)^2$ occurs n times in Equation (e) yields

$$S^2 = \frac{1}{n-1}\left[n\left(\frac{\sum y_i}{n}\right)^2 - 2\frac{\sum y_i}{n}(y_1 + y_2 + \cdots + y_n) + y_1^2 + y_2^2 + \cdots y_n^2\right] \qquad (f)$$

Adding the summation symbol to Equation (f) yields

$$S^2 = \frac{1}{n-1}\left[n\left(\frac{\sum y_i}{n}\right)^2 - \frac{2}{n}\left(\sum y_i\right)^2 + \sum y_i^2\right] \qquad (g)$$

Factoring and regrouping similar summations in Equation (g) produces

$$S^2 = \frac{1}{n-1}\left[\sum y_i^2 - \left(\frac{2}{n} - \frac{1}{n}\right)\left(\sum y_i\right)^2\right] = \frac{1}{n-1}\left[\sum y_i^2 - \frac{1}{n}\left(\sum y_i\right)^2\right] \qquad (h)$$

Multiplying the last term in Equation (h) by n/n yields

$$S^2 = \frac{1}{n-1}\left[\sum y_i^2 - n\left(\frac{\sum y_i}{n}\right)^2\right] \qquad (i)$$

Finally, by substituting Equation (2.1) in Equation (*i*), the following expression for the variance results:

$$S^2 = \frac{\sum y_i^2 - n\bar{y}^2}{n-1} \tag{2.10}$$

Using Equation (2.10), the variance of a sample dataset can be computed by subtracting n times the square of the data's mean from the summation of the squared individual observations. With this equation, the variance and the standard deviation can be computed directly from the data in a single pass. However, it should be stated that with large numerical values, Equation (2.10) may overwhelm a hand-held calculator or a computer working in single precision. The data should be *centered* or Equation (2.5) used when this problem arises. Centering a dataset involves subtracting a constant value (usually, the arithmetic mean or something near the mean) from all values in a dataset. By doing this, the values are modified to a smaller, more manageable size.

In some cases, when working with large point cloud data sets, even with the data centered, there can be precision loss with Equation (2.10). In these situations, an incremental one-pass approach developed by Welford (1962) and modified by Hoemmen (2007)[1] avoids the need to do the summations of potentially large values in the term $\sum y_i^2$. Instead, the approach incrementally computes the relatively small differences between values and updates. Two temporary variables called M_k and Q_k serve as proxies for the mean and variance. They are computed while looping through the values of y

$$M_k = \begin{cases} y_1 & k = 1 \\ M_{k-1} + \dfrac{y_k - M_{k-1}}{k} & k = 2, \ldots, n \end{cases} \tag{2.11a}$$

$$Q_k = \begin{cases} 0 & k = 1 \\ Q_{k-1} + \dfrac{(k-1)(y_k - M_{k-1})^2}{k} & k = 2, \ldots, n \end{cases} \tag{2.11b}$$

Upon completing the computation for all n values, the variance can be computed as

$$S^2 = \frac{Q_n}{n-1} \tag{2.12}$$

Mathematically, this approach is equivalent to Equation (2.10) but minimizes precision loss.

[1]Hoemmen, M. (2007). "Computing the Standard Deviation Efficiently" can be found at https://citeseerx.ist.psu.edu/document?repid=rep1&type=pdf&doi=d28cb3d22420bc144898b71373843f9114139206.

2.9 NUMERICAL EXAMPLES

Example 2.1 Using the data from Table 2.2, determine the sample set's mean, median, and mode and the standard deviation using both Equations (2.7) and (2.10). Also plot its histogram. (Recall that the data in Table 2.2 result from the second's portion of 50 total station directions.)

SOLUTION *Mean:* From Equation (2.1) and using the $\sum y_i$ value from Table 2.4, we have

$$\bar{y} = \frac{\sum_{i=1}^{50} y_i}{50} = \frac{1175}{50} = 23.5''$$

Median: Since there is an even number of observations, the data's midpoint lies between the values that are numerically 25th and 26th from the beginning of the ordered set. These values are 23.4 and 23.5, respectively. Averaging these observations yields 23.45″.

Mode: The mode, which is the most frequently occurring value, is 23.8″. It appears three times in the sample.

Range, *Class Width, Histogram:* These data were developed in Section 2.4, with the histogram plotted in Figure 2.1.

Standard Deviation: Table 2.4 lists the residuals [computed using Equation (2.3)], and their squares, for each observation.

From Equation (2.7) and using the value of 92.36 from Table 2.4 as the sum of the squared residuals, the standard deviation for the sample set is computed as

$$S = \sqrt{\frac{\sum v_i^2}{n-1}} = \sqrt{\frac{92.36}{50-1}} = \pm 1.37''$$

TABLE 2.4 Data Arranged for the Solution of Example 2.1

No.	y	v	v^2	No.	y	v	v^2	No.	y	v	v^2	No.	y	v	v^2
1	20.1	3.4	11.56	13	22.6	0.9	0.81	25	23.4	0.1	0.01	38	24.4	−0.9	0.81
2	20.5	3.0	9.00	14	22.7	0.8	0.64	26	23.5	0.0	0.00	39	24.6	−1.1	1.21
3	21.2	2.3	5.29	15	22.8	0.7	0.49	27	23.6	−0.1	0.01	40	24.7	−1.2	1.44
4	21.7	1.8	3.24	16	22.8	0.7	0.49	28	23.7	−0.2	0.04	41	24.8	−1.3	1.69
5	21.8	1.7	2.89	17	22.9	0.6	0.36	29	23.8	−0.3	0.09	42	25.0	−1.5	2.25
6	21.9	1.6	2.56	18	22.9	0.6	0.36	30	23.8	−0.3	0.09	43	25.2	−1.7	2.89
7	22.0	1.5	2.25	19	23.0	0.5	0.25	31	23.8	−0.3	0.09	44	25.3	−1.8	3.24
8	22.2	1.3	1.69	20	23.1	0.4	0.16	32	23.9	−0.4	0.16	45	25.3	−1.8	3.24
9	22.3	1.2	1.44	21	23.1	0.4	0.16	33	24	−0.5	0.25	46	25.4	−1.9	3.61
10	22.3	1.2	1.44	22	23.2	0.3	0.09	34	24.1	−0.6	0.36	47	25.5	−2.0	4.00
11	22.5	1.0	1.00	23	23.2	0.3	0.09	35	24.1	−0.6	0.36	48	25.9	−2.4	5.76
12	22.6	0.9	0.81	24	23.3	0.2	0.04	36	24.2	−0.7	0.49	49	25.9	−2.4	5.76
								37	24.3	−0.8	0.64	50	26.1	−2.6	6.76
												$\sum$ =	1175	0	92.36

Summing the squared y-values of Table 2.4 yields

$$\sum y_i^2 = 27{,}704.86$$

Using Equation (2.10), the standard deviation for the sample set is

$$S = \sqrt{\frac{\sum y_i^2 - n\bar{y}^2}{n-1}} = \sqrt{\frac{27{,}704.86 - 50(23.5)^2}{50-1}} = \sqrt{\frac{92.36}{49}} = \pm 1.37''$$

By demonstration in Example 2.1, Equations (2.7) and (2.10) will yield the same standard deviation for a sample set. Notice that the number of observations within a single standard deviation from the mean, that is, between $(23.5'' - 1.37'')$ and $(23.5'' + 1.37'')$, or between $22.13''$ and $24.87''$, is 34. This represents $34/50 \times 100\%$ or 68% of all observations in the sample and matches the theory noted earlier. Also note that the algebraic sum of residuals is zero as was earlier demonstrated by Equation (*b*).

The histogram, shown in Figure 2.1, plots class relative frequencies versus class values. Notice how the values tend to be grouped about the central point. This is an example of a precise dataset.

Example 2.2 The dataset shown below also represents the second's portion of 50 observations of a direction. Compute the mean, median, and mode, and use Equation (2.10) to determine the standard deviation. Also construct a histogram. Compare the data of this example with those of Example 2.1.

34.2	33.6	35.2	30.1	38.4	34.0	30.2	34.1	37.7	36.4
37.9	33.0	33.5	35.9	35.9	32.4	39.3	32.2	32.8	36.3
35.3	32.6	34.1	35.6	33.7	39.2	35.1	33.4	34.9	32.6
36.7	34.8	36.4	33.7	36.1	34.8	36.7	30.0	35.3	34.4
33.7	34.1	37.8	38.7	33.6	32.6	34.7	34.7	36.8	31.8

SOLUTION Table 2.5, which arranges each observation and its square in ascending order, is first prepared.

$$\textit{Mean: } \bar{y} = \frac{\sum y_i}{n} = \frac{1737.0}{50} = 34.74''$$

Median: The median is between the 25th and 26th values, which are both $34.7''$. Thus, the median is $34.7''$.

Mode: The data have three different values that occur with a frequency of 3. Thus, the modes for the dataset are the three values $32.6''$, $33.7''$, and $34.1''$.

Range: The range of the data is $39.3'' - 30.0'' = 9.3''$.

Class Width: For comparison purposes, the class width of 0.86 is taken since it was used for the data in Table 2.2. Since it is desired that the histogram be centered

TABLE 2.5 Data Arranged for the Solution of Example 2.2

No.	y	y^2	No.	y	y^2	No.	y	y^2	No.	y	y^2
1	30.0	900.00	13	33.5	1122.25	25	34.7	1204.09	38	36.3	1317.69
2	30.1	906.01	14	33.6	1128.96	26	34.7	1204.09	39	36.4	1324.96
3	30.2	312.04	15	33.6	1128.96	27	34.8	1211.04	40	36.4	1324.96
4	31.8	1011.24	16	33.7	1135.69	28	34.8	1211.04	41	36.7	1346.89
5	32.2	1036.84	17	33.7	1135.69	29	34.9	1218.01	42	36.7	1346.89
6	32.4	1049.76	18	33.7	1135.36	30	35.1	1232.01	43	36.8	1354.24
7	32.6	1062.76	19	34.0	1156.00	31	35.2	1239.04	44	37.7	1421.29
8	32.6	1062.76	20	34.1	1162.81	32	35.3	1246.09	45	37.8	1428.84
9	32.6	1062.76	21	34.1	1162.81	33	35.3	1246.09	46	37.9	1436.41
10	32.8	1075.84	22	34.1	1162.81	34	35.6	1267.36	47	38.4	1474.56
11	33.0	1089.00	23	34.2	1169.64	35	35.9	1288.81	48	38.7	1497.69
12	33.4	1115.56	24	34.4	1183.36	36	35.9	1288.81	49	39.2	1536.64
						37	36.1	1303.21	50	39.3	1544.49
									$\sum = 1737.0$		60,584.48

about the data's mean value, the central interval is determined by adding and subtracting one-half of the class width (0.43″) to the mean. Thus, the central interval is from (34.74″−0.43″), or 34.31″, to (34.74″ + 0.43″), or 35.17″. To compute the remaining class intervals, the class width is subtracted, or added, to the bounds of the central interval as necessary until all the data are contained within the bounds of the intervals. Thus, the interval immediately preceding the central interval will be from (34.31″−0.86″), or 33.45″, to 34.31″, and the interval immediately following the central interval will be from 35.17″ to (35.17″ + 0.86″), or 36.03″. In a similar fashion, the remaining class intervals were determined, and a class frequency chart was constructed as shown in Table 2.6. Using this table, the histogram of Figure 2.3 was constructed.

Variance: By Equation (2.10), using the sum of observations squared in Table 2.5, the sample variance is

$$S^2 = \frac{\sum y_i^2 - n\bar{y}^2}{n-1} = \frac{60{,}584.48 - 50(34.74)^2}{50-1} = 4.92$$

and the sample *standard deviation* is

$$S = \sqrt{4.92} = \pm 2.22''$$

The number of observations that fall within the bounds of the mean $\pm S$ (i.e., between $34.74'' \pm 2.22''$) is 30. This is 60% of all the observations and closely approximates the theoretical value of 68.3%. These bounds and the mean value are shown as dashed lines in Figure 2.3.

Comparison: The dataset of Example 2.2 has a larger standard deviation (±2.22″) than that of Example 2.1 (±1.37″). The range for the data for

TABLE 2.6 Frequency Table for Example 2.2

Class	Class Frequency	Class Relative Frequency
29.15–30.01	1	0.02
30.01–30.87	2	0.04
30.87–31.73	0	0.00
31.73–32.59	3	0.06
32.59–33.45	6	0.12
33.45–34.31	11	0.22
34.31–35.17	7	0.14
35.17–36.03	6	0.12
36.03–36.89	7	0.14
36.89–37.75	1	0.02
37.75–38.61	3	0.06
38.61–39.47	3	0.06
	$\sum = 50$	$\sum = 1.00$

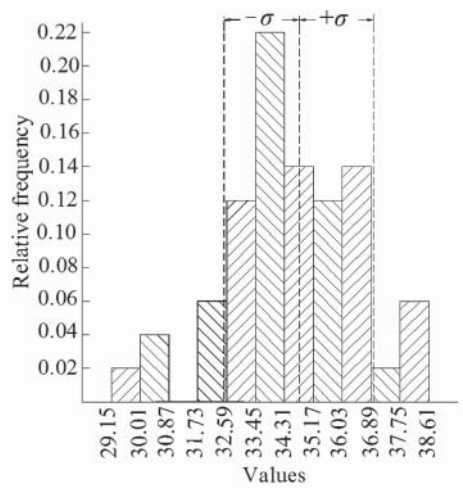


FIGURE 2.3 Histogram for Example 2.2.

Example 2.2 (9.3″) is also larger than that in Example 2.1 (6.0″). Thus, the dataset of Example 2.2 is less precise than that of Example 2.1. A comparison of the two histograms shows this precision difference graphically. For example, note that the histogram in Figure 2.1 is narrower in width and taller at the center than the histogram in Figure 2.3.

2.10 ROOT MEAN SQUARE ERROR AND MAPPING STANDARDS

Today, maps can be obtained in hardcopy and digital form. Thus, modern map accuracy standards such as the ASPRS Positional Accuracy Standards for Digital Geospatial Data (ASPRS, 2024[2]) are often based a statistical quantity known as the *root mean square error* (RMSE). To check the accuracy of a map, one must compute the discrepancy between the coordinates as determined from points on a map versus those same points observed with a higher-order check survey. Obviously, these checkpoints must be well-defined both on the map and on the surface of the earth in order to confidently obtain coordinate values in the same locations during the field survey. The ASPRS standard calls for at least 30 checkpoints distributed throughout the area.

Since the check survey must be of higher accuracy when compared with the map, these discrepancies are often considered to be residuals where the results of the check survey are considered to be without error and thus represent the true values. This concept of values determined from a higher-order survey being considered true values is not without precedence. For example, the distances listed on the coordinate datasheet for a station, or the lengths listed on an EDM calibration baseline report are often considered the true values since the surveys determining these values are considered to be more accurate than what a typical field survey would yield.

Notably, the current accuracies achievable with some mapping technologies such as light detection and ranging (lidar) have approached the achievable accuracies of the checkpoints, which are typically captured with Real-time Kinematic Global Navigation Satellite Systems (RTK GNSS). The ASRPS standard now considers the accuracy of the checkpoints, with a requirement that checkpoint accuracy is at least two times more accurate than the required accuracy for the geospatial product.

RMSE is defined as the square root of the average of squared residuals for values tested. Thus, for map accuracy standards, the differences between coordinates and elevations of points obtained from a map and their values as determined by a check survey are used to determine the accuracies of the map. Mathematically RMSE is computed as

$$\text{RMSE} = \sqrt{\sum_{i=1}^{n} \frac{[f(x_i) - x_i]^2}{n}} \tag{2.13}$$

where n is the number of tested samples from the map; $f(x_i)$ is the position of the point obtained from the map or dataset; x_i is the position obtained from the check survey; and $f(x_i) - x_i$ is the residual error for the point.

[2]ASPRS. (2023). ASPRS Positional Accuracy Standards for Digital Geospatial Data at https://publicdocuments.asprs.org/PositionalAccuracyStd-Ed2-V2.

As an example, consider the data shown in Table 2.7. The map coordinate values for the well-defined points are shown in columns (1)–(3). The check survey coordinate values are shown in columns (4)–(6). The discrepancies shown in columns (7)–(9) are computed as the difference between the mapped and surveyed coordinate values, which are listed as residuals. Thus, the residual for the *easting* coordinate of point 1 is computed as 672,571.819−672,571.777 = 0.042 m. Similarly, all other coordinate differences are determined.

The standard deviations for each coordinate type are computed as

$$S_E = \sqrt{\frac{\sum v_{E_i}^2}{20-1}} = \sqrt{\frac{0.017378}{19}} = \pm 0.030 \text{ m}$$

$$S_N = \sqrt{\frac{\sum v_{N_i}^2}{20-1}} = \sqrt{\frac{0.007031}{19}} = \pm 0.019 \text{ m}$$

$$S_H = \sqrt{\frac{\sum v_{H_i}^2}{20-1}} = \sqrt{\frac{0.033783}{19}} = \pm 0.042 \text{ m}$$

Using Equation (2.13), the RMSE in *easting* (RMSE_E), *northing* (RMSE_N), and *elevation* (RMSE_H) are computed from their corresponding residuals as

$$\text{RMSE}_E = \sqrt{\frac{\sum v_{E_i}^2}{20}} = \sqrt{\frac{0.017378}{20}} = \pm 0.029 \text{ m}$$

$$\text{RMSE}_N = \sqrt{\frac{\sum v_{N_i}^2}{20}} = \sqrt{\frac{0.007031}{20}} = \pm 0.019 \text{ m}$$

$$\text{RMSE}_H = \sqrt{\frac{\sum v_{H_i}^2}{20}} = \sqrt{\frac{0.033783}{20}} = 0.041 \text{ m}$$

From the RMSE_E and RMSE_N, a radial value for the horizontal positional accuracy is determined as

$$\text{RMSE}_r = \sqrt{\text{RMSE}_E^2 + \text{RMSE}_N^2} = \sqrt{0.029^2 + 0.019^2} = 0.035 \text{ m}$$

Following these computations, many standards require a certain confidence level (e.g., 95%) for the horizontal and vertical components of the map. Confidence levels of these values are discussed in Chapter 4.

TABLE 2.7 Map Coordinates Versus Surveyed Checkpoint Coordinates

	Mapped Points			Checkpoints			Residuals		
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Point	E (m)	N (m)	z (m)	E (m)	N (m)	H (m)	v_E	v_N	v_H
1	672,571.819	410,943.912	79.832	672,571.777	410,943.930	79.865	0.042	−0.018	−0.033
2	671,203.830	418,741.450	72.483	671,203.869	418,741.425	72.457	−0.039	0.025	0.026
3	671,203.830	426,812.590	91.565	671,203.847	426,812.566	91.627	−0.017	0.024	−0.062
4	660,396.717	427,222.982	99.340	660,396.666	427,222.983	99.377	0.051	−0.001	−0.037
5	637,824.897	425,170.999	77.340	637,824.849	425,170.984	77.306	0.048	0.015	0.034
6	638,372.093	409,165.527	71.839	638,372.105	409,165.507	71.830	−0.012	0.020	0.009
7	638,782.490	416,963.064	70.133	638,782.487	416,963.064	70.137	0.003	0.000	−0.004
8	651,504.788	426,128.591	78.531	651,504.796	426,128.575	78.571	−0.008	0.016	−0.040
9	643,980.848	422,571.819	81.486	643,980.840	422,571.835	81.414	0.008	−0.016	0.072
10	645,212.038	408,755.130	86.276	645,212.063	408,755.129	86.282	−0.025	0.001	−0.006
11	667,236.662	409,575.923	93.552	667,236.659	409,575.935	93.514	0.003	−0.012	0.038
12	664,911.081	422,982.216	75.410	664,911.068	422,982.213	75.387	0.013	0.003	0.023
13	655,198.358	415,595.075	71.369	655,198.395	415,595.052	71.433	−0.037	0.023	−0.064
14	647,674.419	414,774.282	80.002	647,674.397	414,774.273	80.041	0.022	0.009	−0.039
15	654,787.962	421,340.629	89.366	654,787.971	421,340.648	89.315	−0.009	−0.019	0.051
16	663,269.494	416,279.070	78.303	663,269.486	416,279.044	78.328	0.008	0.026	−0.025
17	656,984.354	408,996.255	81.205	656,984.379	408,996.226	81.209	−0.025	0.029	−0.004
18	668,113.208	431,698.113	76.001	668,113.253	431,698.112	76.087	−0.045	0.001	−0.086
19	655,660.377	431,797.820	72.424	655,660.433	431,797.795	72.446	−0.056	0.025	−0.022
20	643,962.264	430,943.396	84.189	643,962.266	430,943.429	84.179	−0.002	−0.033	0.010
						Sum	0.017378	0.007031	0.033783
						S	±0.030	±0.019	±0.042
						RMSE	±0.029	±0.019	±0.041
						RMSE_r	±0.035		

2.11 DERIVATION OF THE SAMPLE VARIANCE (BESSEL'S CORRECTION)

Recall from Section 2.7 that the denominator of the equation for sample variance was $n-1$, while the denominator of the population variance was n. A simple explanation for this difference is that one observation is necessary to compute the mean ($\bar{y}$), and thus only $n-1$ observations remain for the variance's computation. A derivation of Equation (2.5) will clarify.

Consider a sample size of n drawn from a population with a mean, μ, and standard error of σ. Let y_i be an observation from the sample, then

$$y_i - \mu = (y_i - \bar{y}) + (\bar{y} - \mu) = (y_i - \bar{y}) + \varepsilon \tag{j}$$

where $\varepsilon = (\bar{y} - \mu)$ is the *error* or deviation of the sample mean. Squaring and expanding Equation (j) yields

$$(y_i - \mu)^2 = (y_i - \bar{y})^2 + \varepsilon^2 + 2\varepsilon(y_i - \bar{y})$$

Summing all the observations in the sample from i equaling 1 to n yields

$$\sum_{i=1}^{n}(y_i - \mu)^2 = \sum_{i=1}^{n}(y_i - \bar{y})^2 + n\varepsilon^2 + 2\varepsilon\sum_{i=1}^{n}(y_i - \bar{y}) \tag{k}$$

Since by definition of the sample mean $\bar{y}$

$$\sum_{i=1}^{n}(y_i - \bar{y}) = \sum_{i=1}^{n}y_i - n\bar{y} = \sum_{i=1}^{n}y_i - \sum_{i=1}^{n}y_i = 0 \tag{l}$$

Equation (k) becomes

$$\sum_{i=1}^{n}(y_i - \mu)^2 = \sum_{i=1}^{n}(y_i - \bar{y})^2 + n\varepsilon^2 \tag{m}$$

Repeating this calculation for many samples, the mean value of the left-hand side of Equation (m) will (by definition of σ^2) tend to $n\sigma^2$. Similarly by Equation (2.8), the mean value of $n\varepsilon^2 = n(\mu - \bar{y})^2 n\varepsilon^2 = n(\mu - \bar{y})^2$ will tend to n times the variance of $\bar{y}$ since ε represents the deviation of the sample mean from the population mean. Thus, $n\varepsilon^2 \to n\left(\frac{\sigma^2}{n}\right)n\varepsilon^2 \to n\left(\frac{\sigma^2}{n}\right)$ where $\frac{\sigma^2}{n}$ is the variance in $\bar{y}$ as $n \to \infty$. $n \to \infty$. The above discussion and Equation (m) results in

$$n\sigma^2 \to \sum_{i=1}^{n}(y_i - \bar{y})^2 + \sigma^2 \tag{n}$$

Rearranging Equation (*n*) produces

$$\sum_{i=1}^{n} (y_i - \bar{y})^2 \rightarrow (n-1)\sigma^2 \qquad (o)$$

Thus, from Equation (*o*) and recognizing the left side of the equation as $(n-1)S^2$ for a sample set of data, it follows that

$$S^2 = \frac{\sum_{i=1}^{n} (y_i - \bar{y})^2}{n-1} \rightarrow \sigma^2 \qquad (p)$$

In other words, for a large number of random samples, the value of $\frac{\sum_{i=1}^{n} (y_i - \bar{y})^2}{(n-1)}$ approaches σ^2. That is, S^2 is an unbiased estimate of the population's variance.

2.12 SOFTWARE

A windows-based, statistical software package called STATS is available on the companion website for this book (http://www.wiley.com/college/ghilani). It can be used to quickly perform statistical analysis of datasets as presented in this chapter. The data file used in STATS is simply a listing of the individual observations. For example, in Example 2.1, the data file can be entered as it is shown in Table 2.1. After saving this file, the "Histogram data" option under the programs menu is selected. After entering the appropriate file into the software, the software performs the computations discussed in this chapter and plots a frequency histogram of the data using the user-specified class interval or the desired number of classes.

Additionally, an electronic book is provided on the companion website for this book. To view the electronic book interactively, the Mathcad® software is required. However, for those who do not have a copy of Mathcad®, html files of the electronic book are also on the website. The electronic book demonstrates most of the numerical examples provided in this book. In particular, the electronic book "c2.xmcd" demonstrates the use of Mathcad® to solve Examples 2.1 and 2.2.

Also, a spreadsheet can be used to perform the computations in this chapter. For example, Microsoft Excel® has functions for determining the mean, median, mode, standard deviation, and histogram of data. The *average* function computes the mean for a selected set of data. The *stdev.s* function computes the standard deviation for a selected sample set of data. Similarly, the *mode* and *median* functions determine these values for a set of data, and the *min* and *max* functions determine the minimum and maximum values for the data. Additionally, the software can automatically histogram the data based on the first number of the class intervals, which is known as a *bin* number. These functions are demonstrated for Example 2.1 in the file *c2.xls* on the companion website.

Many of the chapters have programming problems listed at the end of each chapter. The electronic book demonstrates the rudiments of programming these problems. Other programs on the companion website include MATRIX and ADJUST. MATRIX can be used to solve problems involving matrices in this book. ADJUST has working least squares adjustment examples discussed in this book. ADJUST can be used to check solutions to many of the problems in this book.

The installation software for the programs ADJUST, MATRIX, and STATS is available in the zip file on the website. This software is available as an aid for learning the material in this book. Purchasers of this book may install this software on their computers. The spreadsheet and worksheet files discussed in this book can be copied from the companion website to your computer. The Mathcad® e-book should be copied to the *handbook* subdirectory of the Mathcad program. If you do not own Mathcad, html files of the e-book are provided, which can be copied and viewed once you have unpacked the zipped archive from the companion website. Readers should refer to Appendix F for specific details about the software on the website.

PROBLEMS

Note: Partial answers to problems marked with an asterisk are given in Appendix G.

***2.1** The optical micrometer of a precise differential level is set and read 10 times as 6.733, 6.735, 6.734, 6.734, 6.736, 6.731, 6.741, 6.735, 6.733, and 6.736. What value would you assign to the operator's ability to set the micrometer on this instrument?

2.2 A distance measured in units of meters is observed 10 times as 200.532, 200.538, 200.534, 200.537, 200.533, 200.534, 200.533, 200.536, 200.535, and 200.530. What is the:

(a) Range of the data?

(b) Mean?

(c) Median?

***(d)** Mode?

2.3 Using the data in Problem 2.2, tabulate the residuals and compute the variance, standard deviation, and standard deviation of the mean.

2.4 The second's portion of 10 pointings and readings for a particular direction 47.2, 47.9, 46.8, 44.8, 44.4, 43.4, 43.9, 45.7, 37.9, and 48.3. What is the:

***(a)** Largest discrepancy in the data?

(b) Mean?

(c) Median?

(d) Mode?

2.5 Using the data in Problem 2.4, tabulate the residuals and compute the variance, standard deviation, and standard deviation of the mean.

2.6 The second's portion of 32 pointings and readings for a particular direction made using a 1″ total station with a 0.1″ display are:
34.8, 33.3, 34.4, 34.8, 34.9, 35.8, 35.4, 36.0, 35.4, 35.5, 33.8, 36.3, 35.7, 35.3, 35.9, 35.5, 33.6, 34.1, 35.3, 35, 34.4, 34.2, 35.8, 35.2, 34.9, 33.4, 34.6, 35.6, 34.7, 33.4, 34.2, 35.5, 34.7, 33.6, 34.6, 34.2, 33.8, 34.2, 34.4, 34.4, 34.9, 34, 35.7, 36.1, 34.7, 32.8, 34.1, 34.7, 34.3, and 33.3.

(a) What is the mean of the dataset?

(b) Construct a frequency histogram of the data using seven uniform-width class intervals.

***(c)** What are the variance and standard deviation of the data?

(d) What is the standard deviation of the mean?

2.7 An EDM instrument and reflector are set at the ends of a baseline that is 300.014 m long. Its length is measured 23 times, with the following results: 300.164, 300.167, 300.174, 300.170, 300.173, 300.174, 300.173, 300.166, 300.171, 300.166, 300.174, 300.169, 300.168, 300.178, 300.173, 300.163, 300.173, 300.165, 300.173, 300.175, 300.166, 300.177, and 300.169.

(a) What are the mean, median, and standard deviation of the data?

(b) Construct a histogram of the data with seven intervals and describe its properties. On the histogram, lay off the sample standard deviation from both sides of the mean.

(c) How many observations are between $\bar{y} \pm S, \bar{y} \pm S$, and what percentage of observations does this represent?

2.8 Answer Problem 2.7 with the following additional observations. 300.169, 300.169, 300.170, 300.174, and 300.171.

2.9 Answer Problem 2.8 with the following additional observations. 300.166, 300.175, 300.167, and 300.171.

2.10 A distance was measured in two parts with a 100-ft. steel tape and then in its entirety with a 200-ft. steel tape. Five repetitions were made by each method. What are the mean, variance, and standard deviation for each method of measurement?

Distance Measured with 100-ft Tape	Distances Measured with 200-ft Tape
Section 1 100.007, 100.003, 100.002, 100.008, 100.005	136.763, 136.764, 136.757, 136.762, 136.761
Section 2 36.756, 36.759, 36.761, 36.757, 36.758	

2.11 Repeat Problem 2.10 using the following additional data for the 200-ft taped distance. 136.762, 136.761, 136.760, 136.763, and 136.759.

2.12 During a triangulation project, an observer made 16 readings for each direction. The second's portion of the directions to Station Orion are listed as 35.8, 35.2, 34.9, 33.4, 34.6, 35.6, 34.7, 33.4, 34.2, 35.5, 34.7, 33.6, 34.6, 34.2, 33.8, and 34.2.

(a) Using a 0.4″ class interval, plot the histogram using relative frequencies for the ordinates.

(b) Analyze the data and note any abnormalities.

(c) As a supervisor, would you recommend reobservation of the station?

2.13 A line in a survey is measured three times on four separate occasions. The resulting 12 observations in units of meters are 485.319, 485.327, 485.323, 485.316, 485.320, 485.321, 485.320, 485.320, 485.324, 485.319, 485.318, and 485.322.

(a) Compute the mean, median, and mode of the data.

(b) Compute the variance and standard deviation of the data.

(c) Using a class width of 0.004 m, plot a histogram of the data, and note any abnormalities that may be present.

2.14 Repeat Problem 2.13 but use a class width of 0.002 m in part (c).

2.15 During a triangulation project, an observer made 32 readings for each direction using a 3″ total station. The second's portions of the directions are listed below. Using seven class intervals, plot the histogram with relative frequencies for the ordinates. Analyze the data and state whether this set appears to be reasonable.
18, 17, 19, 13, 23, 18, 14, 18, 22, 21, 22, 17, 17, 20, 20, 16, 24, 16, 17, 20, 21, 20, 22, 16, 26, 21, 17, 21, 25, 24, 25, and 20.

2.16 Two students have an argument over who can turn an angle better. To resolve the argument, they agree to each measure a single angle 10 times. The results of the observations are:

Student A	Student B
175°13′12″, 175°13′01″, 175°13′05″, 175°13′15″, 175°13′17″, 175°13′08″, 175°13′15″, 175°13′09″, 175°13′13″, 175°13′18″	175°13′06″, 175°13′05″, 175°13′15″, 175°13′04″, 175°13′10″, 175°13′13″, 175°13′08″, 175°13′12″, 175°13′16″, 175°13′09″

(a) What are the means and variances of both datasets?

(b) Construct a histogram of each dataset using a 3″ class width.

(c) Which student performed the best in this situation?

Use the program STATS to do Problems 2.17–2.21.

2.17 Compute the mean, median, mode, and standard deviation of the data in Table 2.2 and plot a centered histogram of the data using nine intervals.

2.18 Problem 2.6.

2.19 Problem 2.7.

2.20 Problem 2.10.

2.21 Problem 2.16.

2.22 Compute the standard deviation, root mean square error, and horizontal root mean square error for the map and check survey coordinates listed in the following table.

	Map Coordinates			Check Survey		
Point	*e* (m)	*n* (m)	Point	*e* (m)	*n* (m)	Point
1	110,939.576	34,028.488	1	110,939.576	34,028.488	1
2	110,954.298	34,047.097	2	110,954.298	34,047.097	2
3	110,979.632	34,078.005	3	110,979.632	34,078.005	3
4	110,943.509	34,112.142	4	110,943.509	34,112.142	4
5	110,912.703	34,019.536	5	110,912.703	34,019.536	5
6	110,877.952	34,025.923	6	110,877.952	34,025.923	6
7	110,873.027	34,046.191	7	110,873.027	34,046.191	7
8	110,832.314	34,031.750	8	110,832.314	34,031.750	8
9	110,878.363	33,991.541	9	110,878.363	33,991.541	9
10	110,833.958	33,997.846	10	110,833.958	33,997.846	10
11	110,819.256	33,954.075	11	110,819.256	33,954.075	11
12	110,791.832	33,957.849	12	110,791.832	33,957.849	12
13	110,747.492	33,967.866	13	110,747.492	33,967.866	13
14	110,794.562	33,934.575	14	110,794.562	33,934.575	14
15	110,835.511	33,904.681	15	110,835.511	33,904.681	15
16	110,833.157	33,934.369	16	110,833.157	33,934.369	16
17	110,840.519	33,960.096	17	110,840.519	33,960.096	17
18	110,923.004	33,849.047	18	110,923.004	33,849.047	18
19	110,813.632	33,868.653	19	110,813.632	33,868.653	19
20	110,785.320	33,846.409	20	110,785.320	33,846.409	20

PRACTICAL EXERCISES

2.23 Using a total station, point and read horizontal circle to a well-defined target. With the tangent screw or jog-shuttle mechanism, move the instrument

of the point and repoint on the same target. Record this reading. Repeat this process 50 times. Perform the calculations of Problem 2.6 using this dataset.

2.24 Determine your EDM/reflector constant, K, by observing the distances between three points that are online, as shown in the figure. Distance AB should be roughly 60 m long and BC roughly 90 m long with B situated at some location between A and C. From measured values AC, AB, and BC, the constant K can be determined as follows:



Since

$$AC + K = (AB + K) + (BC + K)$$

thus

$$K = AC - (AB + BC)$$

When establishing the line, be sure that $AB \neq BC$ and that all three points are precisely on a straight line. Best results are obtained when the lengths AB and BC are multiples of the fundamental wavelength of the instrument. Use three tripods and tribrachs to minimize setup errors and be sure that all are in adjustment. Measure each line 20 times with the instrument in the metric mode. Be sure to adjust the distances for the appropriate temperature and pressure and for differences in elevation. Determine the 20 values of K and analyze the sample set. What is the mean value for K and what is its standard deviation?

CHAPTER 3

RANDOM ERROR THEORY

3.1 INTRODUCTION

As noted earlier, the adjustment of measured quantities containing random errors is a major concern to people involved in the geospatial sciences. In the remaining chapters, it is assumed that all systematic errors have been removed from the observed values and that only random errors and blunders, which have escaped detection, remain. In this chapter, the general theory of random errors is developed, and some simple methods that can be used to isolate blunders in sets of data are discussed.

3.2 THEORY OF PROBABILITY



Probability is the ratio of the number of times that an event should occur to the total number of possibilities. For example, the probability of tossing a two with a fair die is 1/6 since there are six total possibilities (faces on a die) and only one of these is a two. When an event can occur in m ways and fails to occur in n ways, then the probability of its occurrence is $m/(m+n)$ and the probability of its failure is $n/(m+n)$.

Probability is always a fraction ranging between zero and 1. Zero denotes impossibility and one indicates certainty. Since an event must either occur or fail to occur, the sum of all probabilities for any event is 1, and thus, if 1/6 is the probability of throwing a two with one throw of a die, then $1 - 1/6$, or 5/6 is the probability that a two will not appear.

In probability terminology, a *compound event* is the simultaneous occurrence of two or more independent events. This is the situation encountered most frequently in surveying. For example, random errors from angles and distances (compound events) cause traverse misclosures. The probability of the simultaneous occurrence of two independent events is the product of their individual probabilities.

To illustrate this condition, consider the simple example of having two boxes containing combinations of red and white balls. Box *A* contains four balls, one red and three white. Box *B* contains five balls, two red and three white. What is the probability that two red balls would be drawn if one ball is drawn randomly from each box? The total number of possible pairs is 4×5, or 20, since by drawing one ball from box *A*, any of the five balls in box *B* would complete the pair. Now, there are only two ways to draw two red balls. That is, box *A*'s red ball can be matched with either red ball from box *B*. Therefore, the probability of obtaining two red balls simultaneously is 2/20. The probability of this compound event can be computed as the product of the individual probabilities of drawing a red ball from each box or

$$P = 1/4 \times 2/5 = 2/20.$$

Similarly, the probability of simultaneously drawing two white balls is $3/4 \times 3/5$ or 9/20, and the probability of getting one red and one white is $1 - (2/20 + 9/20)$ or 9/20.

From the foregoing, it is seen that the probability of the simultaneous occurrence of two independent events is the product of the individual probabilities of those two events. This principle is extended to include any number of events

$$P = P_1 \times P_2 \times \cdots \times P_n \tag{3.1}$$

where P is the probability of the simultaneous occurrence of events having individual probabilities $P_1, P_2, \ldots, P_n$.

To develop the principle of how random errors occur, consider a very simple example where a single tape measurement is taken between points *A* and *B*. Assume that this measurement contains a single random error of size 1. Since the error is random, there are two possibilities for the value of the resultant error, +1 or −1. Let t be the number of ways each resultant error can occur, and T be the total number of possibilities, which is two. The probability of obtaining +1, which can occur only one way (i.e., $t = 1$), is t/T or $\frac{1}{2}$. This is also the probability of obtaining −1. Suppose now that in measuring a distance *AE*, the tape must be placed end-to-end so that the result depends on the combination of two of these tape measurements. Then the possible error combinations in the result are −1 and −1, −1 and +1, +1

TABLE 3.1 Occurrence of Random Errors

(1) No. of combining measurements	(2) Value of resultant error	(3) Frequency, t	(4) Total No. of Possibilities, T	(5) Probability
1	+1	1		1/2
	−1	1	2	1/2
2	+2	1		1/4
	0	2	4	1/2
	−2	1		1/4
3	+3	1		1/8
	+1	3		3/8
	−1	3	8	3/8
	−3	1		1/8
4	+4	1		1/16
	+2	4		1/4
	0	6	16	3/8
	−2	4		1/4
	−4	1		1/16
5	+5	1		1/32
	+3	5		5/32
	+1	10		5/16
	−1	10	32	5/16
	−3	5		5/32
	−5	1		1/32

and −1, and +1 and +1, with $T = 4$. The final errors are −2, 0, and +2, and their t values are 1, 2, and 1, respectively. This produces probabilities of 1/4, 1/2, and 1/4, respectively. In general, as n, the number of single combined measurements, is increased, T increases according to the function $T = 2^n$, and thus for three combined measurements, $T = 2^3 = 8$, and for four measurements, $T = 2^4 = 16$.

The analysis of the preceding paragraph can be continued to obtain the results shown in Table 3.1. Figure 3.1(*a*) through (*e*) are histogram plots of the results in

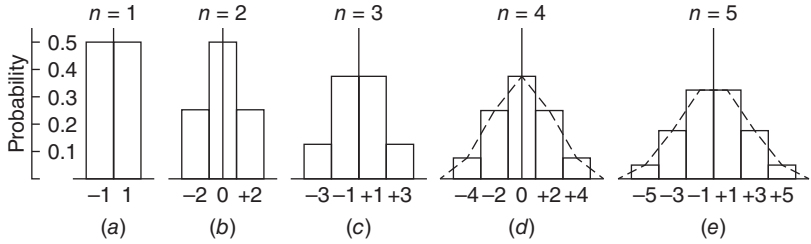


FIGURE 3.1 Plots of probability versus size of errors for examples shown in Table 3.1 for (a) $n = 1$, (b) $n = 2$, (c) $n = 3$, (d) $n = 4$, and (e) $n = 5$ combining measurements.

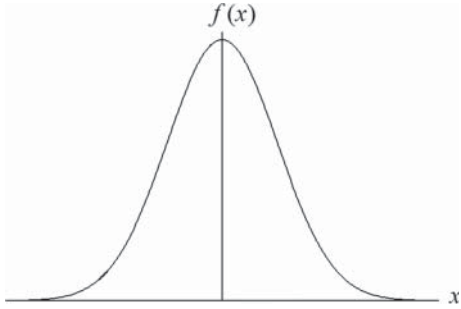


FIGURE 3.2 The normal distribution curve.

Table 3.1, where the values of the resultant errors are plotted as the abscissas, and the probabilities are plotted as ordinates of equal-width bars.

If the number of combining measurements, n, is increased progressively to larger values, the plot of error sizes versus probabilities would approach a smooth curve of the characteristic bell-shape shown in Figure 3.2. This curve is known as the *normal error distribution curve*. It is also called the *probability density function of a normal random variable*. Notice that when n is 4, as illustrated in Figure 3.1(d), and when $n = 5$, as shown in Figure 3.1(e), the dashed lines are already beginning to take on this form.

It is important to notice that the total area of the vertical bars for each plot equals 1. This is true no matter the value of n, and thus *the area under the smooth normal error distribution curve is equal to* 1. If an event has a probability of 1, it is certain to occur, and therefore, *the area under the curve represents the sum of all the probabilities of the occurrence of errors*.

As derived in Section D.1, the equation for the normal distribution curve, also called the *normal probability density function*, is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2} \tag{3.2}$$

where $f(x)$ is the probability density function, e the base of natural logarithms, x the error, and σ the standard error as defined in Chapter 2. Note that Equation 3.2 assumes that the mean is zero. If the mean is not zero, then the curve should be shifted horizontally by the mean. In other words, you would replace x with $x - \mu$ in Equation 3.2.

3.3 PROPERTIES OF THE NORMAL DISTRIBUTION CURVE

In Equation (3.2), $f(x)$ is the probability of the occurrence of an error between x and $x + dx$, where dx is an infinitesimally small value. The error's probability is

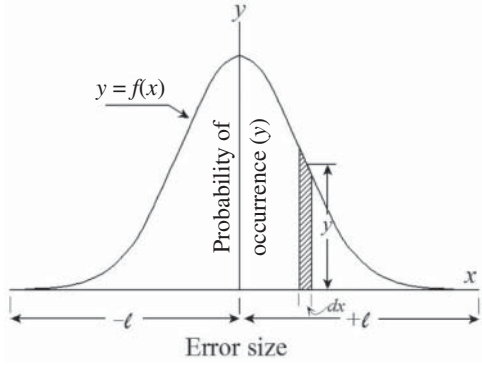


FIGURE 3.3 Normal density function.

equivalent to the area under the curve between the limits of x and $x + dx$, which is shown crosshatched in Figure 3.3. As stated previously, the total area under the probability curve represents the total probability, which is 1. This is represented in equation form as

$$\text{area} = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2} dx = 1 \tag{3.3}$$

Let y represent $f(x)$ in Equation (3.2) and differentiate

$$\frac{dy}{dx} = -\frac{x}{\sigma^2}\left(\frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}\right) \tag{3.4}$$

Recognizing the term in parentheses of Equation (3.4) as y gives

$$\frac{dy}{dx} = -\frac{x}{\sigma^2} y \tag{3.5}$$

Taking the second derivative of Equation (3.2), we obtain

$$\frac{d^2y}{dx^2} = -\frac{x}{\sigma^2}\frac{dy}{dx} - \frac{y}{\sigma^2} \tag{3.6}$$

Substituting Equation (3.5) into Equation (3.6) yields

$$\frac{d^2y}{dx^2} = \frac{x^2}{\sigma^4} y - \frac{y}{\sigma^2} \tag{3.7}$$

Equation (3.7) can be simplified to

$$\frac{d^2y}{dx^2} = \frac{y}{\sigma^2}\left(\frac{x^2}{\sigma^2} - 1\right) \tag{3.8}$$

From calculus, the first derivative of a function gives the slope of the function when evaluated at a point. In Equation (3.5), $dy/dx = 0$ when the values of x or y equal zero. This implies that the curve is parallel to the x axis at the center of the curve when x is zero and is asymptotic to the x axis as y approaches zero.

Also from calculus, a function's second derivative provides the rate of change in a slope when evaluated at a point. The curve's inflection points (points where the algebraic sign of the slope changes) can be located by finding where the function's second derivative equals zero. In Equation (3.8), $d^2y/dx^2 = 0$ when $x^2/\sigma^2 - 1 = 0$, and thus the curve's inflection point occurs when x equals $\pm\sigma$.

Since $e^0 = 1$, if x is set equal to zero in Equation (3.2), then

$$y = \frac{1}{\sigma\sqrt{2\pi}} \tag{3.9}$$

Equation (3.9) yields the curve's central ordinate, and as can be seen, it is inversely proportional to σ. According to Equation (3.9), a group of measurements having small σ must have a large central ordinate. Thus, the area under the curve will be concentrated near the central ordinate, and the errors will be correspondingly small. This indicates that the set of measurements is precise. Since σ bears this relationship to the precision, it is a numerical measure for the precision of a measurement set. In Section 2.7, we defined σ as the *standard error* and gave equations for computing its value.

3.4 STANDARD NORMAL DISTRIBUTION FUNCTION



In Section 3.2, we defined the probability density function of a normal random variable as $f(x) = \frac{1}{(\sigma\sqrt{2\pi})e^{-\frac{x^2}{2\sigma^2}}}$. From this, we develop the *normal distribution function*

$$F_x(t) = \int_{-\infty}^{t} \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}\,dx \tag{3.10}$$

where t is the upper bound of integration, as shown in Figure 3.4.

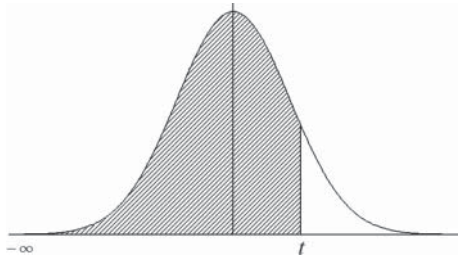


FIGURE 3.4 Area under the normal distribution curve determined by Equation (3.10).

As stated in Section 3.3, the area under the normal density curve represents the probability of occurrence. Furthermore, integration of this function yields the area under the curve. Unfortunately, the integration called for in Equation (3.10) cannot be carried out in closed form, and thus, numerical integration techniques must be used to tabulate values for this function. This has been done for the function when the mean is zero ($\mu = 0$) and the variance is 1 ($\sigma^2 = 1$). The results of this integration are shown in the *standard normal distribution table* of Table D.1. In this table, the leftmost column with a heading of t is the value shown in Figure 3.4 in the units of σ. The top row, with headings 0 through 9, represents the hundredths decimal places for the t values. The tabulated values in the body of the Table D.1 represent areas under the standard normal distribution curve from $-\infty$ to t. For example, to determine the area under the curve from $-\infty$ to 1.68, first find the row with 1.6 in the t column. Then scan along the row to the column with a heading of 8. At the intersection of row 1.6 and column 8 (1.68), the value 0.95352 occurs. This is the area under the standard normal distribution curve from $-\infty$ to a t value of 1.68. Similarly, other areas under the standard normal distribution curve can be found for various values for t. Since the area under the curve represents probability, and its maximum area is 1, this means that there is a 95.352% (i.e., $0.95352 \times 100\%$) probability that t is less than or equal to 1.68. Alternatively, it can be stated that there is a 4.648% [i.e., $(1 - 0.95352) \times 100\%$] probability that t is greater than 1.68.

Once available, Table D.1 can be used to evaluate the distribution function for any mean, μ, and variance, σ^2. For example, if y is a normal random variable with a mean of μ and a variance of σ^2, an equivalent normal random variable $z = (y - \mu)/\sigma$ can be defined that has a mean of zero and a variance of 1. Substituting the definition for z with $\mu = 0$ and $\sigma^2 = 1$ into Equation (3.2), its density function is

$$N_z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \tag{3.11}$$

and its distribution function, which is known as the *standard normal distribution function*, becomes

$$N_z(z) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz \tag{3.12}$$

For any group of normally distributed measurements, the probability of the normal random variable can be computed by analyzing the integration of the distribution function. Again, as stated previously, the area under the curve in Figure 3.4 represents probability. Let z be a normal random variable, then the probability that z is less than some value of t is given by

$$P(z < t) = N_z(t) \tag{3.13}$$

To determine the area (probability) between t values of a and b (the crosshatched areas in Figures 3.5 and 3.6), the difference in the areas between a

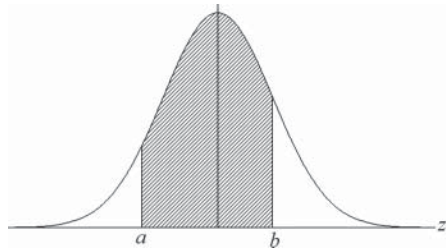


FIGURE 3.5 Area representing the probability in Equation (3.14).

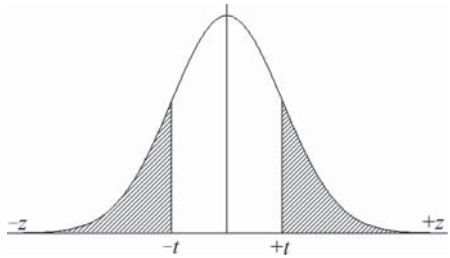


FIGURE 3.6 Area representing the probability in Equation (3.16).

and b, respectively, can be computed. By Equation (3.13), the area from $-\infty$ to b is $P(z < b) = N_z(b)$. By the same equation, the area from $-\infty$ to a is $P(z < a) = N_z(a)$. Thus, the area between a and b is the difference in these values and is expressed as

$$P(a < z < b) = N_z(b) - N_z(a) \tag{3.14}$$

If the bounds are equal in magnitude but opposite in sign, i.e. $-a = b = t$, the probability is

$$P(|z| < t) = N_z(t) - N_z(-t) \tag{3.15}$$

From the symmetry of the normal distribution in Figure 3.6, it is seen that

$$P(z > t) = P(z < -t) \tag{3.16}$$

for any $t > 0$. Also, this symmetry can be seen with Table D.1. The tabular value (area) for a t value of -1.00 is 0.15866. Furthermore, the tabular value for a t value of $+1.00$ is 0.84134. Since the maximum probability (area) is 1, the area above $+1.00$ is $1 - 0.84134$, or 0.15866, which is the same as the area below -1.00. Thus, because the total probability is always 1, we can define the following relationship

$$1 - N_z(t) = N_z(-t) \tag{3.17}$$

Now substituting Equation (3.17) into Equation (3.15), we have

$$P(|z| < t) = 2N_z(t) - 1 \tag{3.18}$$

3.5 PROBABILITY OF THE STANDARD ERROR

The preceding equations can be used to determine the probability of the standard error, which from previous discussion is the area under the normal distribution curve between the limits of $\pm\sigma$. For the standard normal distribution when σ^2 is 1, it is necessary to locate the values of $t = -1$ ($\sigma = -1$) and $t = +1$ ($\sigma = 1$) in Table D.1. As seen previously, the appropriate value from the table for $t = -1.00$ is 0.15866. Also, the tabular value for $t = 1.00$ is 0.84134, and thus, according to Equation (3.15), the area between $-\sigma$ and $+\sigma$ is

$$P(-\sigma < z < +\sigma) = N_z(+\sigma) - N_z(-\sigma)$$

$$P(-\sigma < z < +\sigma) = 0.84134 - 0.15866 = 0.68268$$

From this, it has been determined that approximately 68.3% of all measurements from any data set are expected to lie between $-\sigma$ and $+\sigma$. It also means that for any group of measurements, there is approximately a 68.3% chance that any single observation has an error between $\pm\sigma$. The crosshatched area of Figure 3.7 illustrates that approximately 68.3% of the area exists between $\pm\sigma$. This is true for any set of measurements having normally distributed errors. Note that as discussed in Section 3.3, the inflection points of the normal distribution curve occur at $\pm\sigma$. This is illustrated in Figure 3.7. A detailed treatment describing the core principles of error theory can be found in the Aeronautical Chart and Information Center Technical Report No. 96.[1]

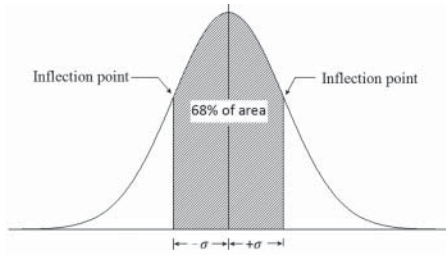


FIGURE 3.7 Normal distribution curve.

[1] Aernautical Chart and Information Center (1962) "Principles of Error Theory and Cartographic Applications" can be found at https://apps.dtic.mil/sti/tr/pdf/AD0276978.pdf.

3.5.1 The 50 Percent Probable Error

For any group of observations, the 50% probable error establishes the limits within which 50% of the errors should fall. In other words, any measurement has the same chance of coming within these limits as it has of falling outside them. Its value can be obtained by multiplying the observations' standard deviation by the appropriate *t value*. Since the 50% probable error has a probability of one-half, Equation (3.18) is set equal to 0.50 and the *t value* corresponding to this area is determined as

$$\begin{aligned} P(|z| < t) = 0.5 &= 2N_z(t) - 1 \\ 1.5 &= 2N_z(t) \\ 0.75 &= N_z(t) \end{aligned}$$

From Table D.1, it is apparent that 0.75 is between a *t value* of 0.67 and 0.68; that is,

$$N_z(0.67) = 0.7486 \text{ and } N_z(0.68) = 0.7517$$

The t value can be found by linear interpolation, as follows:

$$\begin{aligned} \frac{\Delta t}{0.68 - 0.67} &= \frac{0.75 - 0.7486}{0.7517 - 0.7486} = \frac{0.0014}{0.0031} = 0.4516 \\ \Delta t &= 0.01 \times 0.4516 \end{aligned}$$

and $t = 0.67 + 0.0045 = 0.6745$. Thus, the equation for linear interpolation can be expressed as

$$t = t_l + \frac{N_z(t) - N_z(t_l)}{N_z(t_u) - N_z(t_l)}(t_u - t_l) \tag{3.19}$$

where t is the desired t value, t_l and t_u the lower and upper t values that bound t, $N_z(t)$ the desired value from the table, and $N_z(t_l)$ and $N_z(t_u)$, the tabular values that bound $N_z(t)$. Thus, the previous example could be completed in one computation as

$$t = 0.67 + \frac{0.75 - 0.7486}{0.7517 - 0.7486}(0.68 - 0.67) = 0.6745$$

For any set of one-dimensional (1-D) observations, therefore, the 50% probable error can be obtained by computing the standard error and then multiplying it by 0.6745, or

$$E_{50} = 0.6745\sigma \tag{3.20}$$

3.5.2 The 95 Probable Error

The 95% probable error, or E_{95}, is the bound within which, theoretically, 95% of the observation group's errors should fall. This error category is popular with

TABLE 3.2 Multipliers for Various Percent Probable Errors (One-Dimensional)

Symbol	Multiplier	Percent Probable Errors
E_{50}	0.6745σ	50
E_{90}	1.645σ	90
E_{95}	1.960σ	95
E_{99}	2.576σ	99
$E_{99.7}$	2.968σ	99.7
$E_{99.9}$	3.29σ	99.9

surveyors for expressing precision and checking for outliers in data. Using the same reasoning as in developing the equation for the 50% probable error, substituting into Equation (3.18), yields

$$0.95 = P(|z| < t) = 2N_z(t) - 1$$
$$1.95 = 2N_z(t)$$
$$0.975 = N_z(t)$$

Again, from Table D.1, it is determined that 0.975 occurs with a *t value* of 1.960. Thus, to find the 95% probable error for any group of measurements, the following equation is used:

$$E_{95} = 1.960\sigma \tag{3.21}$$

3.5.3 Other Percent Probable Errors

Using the same computation techniques as in Sections 3.5.1 and 3.5.2, other percent probable errors can be calculated. One other percent error worthy of note is $E_{99.7}$. It is obtained by multiplying the standard error by 2.968, or

$$E_{99.7} = 2.968\sigma \tag{3.22}$$

This value is often used for detecting blunders as discussed in Section 3.6. A summary of probable errors with varying percentages, together with their multipliers, is given in Table 3.2.

3.6 USES FOR PERCENT ERRORS

Standard errors and errors of other percent probabilities are commonly used to evaluate measurements for acceptance. Project specifications and contracts often

require that acceptable errors be within specified limits such as the 90% and 95% errors. The 95% error, sometimes called the *two-sigma* (2σ) *error* because it is computed as approximately 2σ, is most often specified. Standard errors are also frequently used. The probable error, E_{50}, is seldom employed.

Higher-percentage errors are used to help isolate outliers (very large random errors) and blunders in data sets. Since outliers seldom occur in a data set, measurements outside a selected high percentage range can be rejected as possible blunders. Generally, any data that differs from the mean by more than 3σ can be considered as blunders and removed from a data set. As seen in Table 3.2, rejecting observations greater than 3σ means that about 99.7% of all observations should be retained. In other words, only about 0.3% of the observations in a set of normally distributed random errors (or 3 observations in 1000) should lie outside the range of $\pm 3\sigma$.

Note that as explained in Chapter 2, standard error and standard deviation are often used interchangeably, when in practice, it is the standard deviation that is computed, not the standard error. Thus, for practical applications, σ in the equations of the preceding sections is replaced by S to distinguish between these two related values.

3.7 PRACTICAL EXAMPLES

Example 3.1 Suppose that the following values (in feet) were obtained in 15 independent distance observations, D_i: 212.22, 212.25, 212.23, 212.15, 212.23, 212.11, 212.29, 212.45, 212.22, 212.24, 212.19, 212.25, 212.27, 212.20, and 212.25. Calculate the mean, S, E_{50}, E_{95}, and check for any observations outside the 99.7% probability level.

SOLUTION From Equation (2.1), the mean is

$$\bar{D} = \frac{\sum D_i}{n} = \frac{3183.55}{15} = 212.237 \text{ ft}$$

From Equation (2.10), S is

$$S = \sqrt{\frac{675{,}666.118 - 15 \times 212.237^2}{15 - 1}} = \sqrt{\frac{0.07773}{14}} = \pm 0.075 \text{ ft}$$

where $\sum D_i = 675{,}666.118$. By scanning the data, it is seen that 12 observations are between 212.24 ± 0.075 or within the range of (212.162, 212.311).[2] This

[2]The expression of (x, y) represents a range between x and y. That is, about 68.3% of the data should lie between 212.16 and 212.28 in this example.

corresponds to $12/15 \times 100$ or 80.0% of the observations. For the set, this is what is expected if it conforms to normal error distribution theory.

From Equation (3.20), E_{50} is

$$E_{50} = 0.6745S = \pm 0.6745\,(0.075) = \pm 0.050 \text{ ft}$$

Again, by scanning the data, 11 observations lie between 212.234 ± 0.050 ft. That is, they are within the range of (212.186, 212.287). This corresponds to $9/15 \times 100$, or 73.3% of the observations. Although this should be 50% and thus is a little high for a normal distribution, it must be remembered that this is only a sample of the population and should not be considered a reason to reject the entire data set. (In Chapter 4, statistical intervals involving sample sets are discussed.)

From Equation (3.20), E_{95} is

$$E_{95} = 1.960S = \pm 1.960\,(0.075) = \pm 0.146 \text{ ft}$$

Note that 13 of the observations lie between 212.236 ± 0.147 (212.091, 212.383) ft or 93.3% of the data are within the range. Similarly, this does not quite reach 95% as we would expect given the relatively small sample size to represent a normal distribution.

At the 99.7% level of confidence, the range of $\pm 2.968S$ corresponds to an interval of ± 0.221 ft. With this criterion for rejection of outliers, all values in the data are within this range. Thus, there is no reason to believe that any observation is a blunder or outlier.

Example 3.2 The second's portions of 50 readings are listed below. Find the mean, standard deviation, and E_{95}. Check the observations at a 99% level of certainty for blunders.

41.9	46.3	44.6	46.1	42.5	45.9	45.0	42.0	47.5	43.2
49.5	45.5	43.3	42.6	44.3	46.1	45.6	52.0	45.5	43.4
43.0	45.7	47.6	42.2	44.3	44.1	42.6	47.2	47.4	44.7
44.2	46.3	49.5	46.0	44.3	42.8	47.1	44.7	45.6	45.5
43.4	45.5	43.1	46.1	43.6	41.8	44.7	46.2	43.2	46.8

SOLUTION The sum of the 50 observations is $2252.0''$, and thus, the mean is $2252.0''/50 = 45.04''$. Using Equation (2.10), the standard deviation is

$$S = \sqrt{\frac{101{,}649.94 - 50(45.04)^2}{50 - 1}} = \pm 2.12''$$

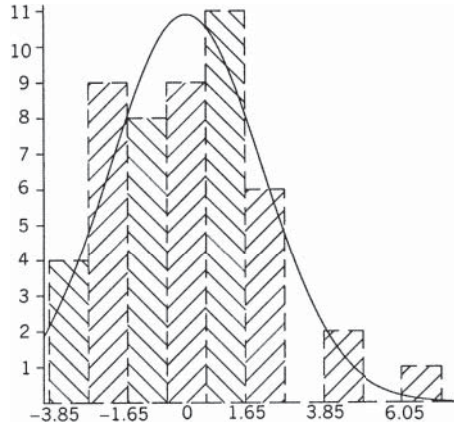


FIGURE 3.8 Skewed data set.

where $\sum y^2 = 101{,}649.94$. There are 35 observations between $45.04'' \pm 2.12''$ or within the range from $42.92''$ to $47.16''$. This corresponds to $35/50 \times 100$ or 70% of the observations and correlates well with the anticipated level of 68.3%.

From Equation (3.21), $E_{95} = \pm 1.960(2.12'') = \pm 4.16''$. The data contain three values that deviate from the mean by more than $4.16''$ (i.e., that are outside the range $40.88''$ to $49.20''$). They are $49.5''$ (2) and $52.0''$. No values are less than $40.88''$, and therefore, $47/50 \times 100$, or 94% of the observations lie in the E_{95} range.

From Equation (3.22), $E_{99} = \pm 2.576(2.12'') = \pm 5.46''$, and thus, 99% of the data should fall between $45.04'' \pm 5.46''$, or $(39.58'', 50.50'')$. One value is greater than $50.50''$, and thus, 98% of all the observations fall in this range.

By the analysis above it is seen that the data set is skewed to the left. That is, values higher than the range always fell on the right side of the data. The histogram shown in Figure 3.8 depicts this skewness. This suggests that it may be wise to reject the value of $52.0''$ as a mistake or blunder. The recomputed values for the data set (minus $52.0''$) are

$$\text{Mean} = (2252'' - 52'')/49 = 44.90''$$

$$\sum y^2 = 101{,}649.94 - 52.0^2 = 98{,}945.94$$

$$S = \sqrt{\frac{98{,}945.94 - 49(44.89795918)^2}{49 - 1}} = \pm 1.88''$$

Now after recomputing errors, 32 observations lie between $\pm S$, which represents 65.3% of the observations; 47 observations lie in the E_{95} range, which represents 95.9% of the data; and no values are outside the E_{99} range. Thus, there is no reason to reject any additional data at the 99% level.

PROBLEMS

Note: Partial answers to problems marked with an asterisk are given in Appendix G.

***3.1** Determine the *t value* for E_{85}.

3.2 Determine the *t value* for $E_{99.7.}$

3.3 Determine the *t* value for E_{90}.

3.4 Determine the *t* value for $E_{99.9}$.

3.5 Define probability.

3.6 Define compound event.

***3.7** If the mean of a population is 2.503 and its variance is 1.051, what is the peak value for the normal distribution curve and the *x* coordinates for the points of inflection?

3.8 If the mean of a population is 36.5 and its variance is 4.3, what is the peak value for the normal distribution curve and the *x* coordinates for the points of inflection?

3.9 If the mean of a population is 15.7 and its variance is 2.6, what is the peak value for the normal distribution curve and the *x* coordinates for the points of inflection?

3.10 Plot the curve in Problem 3.7 using Equation (3.2) to determine ordinate and abscissa values. Remember to shift the curve horizontally based on the mean. Identify and label the points of inflection and the peak. Check those values against your answers in Problem 3.7.

3.11 Plot the curve in Problem 3.8 using Equation (3.2) to determine ordinate and abscissa values. Remember to shift the curve horizontally based on the mean. Identify and label the points of inflection and the peak. Check those values against your answers in Problem 3.8.

3.12 Plot the curve in Problem 3.9 using Equation (3.2) to determine ordinate and abscissa values. Remember to shift the curve horizontally based on the mean. Identify and label the points of inflection and the peak. Check those values against your answers in Problem 3.9.

3.13 The following data represent 21 electronically measured distance observations.

449.99	449.97	450.02	450.00	449.98	449.99	450.03
450.00	450.00	449.99	450.00	449.99	449.98	450.00
449.99	450.01	449.97	450.00	449.98	449.99	449.98

***(a)** Calculate the mean and the standard deviation.

(b) Plot the relative frequency histogram (of residuals) for the data above using five class intervals.

(c) Calculate the E_{50} and E_{90} intervals.

(d) Can any observations be rejected at a 95% level of certainty?

(e) What is the peak value for the normal distribution curve and where are the points of inflection on the curve on the x axis?

3.14 Using the data in Problem 2.12, calculate the $E95$ interval and identify any observations that can be rejected at a 95% level of certainty.

3.15 Repeat Problem 3.14 for the data in Problem 2.15.

3.16 Discuss the normality of each set of data below and whether any observations may be removed at the 95% level of certainty as blunders or outliers. Determine which set is more precise after apparent blunders and outliers are removed. Plot the relative frequency histogram to defend your decisions.

Set 1:

493.50	493.49	493.45	493.52	493.47	493.51	493.49
493.53	493.50	493.48	493.48	493.46	493.54	493.51
493.46	493.51	493.51	493.53	493.52	493.45	493.50

Set 2:

770.323	770.366	770.316	770.313	770.308	770.384	770.356
770.320	770.286	770.325	770.317	770.359	770.311	770.374
770.358	770.314	770.347	770.329	770.317	770.246	770.279

3.17 Using the following data set, answer the questions below.

10.3	17.7	15.7	18.6	5.0	18.7	13.0	8.6	14.0	16.1
15.9	19.6	22.1	14.3	22.6	17.7	9.2	12.7	23.1	22.2
6.2	17.5	9.0	13.6	22.5	13.6	15.7	15.2	21.2	21.5

(a) What are the mean and standard deviation of the data set?

(b) Construct a relative frequency histogram of the data using seven intervals and discuss whether it appears to be a normal data set.

(c) What is the E_{95} interval for this data set?

(d) Would there be any reason to question the validity of any observation at the 95% level?

3.18 Repeat Problem 3.17 using the following data:

148.771	148.770	148.754	148.768	148.756	148.764	148.743
148.748	148.757	148.772	148.739	148.777	148.738	148.751
148.779	148.776	148.757	148.792	148.759	148.760	

(a) What are the mean and standard deviation of the data set?

(b) Construct a relative frequency histogram of the data using five intervals and discuss whether it appears to be a normal data set.

(c) What is the E_{95} interval for this data set?

(d) Would there be any reason to question the validity of any observation at the 95% level?

3.19 What is the E_{95} interval for the following data? Is there any reason to question the validity of any observation at the 95% level?

15.8	14.6	14.6	15.1	16.1	15.5	16.3	15.0	15.6	15.4
13.9	14.7	14.6	15.7	15.1	14.9	15.1	15.1	16.4	15.3

3.20 Using the data from Problem 3.19, what is the E_{99} interval data? Is there any reason to question the validity of any observation at the 99% level?

Use STATS to do Problems 3.21–3.22.

3.21 Problem 3.16

3.22 Problem 3.17

PROGRAMMING PROBLEMS

3.23 Create a spreadsheet to solve Problem 3.13.

3.24 Create a computation package to solve Problem 3.19.

CHAPTER 4

CONFIDENCE INTERVALS

4.1 INTRODUCTION



Table 4.1 contains a discrete population of 100 values. The mean (μ) and variance (σ^2) of the population are 26.1 and 17.5, respectively. By randomly selecting 10 values from Table 4.1, an estimate for the mean and variance of the population can be determined. However, it should not be expected that these estimates ($\bar{y}$ and S^2) would exactly match the mean and variance of the population. Sample sets of 10 values each could continue to be selected from the population to determine additional estimates for the mean and variance of the population. However, it is just as unlikely that these additional values would match those obtained from either the population or the first sample set.

As the sample size is increased, the mean and variance of the sample should approach the values of the population. In fact, as the sample size becomes very, very large, the mean and variance of the samples should be close to those of the population. This procedure was done for several sample sizes starting at 10 values and increasing the sample by 10 values. The results are shown in Table 4.2. Note that the value computed for the mean of the sample approaches the value of the population as the sample size is increased. Similarly, the value computed for the

TABLE 4.1 Population of 100 Values

18.2	26.4	20.1	29.9	29.8	26.6	26.2
25.7	25.2	26.3	26.7	30.6	22.6	22.3
30.0	26.5	28.1	25.6	20.3	35.5	22.9
30.7	32.2	22.2	29.2	26.1	26.8	25.3
24.3	24.4	29.0	25.0	29.9	25.2	20.8
29.0	21.9	25.4	27.3	23.4	38.2	22.6
28.0	24.0	19.4	27.0	32.0	27.3	15.3
26.5	31.5	28.0	22.4	23.4	21.2	27.7
27.1	27.0	25.2	24.0	24.5	23.8	28.2
26.8	27.7	39.8	19.8	29.3	28.5	24.7
22.0	18.4	26.4	24.2	29.9	21.8	36.0
21.3	28.8	22.8	28.5	30.9	19.1	28.1
30.3	26.5	26.9	26.6	28.2	24.2	25.5
30.2	18.9	28.9	27.6	19.6	27.9	24.9
21.3	26.7					

TABLE 4.2 Increasing Sample Sizes

No.	$\bar{y}$	S^2
10	26.9	28.1
20	25.9	21.9
30	25.9	20.0
40	26.5	18.6
50	26.6	20.0
60	26.4	17.6
70	26.3	17.1
80	26.3	18.4
90	26.3	17.8
100	26.1	17.5

variance of the sample also tends to approach the value of the population as the sample size is increased.

Since the mean of a sample set $\bar{y}$ and its variance S^2 are computed from random variables, they are also random variables. This means that even if the size of the sample is kept constant, varying values for the mean and variance will be estimated from the samples with greater confidence given to larger samples. Also, it can be concluded that the statistical values computed from a sample also contain errors. To illustrate this concept, an experiment was run for four randomly selected sets of 10 values from Table 4.1. Table 4.3 lists these samples, their means, and variances. Notice the variation in the computed values for the four sets. As discussed earlier, this variation is expected.

TABLE 4.3 Random Sample Sets from Population

Set 1: 29.9, 18.2, 30.7, 24.4, 36.0, 25.6, 26.5, 29.9, 19.6, 27.9	$\bar{y} = 26.9$	$S^2 = 28.1$
Set 2: 26.9, 28.1, 29.2, 26.2, 30.0, 27.1, 26.5, 30.6, 28.5, 25.5	$\bar{y} = 27.9$	$S^2 = 2.9$
Set 3: 32.2, 22.2, 23.4, 27.9, 27.0, 28.9, 22.6, 27.7, 30.6, 26.9	$\bar{y} = 26.9$	$S^2 = 10.9$
Set 4: 24.2, 36.0, 18.2, 24.3, 24.0, 28.9, 28.8, 30.2, 28.1, 29.0	$\bar{y} = 27.2$	$S^2 = 23.0$

Fluctuations in the computed means and variances from varying sample sets raise questions about the ability of these values to reliably estimate the population values. For example, a higher confidence is likely to be placed on a sample set with a small variance than on one with a large variance. Thus, in Table 4.3, because of its small variance, one is more likely to believe that the mean of the second sample set is a more reliable estimate for the mean of the population than the others. In reality, this is not the case since the means of the other three sets are actually closer to the population mean of 26.1.

As noted earlier, the size of the sample should also be considered when determining the reliability of a computed mean or variance. If the mean were computed from a sample of 5 values and another computed from a sample of 30, then more confidence is likely to be placed on the values derived from the larger sample set than the smaller one, even if both sample sets have the same mean and standard deviation.

In statistics, this relationship between the sample sets, the number of samples, and the statistical values computed for the means and variances is part of *sampling distribution theory*. This theory recognizes that estimates for the mean and variance do vary from sample to sample. *Estimators* are the functions used to compute these estimates. Examples of estimator functions are Equations (2.1) and (2.5) that are used to compute estimates of the mean and variance for a population, respectively. As demonstrated and discussed, these estimates vary from sample to sample and thus have their own population distributions. In Section 4.2, three distributions are defined, which are used for describing or quantifying the reliability of mean and variance estimates. By applying these distributions, statements can be written about the reliability of the computed estimates for any given level of confidence. In other words, a range called the *confidence interval* can be determined within which the population mean and population variance are expected to fall for varying levels of probability.

4.2 DISTRIBUTIONS USED IN SAMPLING THEORY

4.2.1 χ^2 Distribution

The *chi-squared distribution*, symbolized as χ^2, compares the relationship between the population variance and the variance of a sample set based on the number of redundant observations in the sample. It was first published by Karl Pearson in

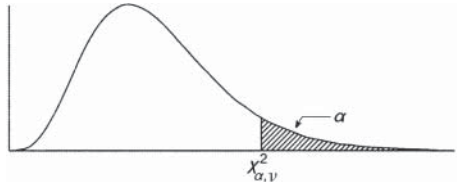


FIGURE 4.1 χ^2 distribution.

1900. If a random sample of n observations, $y_1, y_2, \ldots, y_n$, is selected from a population that has a normal distribution with mean μ and variance σ^2, then by definition the χ^2 sampling distribution is

$$\chi^2 = \frac{\upsilon S^2}{\sigma^2} \tag{4.1}$$

where υ is the number of degrees of freedom in the sample, and the other terms are defined previously.

A plot of the distribution is shown in Figure 4.1. The number of redundancies (degrees of freedom) in sample set statistics such as those for the mean or variance is $\upsilon = n - 1$. In later chapters on least squares, it will be shown that the number of redundancies is based on the number of independent observations and unknown parameters. In the case of the mean, one observation is necessary to determine the mean, thus leaving $n - 1$ values as redundant observations. Table D.2 is a tabulation of χ^2 distribution curves for various degrees of freedom from 1 to 120. To find the area under the upper tail of the curve (right side shown shaded in Figure 4.1) starting at some specific χ^2 value and going to infinity (∞), intersect the row corresponding to the appropriate degrees of freedom, υ, with the column corresponding to the desired area under the curve. For example, to find the specific χ^2 value relating to 1% ($\alpha = 0.010$) of the area under a curve having 10 degrees of freedom, intersect the row headed by 10 with the column headed by the percentage point[1] 0.010 and find a χ^2 value of 23.21. This means that 1% of the area under this curve lies in the upper tail of this curve between the values of 23.21 and ∞.

Due to the asymmetric nature of the distribution, the percentage point (α) of the lower tail (left side of the curve) must be computed from those tabulated for the upper tail. A specific area under the left side (lower tail) of the curve starting at zero and going to some specific χ^2 value is found by subtracting the tabulated α (right-side area) from 1. This can be done since Table D.2 lists α (areas) starting at the χ^2 value and going to ∞, and the total area under the curve is 1. For example, if there are 10 degrees of freedom and the χ^2 value relating to 1% of the area under the left side of the curve is needed, the row corresponding to υ equal to 10 is intersected

[1]Percentage points are decimal equivalents of percent probability; that is, a percent probability of 95% is equivalent to 0.95 percentage points.

with the column headed by $\alpha = 0.990\ (1 - 0.010)$, and a value of 2.56 is obtained. This means that 1% of the area under the curve occurs from 0 to 2.56.

The χ^2 distribution is used in sampling statistics to determine the range in which the variance of the population can be expected to occur based on (1) some specified percentage probability; (2) the variance of a sample set; and (3) the number of degrees of freedom in the sample. In an example given in Section 4.6, this distribution is used to construct probability statements about the variance of the population being in a range centered about the variance S^2 of a sample having v degrees of freedom. In Section 5.4, a statistical test is presented using the χ^2 distribution to check if the variance of a sample is a valid estimate for the population variance.

4.2.2 Student's *t* Distribution

The *t* distribution was developed in 1905 by William Seally Gosset, a statistician at Guinness Brewery in Dublin, Ireland. At the time, he used the pseudonym of Student to publish his paper, and thus the distribution is sometimes called Student's *t* distribution. The distribution is used to compare a population mean with the mean of a sample set based on the number of redundancies (v) in the sample set. It is similar to the normal distribution (discussed in Chapter 3), except that the normal distribution applies to an entire population, whereas the *t* distribution applies to a sample set of the population. This distribution is preferred over the normal distribution when the sample contains fewer than 30 values. Thus, it is an important distribution in analyzing survey data to minimize costs to projects by reducing the number of control or check points to be observed.

If z is a standard normal random variable, as defined in Section 3.4, χ^2 is a chi-squared random variable with v degrees of freedom, and z and χ^2 are both independent variables, then by definition

$$t = \frac{z}{\sqrt{\chi^2/v}} \tag{4.2}$$

The *t* values for selected upper-tail percentage points (shaded area in Figure 4.2) versus the *t distributions* with various degrees of freedom v are listed in Table D.3. For specific degrees of freedom (v) and percentage points (α), the table lists specific

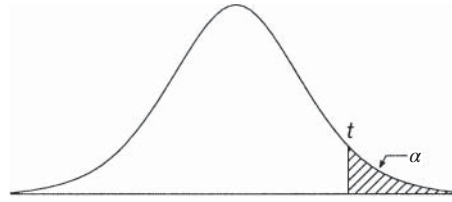


FIGURE 4.2 *t* distribution.

t values that correspond to the areas α under the curve between the tabulated t values and ∞. Similar to the normal distribution, the t distribution is symmetric. Generally, in statistics, only percentage points in the range 0.0005 to 0.4 are necessary since the curve is symmetric. These t values are tabulated in Table D.3. To find the t value relating to $\alpha = 0.01$ for a curve developed with 10 degrees of freedom ($\upsilon = 10$), intersect the row corresponding to $\upsilon = 10$ with the row corresponding to $\alpha = 0.01$. At this intersection, the t value of 2.764 is obtained. This means that 1% ($\alpha = 0.01$) of the area exists under the t distribution curve having 10 degrees of freedom in the interval between 2.764 and ∞. Due to the symmetry of this curve, it can also be stated that 1% ($\alpha = 0.01$) of the area under the curve developed for 10 degrees of freedom also lies between $-\infty$ and -2.764.

As described in Section 4.3, this distribution is used to construct confidence intervals for the population mean (μ) based on the mean ($\bar{y}$) and variance (S^2) of a sample set and the degrees of freedom (υ). An example in that section illustrates the procedure. Furthermore, in Section 5.3, it is shown that this distribution can be used to determine if the sample mean is a reliable estimate for the population mean.

4.2.3 *F* Distribution

This distribution is used when comparing the computed variances from two sample sets. If χ_1^2 and χ_2^2 are two chi-squared random variables with υ_1 and υ_2 degrees of freedom, respectively, and both variables are independent, then by definition

$$F = \frac{\frac{\chi_1^2}{\upsilon_1}}{\frac{\chi_2^2}{\upsilon_2}} \tag{4.3}$$

The distribution was first tabulated by George Snedecor as an aid for Ronald A. Fisher's analysis of variance. Fisher developed the statistic initially as the variance ratio in the 1920s.

Several percentage points (areas under the upper tail of the curve shown shaded in Figure 4.3) of the F distribution are tabulated in Table D.4. Notice that this distribution has υ_1 numerator degrees of freedom and υ_2 denominator degrees of freedom, which correspond to the two sample sets. Thus, unlike the χ^2 and t distributions, each desired α percentage point must be represented in a separate table. In Appendix D, tables for the more commonly used values of α (0.20, 0.10, 0.05, 0.025, 0.01, 0.005, and 0.001) are listed.

To illustrate the use of the tables, suppose that the F value for 1% area under the upper tail of the curve is needed. Also assume that 5 is the numerator degrees of freedom relating to S_1, and 10 is the denominator degrees of freedom relating to S_2. In this example, α equals 0.01 and thus the F table in Table D.4 that is written for $\alpha = 0.01$ must be used. In that table, intersect the row headed by υ_2 equal to 10 with the column headed by υ_1 equal to 5, and find the F value of 5.64. This means

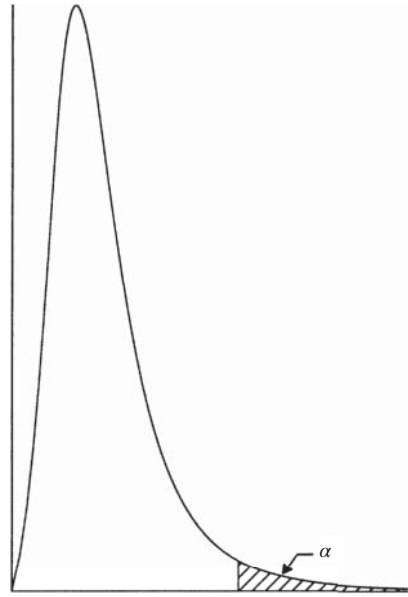


FIGURE 4.3 F distribution.

that 1% of the area under the curve constructed using these degrees of freedom lies in the region going from 5.64 to $+\infty$.

To determine the area in the lower tail of this distribution, use the following functional relationship:

$$F_{\alpha,\upsilon_1,\upsilon_2} = \frac{1}{F_{1-\alpha,\upsilon_2,\upsilon_1}} \tag{4.4}$$

The critical F value for the data in the previous paragraph (υ_1 equal to 5 and υ_2 equal to 10) with α equal to 0.99 (0.01 in the lower tail) is determined by going to the intersection of the row headed by 5 with the column headed by 10 on the α equal to 0.01 page. The intersection is at F equal to 10.05. According to Equation (4.4), the critical $F_{0.99,5,10}$ is $1/F_{0.01,10,5} = 1/10.05 = 0.10$. Thus, 1% of the area is under the F distribution curve going from zero to 0.10.

The F distribution is used to answer the question of whether two sample sets come from the same population of data. For example, suppose two samples have variances of S_1^2 and S_2^2. If these two sample variances represent the same population variance, the ratio of their population variances $\left(\sigma_1^2/\sigma_2^2\right)$ should equal 1 (i.e., $\sigma_1^2 = \sigma_2^2$). As discussed in Section 4.7, this distribution enables confidence intervals to be established for the ratio of the population variances. Also, as discussed in

Section 5.5, the distribution can be used to test whether the ratio of the two variances is statistically equal to 1.

4.3 CONFIDENCE INTERVAL FOR THE MEAN: *t* STATISTIC



In Chapter 3, the standard normal distribution was used to predict the range in which the mean of a population may exist. This was based on the mean and standard deviation for a sample set. However, as noted previously, the normal distribution is based on an entire population, and as was demonstrated, variations from the normal distribution are expected from sample sets having a small number of values. From this expectation, the *t distribution* was developed. As is demonstrated later in this section by an example, the *t* distribution (in Table D.3) for samples having an infinite number of values uses the same *t* values as those listed in Table 3.2 for the normal distribution. It is generally accepted that when the number of observations is greater than about 30, the values in Table 3.2 are valid for constructing intervals about the population mean. However, when the sample set has less than 30 values, a *t* value from the *t* distribution should be used to construct the confidence interval for the population mean.

To derive an expression for a confidence interval of the population mean, a sample mean ($\bar{y}$) is computed from a sample set of a normally distributed population having a mean of μ and a variance of the mean of σ^2/n. Let $z = (\bar{y} - \mu)/(\sigma/\sqrt{n})$ be a normal random variable. Substituting it and Equation (4.1) into Equation (4.2) yields

$$t = \frac{z}{\sqrt{\frac{\chi^2}{v}}} = \frac{\frac{(\bar{y}-\mu)}{\left(\frac{\sigma}{\sqrt{n}}\right)}}{\sqrt{\frac{\left(\frac{vS^2}{\sigma^2}\right)}{v}}} = \frac{\frac{(\bar{y}-\mu)}{\left(\frac{\sigma}{\sqrt{n}}\right)}}{\frac{S}{\sigma}} = \frac{\bar{y}-\mu}{\frac{S}{\sqrt{n}}} \tag{4.5}$$

To compute a confidence interval for the population mean (μ) given a sample set mean and variance, it is necessary to determine the area of a $(1-\alpha)$ region. For example, in a 95% confidence interval (unshaded area in Figure 4.4), center the percentage point of 0.95 on the *t* distribution. This leaves 0.025 in each of the upper- and lower-tail areas (shaded areas in Figure 4.4). The *t* value that locates an

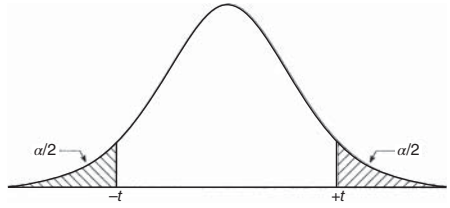


FIGURE 4.4 $t_{\alpha/2}$ plot.

$\alpha/2$ area in both the upper and lower tails of the distribution is given in Table D.3 as $t_{\alpha/2,\upsilon}$. For sample sets having a mean of and variance of S^2, the correct probability statement to locate this area is

$$P(|z| < t) = 1 - \alpha \qquad (a)$$

Substituting Equation (4.5) into Equation (a) yields

$$P\left(\left|\frac{\bar{y} - \mu}{\frac{S}{\sqrt{n}}}\right| < t\right) = 1 - \alpha$$

which after rearranging yields

$$P\left(\bar{y} - t_{a/2}\frac{S}{\sqrt{n}} < \mu < \bar{y} + t_{a/2}\frac{S}{\sqrt{n}}\right) = 1 - \alpha \qquad (4.6)$$

Thus, given $\bar{y}$, $t_{\alpha/2,\upsilon}$, n, and S, it is seen from Equation (4.6) that a $(1 - \alpha)$ probable error interval for the population mean μ is computed as

$$\bar{y} - t_{\frac{\alpha}{2}}\frac{S}{\sqrt{n}} < \mu < \bar{y} + t_{\frac{\alpha}{2}}\frac{S}{\sqrt{n}} \qquad (4.7)$$

where $t_{\alpha/2}$ is the critical t value from the t distribution based on υ degrees of freedom and $\alpha/2$ percentage points.

The following example illustrates the use of Equation (4.7) and Table D.3 for determining the 95% confidence interval for the population mean based on a sample set having a small number of values (n) with a mean of $\bar{y}$ and a variance of S.

Example 4.1 In conducting a control survey, 16 directional readings were observed for a single line. The mean (seconds portion only) of the readings was $25.4''$ with a standard deviation of $\pm 1.3''$. Determine the 95% confidence interval for the population mean and compare this with the interval determined by using a t value determined from the standard normal distribution tables (Table 3.2).

SOLUTION In this example, the confidence level $(1 - \alpha)$ is 0.95, and thus α is 0.05. Since the interval is to be centered about the population mean μ, a value of $\alpha/2$ in Table D.3 is used. This yields equal areas in both the lower and upper tails of the distribution, as shown in Figure 4.4. Thus, for this example, $\alpha/2$ is 0.025. The appropriate t value for this percentage point with υ equal to 15 $(16 - 1)$ degrees of freedom is found in Table D.3 as follows:

Step 1: In the leftmost column of Table D.3, find the row with the correct number of degrees of freedom (υ) for the sample. In this case, it is $16 - 1$, or 15.

Step 2: Find the column headed by 0.025 for $\alpha/2$.

Step 3: Locate the intersection of this row and column, which is 2.131. This is the critical t value.

Step 4: Then, by Equation (4.7), the appropriate 95% confidence interval is

$$24.7 = 25.4 - 2.131\left(\frac{1.3}{\sqrt{16}}\right) = \bar{y} - t_{0.025}\frac{S}{\sqrt{n}} < \mu < \bar{y} + t_{0.025}\frac{S}{\sqrt{n}}$$
$$= 25.4 + 2.131\left(\frac{1.3}{\sqrt{16}}\right) = 26.1$$

This computation can be written more compactly as

$$\bar{y} \pm t_{0.025}\frac{S}{\sqrt{n}} \text{ or } 25.4 \pm 2.131\left(\frac{1.3}{\sqrt{16}}\right) = 25.4 \pm 0.7$$

After making the above calculation, it can be stated that for this sample, with 95% confidence, the population mean (μ) lies in the range (24.7, 26.1). If this was a large sample, the E_{95} value from Table 3.2 could be used for 95%. That E_{95} value is 1.960, and the standard error in the mean then would be $\pm 1.3/\sqrt{16} = \pm 0.325$. Thus, the population's mean would be in the range 5.4 ± 1.960 (0.325″), or (24.8, 26.0). Notice that due to the small sample size, the t distribution gives a larger range for the population mean than does the standard normal distribution, which allows for error in the sample statistics. Also notice that in the t distribution of Table D.3, for a sample of infinite size (i.e., $\upsilon = \infty$), the tabulated t value for α equal to 0.025 is 1.960, which matches Table 3.2.

The t distribution is often used to isolate outliers or blunders in observations. To do this, a percent confidence interval is developed about the mean for a single observation as

$$\bar{y} - t_{\alpha/2}S \le y_i \le \bar{y} + t_{\alpha/2}S \tag{4.8}$$

Using the data from Example 4.1 and Equation (4.8), the 95% range for the 16 directional readings is

$$25.4'' - 2.131(1.3'') = 22.63'' \le y_i \le 28.17'' = 25.4'' + 2.131(1.3'')$$

Thus, 95% of the data should be in the range of (22.6″, 28.2″). Any data values outside of this range can be considered as outliers and rejected with a 95% level of confidence. It is important to note that if the normal distribution value of 1.960 was used to compute this interval, the range would be smaller at (22.85″, 27.95″). Using the normal distribution could result in discarding more observations than is justified

when using the sample estimates of the mean and variance. It is important to note that this will become more significant as the number of observations in the sample becomes smaller. For example, if only four directional readings are obtained, then the t distribution multiplier would become 3.183. The resulting 95% confidence interval for a single observation would be 1.6 times larger than that derived using normal distribution t value.

4.4 TESTING THE VALIDITY OF THE CONFIDENCE INTERVAL

A test that demonstrates the validity of the theory of the confidence interval is illustrated as follows. Using software to generate random numbers following a normal distribution, 1000 sample datasets of 16 values each were collected randomly from a population with mean $\mu = 25.4$ and standard error $\sigma = \pm 1.3$. Using a 95% confidence interval ($\alpha = 0.05$) and Equation (4.7), the interval for the population mean derived for each sample set was computed and compared with the actual population mean. If the theory is valid, the interval constructed is expected to contain the population's mean 95% of the time based on the confidence level of 0.05. Appendix E shows the 95% intervals computed for the 1000 samples. The intervals not containing the population mean of 25.4 are marked with an asterisk. From the data tabulated, it is seen that 50 of 1000 sample sets failed to contain the population mean. This corresponds to exactly 5% of the samples. In other words, the proportion of samples that enclose the mean is exactly 95%. This demonstrates that the bounds calculated by Equation (4.7), in fact, enclose the population mean at the confidence level selected.

4.5 SELECTING A SAMPLE SIZE

A common problem encountered in surveying practice is to determine the necessary number of repeated observations to meet a specific precision. In practice, the size of S cannot be controlled absolutely. Rather, as seen in Equation (4.7), the confidence interval can be controlled only by varying the number of repeated observations. In general, the larger the sample size, the smaller the confidence interval. From Equation (4.7), the range in which the population mean (μ) resides at a selected level of confidence (α) is

$$\bar{y} \pm t_{\alpha/2} \frac{S}{\sqrt{n}} \qquad (b)$$

Now let I represent one-half of the interval in which the population mean lies. Then, from Equation (b), I is

$$I = t_{\alpha/2} \frac{S}{\sqrt{n}} \qquad (4.9)$$

Rearranging Equation (4.9),

$$n = \left(\frac{t_{\alpha/2} S}{I} \right)^2 \tag{4.10}$$

In Equation (4.10), n is the number of repeated measurements; I is the desired confidence interval; $t_{\alpha/2}$ is the t value based on the number of degrees of freedom (υ); and S is the sample set standard deviation. In the practical application of Equation (4.10), $t_{\alpha/2}$ and S are unknown since the dataset has yet to be collected. Also, the number of observations, and thus the number of redundancies, is unknown, since they are the computational objectives in this problem. Therefore, Equation (4.10) must be modified to use the standard normal random variable, z and its critical value of E, which is not dependent on υ or n, that is

$$n = \left(\frac{t_{\alpha/2} \sigma}{I} \right)^2 \tag{4.11}$$

where n is the number of repetitions; $t_{\alpha/2}$ is the E value determined from the standard normal distribution table (Table D.1); σ is an estimated value for the standard error of the measurement; and I is the desired confidence interval.

Example 4.2 From the pre-analysis of a horizontal control network, it is known that all angles must be measured to within $\pm 2''$ at the 95% confidence level. How many repetitions will be needed if the standard deviation for a single angle observation has been determined to be $\pm 2.6''$?

SOLUTION In this problem, a final 95% confidence interval of $\pm 2''$ is desired. From previous experience or analysis,[2] the standard error for a single angle observation is estimated as $\pm 2.6''$. From Table 3.2, the multiplier (or E value) for a 95% confidence level is found to be 1.960. Substituting this into Equation (4.11) yields

$$n = \left(\frac{1.960 \times 2.6}{2} \right)^2 = 6.49$$

Thus, eight repetitions are selected since this is the closest even number above 6.49. [Note that it is necessary to select an even number of repetitions because an equal number of face-left (direct) and face-right (reverse) readings must be taken to eliminate instrumental systematic errors.]

[2] See Chapter 6 for procedures used to estimate the variances in angle observations.

4.6 CONFIDENCE INTERVAL FOR A POPULATION VARIANCE



From Equation (4.1), $\chi^2 = \frac{\upsilon S^2}{\sigma^2}$, and thus confidence intervals for the variance of the population, σ^2, are based on the χ^2 statistic, sample variance, and its degrees of freedom. Percentage points (areas) for the upper and lower tails of the χ^2 distribution are tabulated in Table D.2. This table lists values (denoted by χ^2_α) that determine the upper boundary for areas from χ^2_α to $+\infty$ of the distribution, such that

$$P(\chi^2 > \chi^2_\alpha) = \alpha$$

for a given number of redundancies, υ. Unlike the normal distribution and the t distribution, the χ^2 distribution is not symmetric about zero. To locate an area in the lower tail of the distribution, the appropriate value of $\chi^2_{1-\alpha}$ must be found, where $P\left(\chi^2 > \chi^2_{1-\alpha}\right) = 1 - \alpha$. These facts are used to construct a probability statement for χ^2 as:

$$P(\chi^2_{1-\alpha/2} < \chi^2 < \chi^2_{\alpha/2}) = 1 - \alpha \tag{4.12}$$

where $\chi^2_{1-\alpha/2}$ and $\chi^2_{\alpha/2}$ are tabulated in Table D.2 by the number of redundant observations. Substituting Equation (4.1) into Equation (4.12) yields

$$P\left(\chi^2_{1-\alpha/2} < \frac{\upsilon S^2}{\sigma^2} < \chi^2_{\alpha/2}\right) = P\left(\frac{\chi^2_{1-\alpha/2}}{\upsilon S^2} < \frac{1}{\sigma^2} < \frac{\chi^2_{\alpha/2}}{\upsilon S^2}\right) \tag{4.13}$$

Recalling a property of mathematical inequalities, that taking the reciprocal of a function the inequality is reversed, it follows that

$$P\left(\frac{\upsilon S^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{\upsilon S^2}{\chi^2_{1-\alpha/2}}\right) = 1 - \alpha \tag{4.14}$$

Thus, the $(1 - \alpha)$ confidence interval for the population variance (σ^2) is

$$\frac{\upsilon S^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{\upsilon S^2}{\chi^2_{1-\alpha/2}} \tag{4.15}$$

Example 4.3 An observer's pointing and reading error is estimated by collecting 20 readings while pointing at a distant, well-defined target. The sample standard deviation for the operator is determined to be $\pm 1.8''$. What is the 95% confidence interval for σ^2?

SOLUTION For this example, the desired area enclosed by the confidence interval $(1 - \alpha)$ is 0.95. Thus, α is 0.05, which is divided into the lower and upper tails of the distribution, and $\alpha/2$ equals 0.025. The tabulated values of $\chi^2_{0.025}$ and $\chi^2_{0.975}$ with υ equal to 19 degrees of freedom are needed. They are found in the χ^2 table (Table D.2) as follows:

Step 1: Find the row with 19 degrees of freedom and intersect it with the column headed by 0.975. The critical value at the intersection is 8.91.

Step 2: Follow this procedure for 19 degrees of freedom and 0.025. The critical value is 32.85. Using Equation (4.15), the 95% confidence interval for σ^2 is

$$\frac{(20-1)1.8^2}{32.85} < \sigma^2 < \frac{(20-1)1.8^2}{8.91}$$

$$1.87 < \sigma^2 < 6.91$$

Thus, 95% of the time the population's variance should lie between 1.87 and 6.91.

The χ^2 distribution is often used to see if the reference variance from a least-squares adjustment is statistically equal to one. As you will learn in Chapters 5 and 16, a two-tailed statistical test, which is equivalent to this confidence interval, is often used. This test is known as the *goodness-of-fit* test. However, some software packages simply list the reference variance and the range of the confidence interval provided by Equation (4.15) at a user-specified confidence level. The user must then check to see if the reference variance computed from the adjustment is within the confidence interval provided.

4.7 CONFIDENCE INTERVAL FOR THE RATIO OF TWO POPULATION VARIANCES

As will be seen in Section 16.7, another common statistical procedure used is to compare the ratio of two population variances. The sampling distribution of the ratio of σ_1^2/σ_2^2 is well known when samples are collected randomly from a normal population. The confidence interval for σ_1^2/σ_2^2 is based on the F distribution using Equation (4.3) as

$$F = \frac{\dfrac{\chi_1^2}{v_1}}{\dfrac{\chi_2^2}{v_2}}$$

Substituting Equation (4.1) and reducing yields

$$F = \frac{\dfrac{\dfrac{(v_1 S_1^2)}{\sigma_1^2}}{v_1}}{\dfrac{\dfrac{(v_2 S_2^2)}{\sigma_2^2}}{v_2}} = \frac{\dfrac{S_1^2}{\sigma_1^2}}{\dfrac{S_2^2}{\sigma_2^2}} = \frac{S_1^2}{S_2^2}\left(\frac{\sigma_2^2}{\sigma_1^2}\right) \tag{4.16}$$

To establish a confidence interval for the ratio, the lower and upper values corresponding to the tails of the distribution must be found. A probability statement to find the confidence interval for the ratio is constructed as follows:

$$P(F_{1-\alpha/2,\upsilon_1,\upsilon_2} < F < F_{\alpha/2,\upsilon_1,\upsilon_2}) = 1 - \alpha$$

Rearranging yields

$$\begin{aligned} P(F_l < F < F_u) &= P\left[F_l < \frac{S_1^2}{S_2^2}\left(\frac{\sigma_2^2}{\sigma_1^2}\right) < F_u\right] \\ &= P\left(F_l \frac{S_2^2}{S_1^2} < \frac{\sigma_2^2}{\sigma_1^2} < \frac{S_2^2}{S_1^2} F_u\right) \\ &= P\left(\frac{1}{F_u}\frac{S_1^2}{S_2^2} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{S_1^2}{S_2^2}\frac{1}{F_l}\right) = 1 - \alpha \end{aligned} \tag{4.17}$$

where $F_l = F_{1-\alpha/2,\upsilon_1,\upsilon_2}$ and $F_u = F_{\alpha/2,\upsilon_1,\upsilon_2}$. Substituting Equation (4.4) into (4.17) yields

$$\begin{aligned} &P\left(\frac{1}{F_{\alpha/2,\upsilon_1,\upsilon_2}}\frac{S_1^2}{S_2^2} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{S_1^2}{S_2^2}\frac{1}{F_{1-\alpha/2,\upsilon_1,\upsilon_2}}\right) \\ &\quad = P\left(\frac{1}{F_{\alpha/2,\upsilon_1,\upsilon_2}}\frac{S_1^2}{S_2^2} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{S_1^2}{S_2^2} F_{\alpha/2,\upsilon_2,\upsilon_1}\right) = 1 - \alpha \end{aligned} \tag{4.18}$$

Thus, from Equation (4.18), the $(1 - \alpha)$ confidence interval for the σ_1^2/σ_2^2 ratio is

$$\frac{1}{F_{\alpha/2,\upsilon_1,\upsilon_2}}\frac{S_1^2}{S_2^2} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{S_1^2}{S_2^2} F_{\alpha/2,\upsilon_2,\upsilon_1} \tag{4.19}$$

Notice that the degrees of freedom for the upper and lower limits in Equation (4.19) are opposite to each other. Thus υ_2 is the numerator degrees of freedom, and υ_1 is the denominator degrees of freedom in the upper limit.

An important situation where Equation (4.19) can be applied occurs in the analysis and adjustment of horizontal control surveys. During least-squares adjustments of these types of surveys, control stations fix the data in space both positionally and rotationally. When observations tie into more than a minimal amount of control stations, the control coordinates must be mutually consistent. If they are not, any attempt to adjust the observations to the control will warp the data to fit the discrepancies in the control. A method for isolating control stations that are not consistent is to first do a least-squares adjustment using only enough control to fix

the data both positionally and rotationally in space. This is known as a *minimally constrained* adjustment, which is sometimes called a *free* adjustment. In traverse surveys, this means that one station must have fixed coordinates, and one line must be fixed in direction. This adjustment is then followed with an adjustment using all available control. If the control information is consistent, the reference variance (S_1^2) from the minimally constrained adjustment should be statistically equivalent to the reference variance (S_2^2) obtained when using all control information (a constrained adjustment). That is, the ratio of S_1^2/S_2^2 should be approximately equal to 1.

Example 4.4 Assume that a minimally constrained trilateration network adjustment has 24 degrees of freedom and a reference variance of 0.49. A constrained network adjustment of the same data has 30 degrees of freedom with a reference variance of 2.25. What is the 95% $(1-\alpha)$ confidence interval for the ratio of the variances, and does this interval contain the numerical value of 1? Stated in another way, is there reason to be concerned about the control having values that are not consistent with the observational data?

SOLUTION In this example, the objective is to determine whether the two reference variances are statistically equal to ensure data are consistent. To solve the problem, let the variance in the numerator be 2.25 and in the denominator be 0.49. Thus, the numerator has 30 degrees of freedom ($\upsilon_1 = 30$) and corresponds to the adjustment using all the control. The denominator has 24 degrees of freedom ($\upsilon_2 = 24$) and corresponds to the minimally constrained adjustment.[3] With α equal to 0.05 and using Equation (4.19), the 95% confidence interval for this ratio is

$$2.08 = \frac{2.25}{0.49}\left(\frac{1}{2.21}\right) < \frac{\sigma_1^2}{\sigma_2^2} < \frac{2.25}{0.49}(2.14) = 9.83$$

Note from the calculations above that 95% of the time, the ratio of the population variances is in the range 2.08 to 9.83. Since this interval does not contain 1, it can be stated that $\sigma_1^2/\sigma_2^2 \neq 1$, or more precisely $\sigma_1^2 \neq \sigma_2^2$, at a 95% level of confidence. Recalling from Equation (2.4) that the size of the variance depends on the size of the errors, it can be stated that the constrained adjustment revealed discrepancies between the observations and the control. This could be caused by inconsistencies in the coordinates of the control stations or by the presence of uncorrected systematic errors in the observations. An example of an uncorrected systematic error is the failure to reduce ground distance observations to the mapping projection surface.

[3] For confidence intervals, it is not important which variance is selected as the numerator. In this case, the larger variance was selected arbitrarily as the numerator to match statistical testing methods discussed in Chapter 5.

4.8 SOFTWARE

The tables in Appendix D are limited in relationship to the vast number of degrees of freedom and percentage points possible. Fortunately, these limitations can often be overcome with software. For example, the *t* distribution does not contain the required column for determining a 99.7% confidence interval or a row for 43 degrees of freedom. The STATS program, which is available on the companion website for this book, can determine the critical *t* value for this interval. To do this using STATS, select the *t* distribution option from the statistics menu, as shown in Figure 4.5. The software will request the necessary information from the user, as shown in Figure 4.6, and return the critical *t* value, as shown in

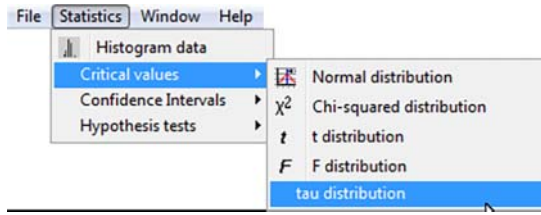


FIGURE 4.5 Selecting the *t* distribution from the ADJUST Statistics menu.

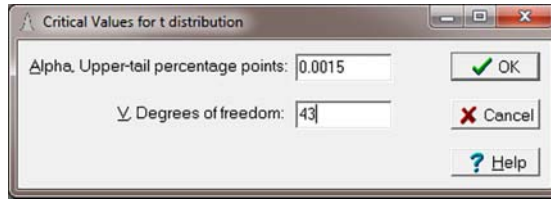


FIGURE 4.6 Entering the upper-tail percentage points and degrees of freedom for the *t* distribution critical value.

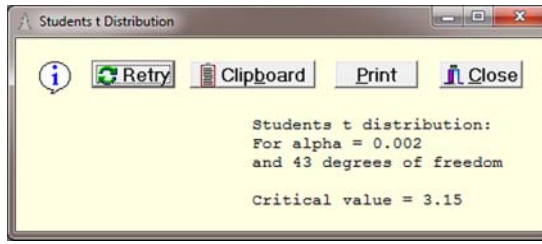


FIGURE 4.7 Computed critical value from *t* distribution for a 99.7% confidence interval with 43 degrees of freedom.

Figure 4.7. Similar procedures can be used to determine the critical values for the χ^2 and F distributions. STATS can also compute confidence intervals at selected probability levels. Figure 4.8 shows the entry screen for Example 4.1. As can be seen from the sample mean, standard deviation, number of observations, and confidence level are entered. Notice that the confidence level is entered as a percentage. In this example, 95 is entered for the confidence level. Upon selecting the OK button, the software displays the computed confidence interval, as shown in Figure 4.9. Similarly, confidence intervals for observations, the population variance, and the ratio of two population variances can be computed using STATS.

Spreadsheets can also be used to determine the critical values for distributions. For example, in Microsoft Excel®, the function *T.INV.2T*(*α, degrees of freedom*) can be used to determine the critical *t* value (formerly *tinv*). However, this function requires the overall percentage points from $+\infty$ to be entered. Thus, to find the critical *t* value for a 99.7% confidence interval with 43 degrees of freedom, the appropriate expression would be *T.INV.2T*(*0.003, 43*) where 0.003 represents the sum of the lower- and upper-tail areas. This results in a value of 3.146. In Excel®, the χ^2 distribution function is *CHISQ.INV.RT* (*α, degrees of freedom*) or *chiinv* in earlier versions of Excel®. However, this function requires the upper and lower percentage points for α, which is 0.0015 and 0.9985 for a 99.7% confidence interval. With this example, a range of 20.529 to 75.762 is computed. Finally, the F distribution critical values can be determined using the function *F.INV.RT*(α, v_1, v_2), where the

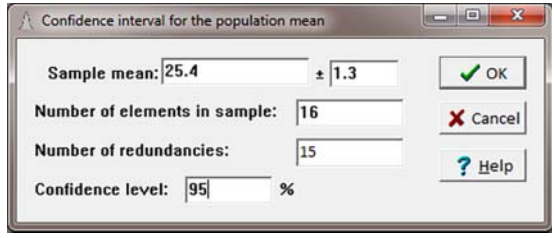


FIGURE 4.8 Entry of data from Example 4.1 into STATS to compute confidence interval.

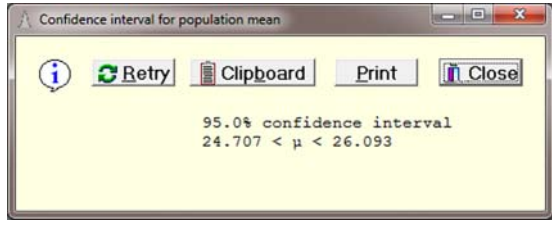


FIGURE 4.9 Computed confidence interval for Example 4.1 from STATS.

function requires the appropriate percentage points for α, which is 0.0015 in this example, v_1 are the number of degrees of freedom for the numerator variance, and v_2 are the number of degrees of freedom for the denominator variance. In earlier versions, *finv* is used. These three functions can be used to reproduce the tables in Appendix D.

Mathcad® can also be used to generate the various critical values not shown in the Appendix D tables. Chapter 4 of the e-book written for Mathcad® that accompanies this book demonstrates the use of these functions while recreating the examples presented in this chapter. Unlike the spreadsheet and the tables in this book, Mathcad® takes the probability and degrees of freedom for its input. For example, the critical value from the *t* distribution for a 99.7% confidence interval with 43 degrees of freedom is computed as $t := qt(0.9985, 43)$, where 0.9985 is determined from the area under the *t* distribution curve going from $-\infty$ to the upper-tail bounds; that is, it is $1 - \alpha/2$, where α is 0.003. Similarly, the χ^2 critical values are computed by calling the function *qchisq*(*p, df*), where *p* is the probability as defined previously and *df* are the number of degrees of freedom for the variance. *F*-distribution critical values are determined by calling the function $qF(p, df_1, df_2)$, where *p* is the probability as defined previously, df_1 are the number of degrees of freedom for the numerator variance, and df_2 are the number of degrees of freedom for the denominator variance.

PROBLEMS

Note: Partial answers to problems marked with an asterisk are given in Appendix G.

4.1 What is the estimator function for the population mean?

4.2 Use the χ^2 distribution table (Table D.2) to determine the critical values from the χ^2 distribution, which would be used to construct confidence intervals for a population variance for the following combinations:

***(a)** $\alpha = 0.10, v = 4$

(b) $\alpha = 0.05, v = 10$

(c) $\alpha = 0.01, v = 20$

(d) $\alpha = 0.005, v = 20$

4.3 Use the *t* distribution table (Table D.3) to determine the critical values from the distribution which would be used to construct confidence intervals for a population mean for each of the following combinations:

***(a)** $\alpha = 0.10, v = 4$

(b) $\alpha = 0.05, v = 10$

(c) $\alpha = 0.01, v = 20$

(d) $\alpha = 0.005, v = 20$

4.4 Use the F distribution table (Table D.4) to determine the critical values from the F distribution, which would be used to construct confidence intervals for the ratio of two sample variances for each of the following combinations:

***(a)** $\alpha = 0.10, \upsilon_1 = 20, \upsilon_2 = 5$

(b) $\alpha = 0.05, \upsilon_1 = 10, \upsilon_2 = 8$

(c) $\alpha = 0.01, \upsilon_1 = 20, \upsilon_2 = 10$

(d) $\alpha = 0.005, \upsilon_1 = 30, \upsilon_2 = 20$

4.5 What is the computed χ^2 value for the following conditions?

***(a)** $\upsilon = 4, S^2 = 1.56$, and $\sigma^2 = 1.00$

(b) $\upsilon = 9, S^2 = 0.75$, and $\sigma^2 = 1.00$

(c) $\upsilon = 15, S^2 = 0.12$, and $\sigma^2 = 1.00$

(d) $\upsilon = 20, S^2 = 1.92$, and $\sigma^2 = 1.00$

4.6 A least-squares adjustment is computed twice on a dataset. When the data are minimally constrained with 14 degrees of freedom, a variance of 1.09 is obtained. In the second run, the fully constrained network has 20 degrees of freedom with a variance of 1.34. The a priori estimate for the reference variances in both adjustments are one; that is, $\sigma_1^2 = \sigma_2^2 = 1$.

(a) What is the 95% confidence interval for the reference variance in the minimally constrained adjustment? The population variance is one. Does this interval contain one?

(b) What is the 95% confidence interval for the reference variance in the fully constrained adjustment? The population variance is one. Does this interval contain one?

(c) What is the 95% confidence interval for the ratio of the two variances? Is there any reason to be concerned about the consistency of the control? Statistically justify your response.

4.7 Repeat Problem 4.6 where the minimally constrained adjustment has 19 degrees of freedom, a standard deviation of 0.63 is obtained and the fully constrained adjustment has 24 degrees of freedom with a standard deviation of 1.99. Use a 99% confidence interval.

(a) What is the 99% confidence interval for the reference variance in the minimally constrained adjustment? The population variance is one. Does this interval contain one?

(b) What is the 99% confidence interval for the reference variance in the fully constrained adjustment? The population variance is one. Does this interval contain one?

(c) What is the 99% confidence interval for the ratio of the two variances? Is there any reason to be concerned about the consistency of the control? Statistically justify your response.

4.8 The calibrated length of a baseline is 132.182 m. A properly reduced average distance of 132.186 m with a standard deviation of ±0.0028 m is computed after the line is observed six times with an EDM.

***(a)** What is the 95% confidence interval for the population mean?

(b) At a 95% level of confidence, can you state that the EDM is working properly? Statistically justify your response.

4.9 Repeat Problem 4.8 using a calibrated baseline of 1200.048 m and an observed length of 1200.045 with a standard deviation of ±0.0035 m after the line is observed 12 times with an EDM.

(a) What is the 95% confidence interval for the population mean?

(b) At a 95% level of confidence, can you state that the EDM is working properly? Statistically justify your response.

4.10 Repeat Problem 4.8 at 99% confidence.

(a) What is the 99% confidence interval for the population mean?

(b) At a 99% level of confidence, can you state that the EDM is working properly? Statistically justify your response.

4.11 Repeat Problem 4.9 at 99%.

(a) What is the 99% confidence interval for the population mean?

(b) At a 99% level of confidence, can you state that the EDM is working properly? Statistically justify your response.

4.12 Develop a 95% confidence interval for the data in Problem 3.18 and determine if any observations may be rejected as outliers.

4.13 Develop a 95% confidence interval for the data in Problem 3.19 and determine if any observations may be rejected as outliers.

***4.14** An observer's pointing and reading standard deviation is determined to be ±3.4″ after pointing and reading the circles of a particular instrument 10 times ($n = 10$). What is the 99% confidence interval for the population variance?

4.15 An observer's pointing and reading standard deviation is determined to be ±3.2″ after pointing and reading the circles of a particular instrument 10 times ($n = 10$). What is the 95% confidence interval for the population variance?

4.16 Repeat Problem 4.14 at 95%.

4.17 Repeat Problem 4.15 at 99%.

4.18 Was the observer in Problem 4.15 statistically better than the observer in Problem 4.14 at a

***(a)** 90% level of confidence?

(b) 99% level of confidence?

4.19 Using sample statistics and the data in Example 3.1 construct a 99% confidence interval

(a) For the data and identify any observations that may be outliers.

(b) For the population variance.

4.20 Using sample statistics and the data in Example 3.2 construct a 99% confidence interval

(a) For the data and identify any observations that may be outliers.

(b) For the population variance. $\bar{s} = 45.04, S = \pm 2.12, n = 50, v = 49$.

4.21 For the data in Problem 3.16 construct a 99% confidence interval for the ratio of the two variances for Sets 1 and 2. Are the variances statistically equal at this level of confidence? Justify your response.

4.22 Using sample statistics and the data from Problem 3.17 construct a 95% confidence interval and

(a) Identify any observations that may be identified as possible outliers in the data.

(b) For the mean.

4.23 Using sample statistics (in feet) and the data from Problem 3.13 construct a 95% confidence interval and

(a) Identify any observations that may be identified as possible outliers in the data.

(b) For the population variance

(c) If the length observed in Problem 3.13 was part of a calibration baseline with a published length of 449.972 ft, and the reported distances were properly reduced to their ground marks, is the instrument working properly at a 95% level of confidence?

(d) If the manufacturer specifications for this instrument indicate that at a distance of this length, the measurements should have a variance of 0.000025, is the precision of the instrument meeting the specifications at a 95% level of confidence?

4.24 Use STATS to determine the critical values from the χ^2 distribution that would be used to construct a 95% confidence interval for a population variance when there are 44 degrees of freedom.

4.25 Use STATS to determine the critical value from the t distribution that would be used to construct a 99.7% confidence interval for a population mean when there are 21 degrees of freedom.

4.26 Use STATS to determine the critical values from the F distribution that would be used to construct a 99.7% confidence interval for the ratio of two sample variances when $\upsilon_1 = 20$ and $\upsilon_2 = 22$.

4.27 Describe a method used to determine if the control station coordinates are inconsistent with the observations.

CHAPTER 5

STATISTICAL TESTING

5.1 HYPOTHESIS TESTING



In Example 4.4, we were not concerned about the actual bounds of the constructed interval, but rather whether the constructed interval contained the expected ratio of the variances. This is often the case in statistics. That is, the actual values of the interval are not as important as is answering the question: Is the sample statistic consistent with what is expected from the population? The procedures used to test the validity of a statistic are known as *hypothesis testing*. The basic elements of a hypothesis test are as follows:

1. The *null hypothesis*, H_0, is a statement that compares a population statistic, which is usually derived from project or instrument specifications, with the population of a sample. This implies that the sample statistic is what is "expected" from the population. In Example 4.4, this would be when the ratio of the variances is statistically one.
2. The *alternative hypothesis*, H_a, is what is accepted when a decision is made to reject the null hypothesis and thus represents an alternative population of data from which the sample statistic was derived. In Example 4.4, the alternative

hypothesis would be the case where the ratio of the variances is not equal to 1, and thus, the variance did not come from the same population of data.

3. The *test statistic* is computed from the sample data and is the value used to determine whether the null hypothesis should be rejected. When the null hypothesis is rejected, it can be said that the sample statistic computed is not consistent with what is expected from the population. In Example 4.4, a rejection of the null hypothesis would occur when the ratio of the variances is not statistically equivalent to 1.
4. The *rejection criterion* is the expression for the test statistic where the null hypothesis is rejected. In reference to confidence intervals, this number takes the place of the confidence interval bounds. That is, when the computed test statistic is greater than the rejection criterion defining the rejection region, it is equivalent to the sample statistic of the null hypothesis being outside the bounds of the confidence interval. Thus, when the rejection criterion is true, the null hypothesis is rejected.
5. The *test decision* is the practical result of the test. This is a statement about whether the observation or measurement statistically agrees with normal distribution theory. If the test fails to reject the null hypothesis then there is no statistical reason to question the normality of the data. On the other hand, if the test rejects the null hypothesis then there is a statistical reason to question the normality of the observations.

Whenever a test decision is made concerning the null hypothesis, there is always a possibility of making a wrong decision since there can never be 100% certainty about a statistic or a test. Returning to Example 4.4, a confidence interval of 95% was constructed. With this interval, there is a 5% chance that the decision is incorrect. That is, it is possible that the larger than expected ratio of the variances is consistent with the population of observations. This reasoning suggests that further analysis of statistical testing is needed.

Two basic errors can occur when a decision is made about a statistic. A valid statistic could be rejected, or an invalid statistic could be accepted. These two errors can be stated in terms of statistical testing elements as *Type I* and *Type II* errors. If the null hypothesis, H_0, is rejected when in fact it is true, a Type I error is committed. If the null hypothesis is not rejected when in fact it is false, a Type II error occurs. Since these errors are not from the same population, the probability of committing each error is not directly related. A decision must be made as to the type of error that is more serious for the situation and thus based on the consequences of committing each error. For instance, if a contract calls for positional accuracies on 95% of the stations to be within ± 0.3 ft, the surveyor is more inclined to accept a Type I error to ensure that the contract specifications are met. However, the same surveyor needing only 1-ft accuracy on control to support a small-scale mapping project may be more inclined to accept a Type II error. In either case, it is important to compute the probabilities of committing both Type I and Type II errors to assess

the reliability of the inferences derived from a hypothesis test. For emphasis, the two basic hypothesis-testing errors are repeated.

Type I Error: Rejecting the null hypothesis H_0 when it is in fact true (symbolized by α).

Type II Error: Not rejecting the null hypothesis H_0 when it is in fact false (symbolized by β).

Table 5.1 shows the relationship between the decision, the probabilities of α and β, and the acceptance or rejection of the null hypothesis, H_0. In Figure 5.1, the left distribution represents the data from which the null hypothesis is derived. That is, this distribution represents a true null hypothesis. Similarly, the distribution on the right represents the distribution of data for the true alternative. These two distributions can be attributed to measurements that contain only random errors (left distribution) versus measurements containing blunders (right distribution). In the figure, it is seen that valid measurements in the α region of the left distribution are being rejected at a significance level of α. Thus, α represents the probability of committing a Type I error. This is known as the *significance level of the test*. Furthermore, data from the right distribution are being accepted at a β level of significance. The *power of the test* is $1 - \beta$ and corresponds to a true alternative

TABLE 5.1 Relationship in Statistical Testing

	Decision	
Situation	Accept H_0	Reject H_0
H_0 true	Correct decision: $P = 1 - \alpha$ (confidence level)	Type I error: $P = \alpha$ (significance level)
H_0 false (H_a true)	Type II error: $P = \beta$	Correct decision: $P = 1 - \beta$ (power of test)

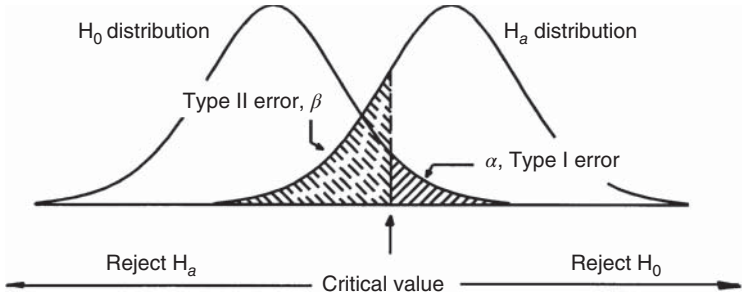


FIGURE 5.1 Graphical interpretation of Type I and Type II errors.

hypothesis H_a. Methods of computing β or $1 - \beta$ are not clear and are often difficult since nothing is generally known about the distribution of the alternative hypothesis. Consequently, in statistical testing, the objective is to prove the alternative hypothesis H_a true by showing that the data does not support the statistic coming from the null hypothesis distribution. In doing this, only a Type I error can be made, for which a known probability of making a wrong decision is α.

Example 5.1 As an example, assume that for a population of 10,000 people, a flu virus test has a 95% level of confidence and thus a significance level, α, of 0.05. Suppose that 9200 people test negative for the flu virus and 800 test positive. Of the 800 people who tested positive, 5% or 40 people will test incorrectly (false-positive). That is, they will test positive for the flu but do not have it. This is an example of committing a Type I error at an α level of significance. Similarly, assume that 460 people test negative for the flu when, in fact, they do have it (a false-negative case). This is an example of a Type II error at a probability of β that is equal to 0.046 (460/10,000). Furthermore, in this example, the power of the test is $1 - \beta$ or 0.954.

From the foregoing, it is seen that it is possible to set the probability of committing a Type I error for a given H_0. However, for a fixed level of α and sample size n, the probability of a Type II error, β, is generally unknown. If the null hypothesis H_0 and α are fixed, the power of the test can only be increased by increasing the sample size, n. Since the power of the test may be low or unknown, statisticians always state that the test *failed to reject* the null hypothesis H_0 rather than making any statement about its acceptance. This is an important statistical concept. That is, it should never be stated that the null hypothesis H_0 is accepted since the power of the test is unknown. It should only be said that "there is no statistical evidence to reject the null hypothesis." Because of this small but important distinction, it is important to construct a test that rejects the null hypothesis whenever it is possible.

A similar situation exists with surveying measurements. If a distance measurement contains a large systematic error, it is possible to detect this with a fully constrained adjustment and thus reject the null hypothesis. However, if a distance contains a very small systematic error, the ability to detect the systematic error may be low. Thus, while some confidence can be placed in the rejection of the null hypothesis, it can never be said that the null hypothesis H_0 should be accepted since the probability of undetected small systematic errors or blunders cannot be determined. What we strive to do is minimize the size of these errors so that they have little effect on the computed results.

5.2 SYSTEMATIC DEVELOPMENT OF A TEST

When developing a statistical test, the statistician must determine the test variables and the type of test to perform. This chapter will explore statistical tests for

the mean, variance, and ratio of two sample variances. The t test is used when comparing the sample mean of one population against another population mean. In surveying, the test often compares the mean of a sample set of observations against a known calibration value. The χ^2 test is used when comparing the sample variance of one population against another population variance. As discussed in Section 16.7, this test is used in a least-squares adjustment when comparing the reference variance from an adjustment against its known population value. Finally, when comparing sample variances from two different populations, the F test is used. As discussed in Section 21.7, this test is used in least-squares adjustments when comparing the reference variances from a minimally constrained and over-constrained adjustment. Table 5.2 lists the test variables for these three statistical tests.

A test can take two forms based on their distributions. The one-tailed test uses the critical value from either the left or right side of the distribution, whereas the two-tailed test is much like a confidence interval, with critical value divided equally on both sides of the distribution.

In the *one-tailed test*, the concern is whether the sample statistic is either greater or less than the statistic being tested. In the *two-tailed test*, the concern is whether the sample statistic is different from the statistic being tested. For example, when checking the angle reading capabilities of a total station against the manufacturer's specifications, the surveyor would not be concerned if the instrument is working at a level better than the manufacturer's stated accuracy. However, the surveyor would probably send the instrument in for repairs if it was performing worse than the manufacturer's stated accuracy. In this case, it would be appropriate to perform a one-tailed test. On the other hand, when checking the mean observed distance from an EDM against a known calibration baseline length, the surveyor wants to know if the mean length is statistically different from the calibration length. In this case, it is appropriate to perform a two-tailed test.

Statistical tests can be applied to evaluate many different types of survey instruments in addition to a total station. For example, a test could determine if certain observations with GNSS are likely affected by multipath or other issues

TABLE 5.2 Test Variables and Statistical Tests

Variable 1, Test Statistic	Variable 2, Sample Population	Null Hypothesis	Test Statistic
Population mean, μ_0	Mean, μ	$\mu = \mu_0$	t
Population variance, σ_0^2	Variance, σ^2	H_0: $\sigma^2 = \sigma_0^2$	χ^2
Ratio of sample variances equals 1	$\frac{\sigma_1^2}{\sigma_2^2}$	$H_0 : \frac{\sigma_1^2}{\sigma_2^2} = 1$	F

by repeating observations on a control point at different times and comparing with manufacturer precision estimates. For laser scanning or photogrammetry, a statistical test can be used for construction quality control to identify if there are significant deviations in the surface by comparing the variance of points used to fit planes or other geometric shapes to the surface with the estimated precision of the measurements. However, in such a case, one must be careful in designing the survey plan to minimize additional measurement errors resulting from poor scanning geometry (e.g., oblique view of the surface).

In the following sections, it is important to identify (1) the appropriate test statistic and (2) the type of test to perform. In all forms of statistical testing, a test statistic is developed from the sample set. The test statistic is then compared against a critical value from the distribution. If the rejection region statement is true, the null hypothesis H_0 is rejected at the selected level of significance. As stated earlier, this is the goal of a well-developed test since only the Type I error occurs at a known level of significance, which is α. If the rejection region statement is false, the test fails to reject the null hypothesis, H_0. Because of the possibility of a Type II error, and due to the lack of knowledge about the alternative distribution, no statement about the validity of the null hypothesis can be made. At best, it can be stated that there is no reason to reject the null hypothesis, H_0.

5.3 TEST OF HYPOTHESIS FOR THE POPULATION MEAN

To ensure that the quality of their beer was consistent, Guinness Brewery approached their process scientifically. Questions such as "Where do the best ingredients in the brewing process come from, or how much bitters, sugars, etc. develop the best brew?" needed to be answered. Of course, Guiness, like surveyors, were dealing with small sample sizes. Guinness hired William Sealy Gosset, a mathematician, to answer these questions. In the early twentieth century, Gosset derived the *t* distribution and *t* test as a method of answering the questions. He wanted to publish the paper on his scientific method, but Guinness did not wish to have competitors realize that Guinness was using scientific methods in their brewing recipe. So, he allowed Gosset to publish his findings about the *t* distribution under the pseudonym of Student. Today this is often referred to as the Student's *t* test for this reason.



At times, it may be desirable to test the population of a sample mean against a known value. The *t* distribution is used to build this test. The null hypothesis, H_0, for this test can take two forms: one- or two-tailed tests. In the *one-tailed test*, the concern is whether the sample mean is either statistically greater or less than the population mean. In the *two-tailed test*, the concern is whether the population of the sample mean, μ_0, is statistically different from another population mean, μ; this is equivalent to seeing if the population mean is in a confidence interval as was done in Chapter 4. These two tests are shown below.

	One-Tailed Test	Two-Tailed Test
Null hypothesis:	H_0: $\mu = \mu_0$	H_0: $\mu = \mu_0$
Alternative hypothesis:	H_a : $\mu > \mu_0$ ($\mu < \mu_0$)	H_a : $\mu \neq \mu_0$
Test statistic:[1]	$t = \dfrac{\bar{y} - \mu_0}{\frac{S}{\sqrt{n}}}$	(5.1)
The region where the null hypothesis H_0 is rejected is	$t > t_\alpha$ (or $t < t_\alpha$)	$\lvert t \rvert > t_{\alpha/2}$

It should be stated that for large samples ($n > 30$), the *t value* can be replaced by the standard normal variate, z.

Example 5.2 A baseline of calibrated length 400.008 m is observed 20 times with an EDM instrument. The average of the 20 observed distances that are properly reduced is 400.012 m with a standard deviation of ±0.002 m. Is the reduced distance different from the calibrated distance at a 0.05 level of significance?

SOLUTION Assuming that proper field and office procedures were followed, the fundamental question is whether the EDM is working within its specifications; that is, is the EDM providing distance observations in the population of calibrated values. To answer this question, a two-tailed test is used to determine whether the distance is either the same or different than the calibrated distance at a 0.05 level of significance. That is, the mean of the observed distances will be rejected if it is either statistically too short or too long to be considered the same as the calibration value. The rationale behind using a two-tailed test is similar to that used when constructing a confidence interval, as in Example 4.1. That is, 2.5% of the area from the lower and upper tails of the t distribution is to be excluded from the interval constructed, or in this case, the test.

Null hypothesis:	H_0: $\mu = 400.012$
Alternative hypothesis:	H_a: $\mu \neq 400.012$
By Equation (5.1), the test statistic is	$t = \dfrac{\bar{y} - \mu}{\frac{S}{\sqrt{n}}} = \dfrac{\lvert 400.012 - 400.008 \rvert}{\frac{0.002}{\sqrt{20}}} = 8.944$
Rejection criterion:[2]	$t = 8.944 > 2.093 = t_{\alpha/2}$

Since a two-tailed test is being performed, the $\alpha/2$ (0.025) column in the t distribution table is intersected with the 19 degrees of freedom row ($\upsilon = n - 1$). From the

[1] When performing a two-tailed test, you should take the absolute value of t.
[2] Observed distances must be reduced to the marks on the ground when using a calibration baseline.

t distribution (Table D.3), $t_{0.025,19}$ is found to be 2.093. Since the absolute value of the computed t, which is 8.944, is greater than the tabulated value of 2.093, the rejection criterion is satisfied. Since this rejection criterion is true, the null hypothesis H_0 can be rejected at a 5% level of significance or 95% level of confidence. This implies that at least 5% of the time, the decision will be wrong and that a Type I error will occur.

As stated earlier, a 95% confidence interval for the population mean could have been constructed also to derive the same results. Using Equation (4.7), the confidence interval is

$$400.011 = 400.012 - 2.093\left(\frac{0.002}{\sqrt{20}}\right) \le \mu \le 400.012 + 2.093\left(\frac{0.002}{\sqrt{20}}\right)$$
$$= 400.013$$

Note that this 95% confidence interval fails to contain the baseline value of 400.008, and similarly, there is reason to be concerned about the calibration status of the instrument. That is, the EDM may not be functioning properly and should be retested or sent in for repair.

5.4 TEST OF HYPOTHESIS FOR THE POPULATION VARIANCE



After checking whether an observed length compares favorably with a calibrated value, as shown with the procedure in Example 5.2, the surveyor may also want to check if the instrument is measuring at its published precision. The χ^2 distribution can compare the variance of one population, σ_0^2, against the variance from another population, σ^2.

As shown in Table 5.2, the χ^2 distribution checks the sample variance from one population against another population variance, which is typically a specification of an instrument as reported by a manufacturer or a requirement for the survey to meet a certain accuracy. By using Equation (4.1), the following statistical test is written.

	One-Tailed Test	Two-Tailed Test
Null hypothesis:	H_0: $\sigma^2 = \sigma_0^2$	H_0: $\sigma^2 = \sigma_0^2$
Alternative hypothesis:	H_a: $\sigma^2 > \sigma_0^2$ (or H_a: $\sigma^2 < \sigma_0^2$)	H_a: $\sigma^2 \neq \sigma_0^2$
Test statistic:	$\chi^2 = \dfrac{\upsilon S^2}{\sigma_0^2}$ (5.2)	
The null hypothesis H_0 is rejected when the rejection criterion is satisfied		
	$\chi^2 > \chi_a^2$ (or $\chi^2 > \chi_{1-a}^2$)	$\chi^2 < \chi_{1-\alpha/2}^2$ or $\chi^2 > \chi_{\alpha/2}^2$

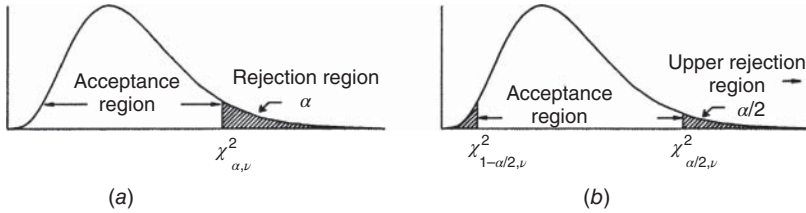


FIGURE 5.2 Graphical interpretation of (*a*) one- and (*b*) two-tailed tests.

Graphically, the null hypothesis H_0 is rejected in the one-tailed test when the computed χ^2 value is greater than the tabulated value when the upper-tail is being checked or when the χ^2 value is less than the tabulated when the lower-tail is being checked. The upper-tail rejection region is the shaded region shown in Figure 5.2(*a*). In the two-tailed test, the null hypothesis H_0 is rejected when the computed value is either less than $\chi^2_{1-\alpha/2}$ or greater than $\chi^2_{\alpha/2}$. This is similar to the population variance being outside the confidence interval constructed from the sample variance using Equation (4.15). Again, in the two-tailed test, the probability selected is evenly divided between the upper and lower tails of the distribution such that the acceptance region is centered on the distribution. These rejection criteria are shown graphically in Figure 5.2.

Example 5.3 The owner of a surveying firm wants all surveying technicians to be able to read a particular instrument to within $\pm 1.5''$. To test this value, the owner asks the senior field crew chief to perform a reading test with the instrument. The crew chief reads the circle 30 times and obtains a $\sigma_r = \pm 0.9''$. Does this support the $\pm 1.5''$ limit at a 5% level of significance?

SOLUTION In this case, the owner wishes to test the null hypothesis that σ^2 equals 1.5^2, against the alternative of σ^2 being greater than 1.5^2. That is, all standard deviations that are equal to or less than $1.5''$ will be accepted. Thus, a one-tailed test is constructed as follows (note that $\upsilon = 30 - 1$, or 29):

Null hypothesis:	$H_0: \sigma^2 = 1.5^2$
Alternative hypothesis:	$H_a: \sigma^2 > 1.5^2$
Test statistic from Equation (5.2):	$\chi^2 = \dfrac{(30-1)0.9^2}{1.5^2} = 10.44$

The null hypothesis, H_0, is rejected when the computed test statistic exceeds the tabulated critical value; that is, if the following statement is true.

$$\chi^2 = 10.44 > 42.56 = \chi^2_{0.05,29}$$

The critical value 42.56 comes from Table D.2 for $\chi^2_{0.05,29}$. Since the computed χ^2 value (10.44) is less than the tabulated critical value (42.56), the null hypothesis H_0 cannot be rejected.

However, simply failing to reject the null hypothesis does not mean that the value of $\pm 1.5''$ is valid. This example demonstrates a common problem in statistical testing when results are interpreted incorrectly. A valid sample set from the population of all surveying employees cannot be obtained by selecting only one employee. Furthermore, the test is flawed since every instrument reads differently and is in a different state of calibration. Additionally, a new employee will have problems reading an instrument initially due to their lack of experience with the instrument. To account properly for this lack of experience, the employer could test a random sample of prospective employees during the interview process and again after several months of employment. The owner could then check for a correlation between the company's satisfaction with the employee and the employee's initial ability to read the instrument. Nevertheless, it is unlikely that any correlation would be found. Ultimately, this is an example of misusing statistics.

The previous example illustrates an important point to be made when using statistics. The interpretation of statistical testing requires judgment by the person performing the test. It should always be remembered that with a test, the objective is to reject and not accept the null hypothesis. Furthermore, a statistical test should only be used where appropriate.

Example 5.4 The manufacturer specified accuracy for an EDM measuring a 1500-m distance is ± 6 mm. A standard deviation of ± 8 mm is obtained with this instrument from 10 observations of a 1500-m distance. Is this instrument repeating measurements within its specifications at a 1% level of significance?

SOLUTION In this case, we want to test if σ^2 equals 6^2 versus if σ^2 is greater than 6^2. In other words, the question being posed is whether the population variance of the sample is statistically higher than the specified accuracy of the instrument from the manufacturer. Thus, a one-tailed test with α equal to 0.01 is performed as:

Null hypothesis: $H_0 : \sigma^2 = 6^2$

Alternative hypothesis: $H_a : \sigma^2 > 6^2$

Test statistic from Equation (5.2): $\chi^2 = \dfrac{9(8^2)}{6^2} = 16$

The null hypothesis H_0 will be rejected if the computed χ^2 value is greater than the tabulated value of $\chi^2_{0.01,9}$ or when the following statement is true:

$$\chi^2 = 16 > 21.67 = \chi^2_{0.01,9}$$

Since this statement is false, that is, 16 is not greater than 21.67, the null hypothesis H_0 cannot be rejected at a 1% level of significance. However, this does not mean that we can accept the fact that the instrument is repeating satisfactorily since we do not have any knowledge about the alternative distribution. Thus, we can simply state that due to this test, we see no reason at this time to question the repeatability of this instrument or send the instrument in for repair since we know nothing about the possible presence of a Type II error.

Example 5.4 demonstrates the problem of a Type II error; that is, a negative result (failing to reject the null hypothesis) prevents one from accepting the null hypothesis H_0 as true. This problem is often overcome in statistical testing by repeated experiments. That is, in this example, the same process could be repeated on different days at different times to introduce the variability of the environment into the test. If the test continues to fail to reject the null hypothesis, then the instrument could be assumed to be working properly. In medical testing, a positive result on a test often means that the client has tested positive for some disease. To ensure that a Type I or Type II is not present, the test is often repeated several times to check if the same result is obtained. Even then, some incorrect decisions are made.

5.5 TEST OF HYPOTHESIS FOR THE RATIO OF TWO POPULATION VARIANCES



Surveyors have generally considered control to be absolute and without error when adjusting data. However, it is a known fact that control, like any other quantities derived from observations, may contain errors. As discussed in Section 4.7, one method to detect for potential errors in control and systematic errors in observations is to perform both minimally constrained and fully constrained least-squares adjustments. After performing both adjustments, the ratio of the post-adjustment reference variances can be compared. If the control is without error or has insignificant errors[3] and no systematic errors are present in the data, the ratio of the two reference variances should be close to 1. Using Equation (4.18), a

[3]Chapter 20 presents a method of adjusting the control so that their errors are correctly taken into consideration.

hypothesis test can be constructed to compare the ratio of two population variances using their two sample variances as follows:

	One-Tailed Test	Two-Tailed Test
Null hypothesis:	$H_0: \dfrac{\sigma_1^2}{\sigma_2^2} = 1 \left(\text{i.e.}, \sigma_1^2 = \sigma_2^2\right)$	$H_0: \dfrac{\sigma_1^2}{\sigma_2^2} = 1 \left(\text{i.e.}, \sigma_1^2 = \sigma_2^2\right)$
Alternative hypothesis:	$H_a: \dfrac{\sigma_1^2}{\sigma_2^2} > 1 \left(\text{i.e.}, \sigma_1^2 > \sigma_2^2\right)$ or $H_a: \dfrac{\sigma_1^2}{\sigma_2^2} < 1 \left(\text{i.e.}, \sigma_1^2 < \sigma_2^2\right)$	$H_a: \dfrac{\sigma_1^2}{\sigma_2^2} \neq 1 \left(\text{i.e.}, \sigma_1^2 \neq \sigma_2^2\right)$
Test statistic:	$F = \dfrac{S_1^2}{S_2^2}$ or $F = \dfrac{S_2^2}{S_1^2}$	$F = \dfrac{\text{larger sample variance}}{\text{smaller sample variance}}$
The null hypothesis should be rejected when the following statement is satisfied:		
	$F > F_\alpha$	$F > F_{\alpha/2}$

F_α and $F_{\alpha/2}$ are critical values that locate the α and $\alpha/2$ areas, respectively, in the upper tail of the F distribution with υ_1 numerator degrees of freedom and υ_2 denominator degrees of freedom. Notice that in the two-tailed test, the degrees of freedom of the numerator are taken from the numerically larger sample variance and the degrees of freedom of the denominator are from the smaller variance.

Example 5.5 Using the same data as presented in Example 4.4, would the null hypothesis be rejected?

SOLUTION In this example, a two-tailed test is appropriate since the only concern is whether the two reference variances are statistically equal or not. The interval is centered on the F distribution with an $\alpha/2$ area in the lower and upper tails. In the analysis, the 30 degrees of freedom in the numerator correspond to the larger sample variance, and the degrees of freedom in the denominator are 24 so that the following test is constructed:

Null hypothesis: $\mathrm{H}_0: \dfrac{\sigma_1^2}{\sigma_2^2} = 1$

Alternative hypothesis: $\mathrm{H}_a: \dfrac{\sigma_1^2}{\sigma_2^2} \neq 1$

Test statistic: $F = \dfrac{2.25}{0.49} = 4.59$

Rejection of the null hypothesis occurs when the following statement is true:

$$F = 4.59 > 2.21 = F_{0.025,30,24}$$

Here it is seen that the computed F value (4.59) is greater than its tabulated critical value (2.21) from Table D.4 with υ_1 equal to 30 and υ_2 equal to 24. Thus, the null hypothesis H_0 can be rejected with a 5% chance of committing a Type I error. In other words, the fully constrained adjustment does not have the same variance as its minimally constrained counterpart at the selected 0.05 level of significance. Notice that the same result was obtained here as was obtained in Example 4.4 with the 95% confidence interval. Again, the network should be inspected for the presence of systematic errors, followed by an analysis of possible errors in the control stations. This post-adjustment analysis is revisited in greater detail in Chapters 20 and 25.

Example 5.6 Ron and Kathi continually debate who observes angles more precisely with a particular total station. Their supervisor, after hearing enough, describes a test where each is to measure a particular direction by pointing and reading the instrument 51 times. They must then compute the variance for their data. At the end of the 51 readings, Kathi determines her variance to be 0.81, and Ron finds his to be 1.21. Is Kathi a better instrument operator at a 0.01 level of significance?

SOLUTION In this situation, even though Kathi's variance implies that her observations come from a more precise population than Ron's, a determination must be made to see if the reference variances are statistically equal versus Kathi's being better than Ron's. This test requires a one-tailed F test with a significance level of $\alpha = 0.01$.

Null hypothesis: $$H_0 : \frac{\sigma_R^2}{\sigma_K^2} = 1 \left(\sigma_R^2 = \sigma_K^2\right)$$

Alternative hypothesis: $$\mathrm{H}_a : \frac{\sigma_R^2}{\sigma_K^2} > 1 \left(\sigma_R^2 > \sigma_K^2\right)$$

Test statistic: $$F = \frac{1.21}{0.81} = 1.49$$

The null hypothesis is rejected when the computed value for F (1.49) is greater than the tabulated value of $F_{0.01,50,50}$ (1.95) where $\upsilon_1 = \upsilon_2 = 50$ $(51 - 1)$. Here it is seen that the computed value for F is less than its tabulated critical value, and thus, the test statistic does not satisfy the rejection region. That is, the rejection criterion is:

$$F = 1.49 > 1.95 = F_{\alpha,50,50}$$

Here the rejection criterion is false; that is, $1.49 < 1.95$, and thus, there is no statistical reason to believe that Kathi is better than Ron at a 0.01 level of significance.

Example 5.7 A baseline is repeatedly observed using an EDM instrument over a period of time. Each day, 10 observations are taken and averaged. The variances for the observations are listed below. At a significance level of 0.05, are the results of Day 2 significantly different from those of Day 5?

Day	1	2	3	4	5
Variance, σ^2 (mm^2)	50.0	61.0	51.0	53.0	54.0

SOLUTION This problem involves checking whether the variances of Day 2 and Day 5 are statistically equal versus their being different. This is the same as constructing a confidence interval involving the ratio of the variances. Because the concern is about equality or inequality, this will require a two-tailed test. Since there were 10 observations collected each day, both variances are based on 9 degrees of freedom (υ_1 and υ_2). Assume that the variance for Day 2 is σ_2^2 and the variance for Day 5 is σ_5^2. The test is constructed as follows:

Null hypothesis: $$H_0 : \frac{\sigma_2^2}{\sigma_5^2} = 1 \left(\sigma_2^2 = \sigma_5^2\right)$$

Alternative hypothesis: $$H_a : \frac{\sigma_2^2}{\sigma_5^2} \neq 1 \left(\sigma_2^2 \neq \sigma_5^2\right)$$

Test statistic: $$F = \frac{61}{54} = 1.13$$

The null hypothesis is rejected when the computed F value (1.13) is greater than the tabulated critical value (4.03) in Table D.4. In this case, the rejection criterion of $F = 1.13 > 4.03 = F_{0.025,9,9}$ is not satisfied; that is, $1.13 < 4.03$. Consequently, the test fails to reject the null hypothesis H_0, and there is no statistical reason to believe that the data of Day 2 are statistically different from those of Day 5.

5.6 USING SOFTWARE

STATS has the capability to perform hypothesis tests for the population mean, variance, and the ratio of two variances. This option, found in the statistics menu, provides a check on computations for the users of this book. Figure 5.3 shows the entry screen for the values from Example 5.2. Notice that the level of significance should be entered in percentage points and that the user can select that the test be performed in the lower tail or upper tail of the distribution, or perform the two-tailed test, which was demonstrated in Example 5.2. As shown in the figure, the user enters the level of significance for the test. When entering the level of significance, users do not need to enter difference levels of significance for one-tailed

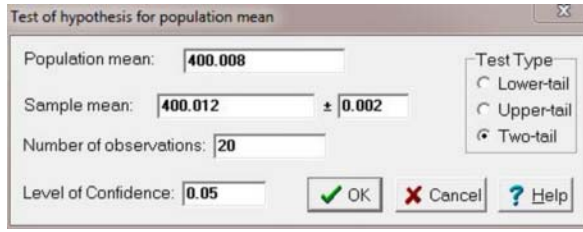


FIGURE 5.3 Entry screen for performing the *t* test as shown in Example 5.2 in STATS.

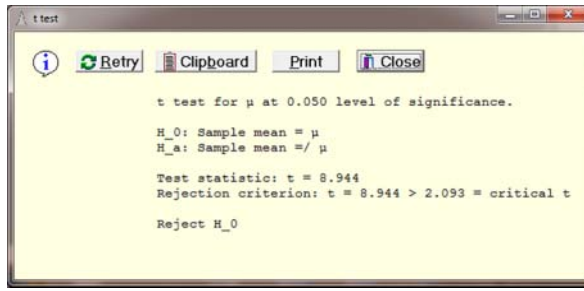


FIGURE 5.4 Results for *t* test discussed in Example 5.2 in STATS.

and two-tailed tests. Thus, to perform the tests at a 95% level of confidence, a 0.05 level of significance is entered for both the one-tailed and two-tailed tests. The results of the *t* test are shown in Figure 5.4. Unfortunately, the software does not have access to the ≠ symbol and thus chose to represent it as =/. The software can perform the χ^2 and *F* test similarly.

PROBLEMS

Note: Answer to problems marked with * can be found in Appendix G.

5.1 At what percent can a wrong decision be made about H_0 in a statistical test?

5.2 Explain why medical tests on patients are performed several times in a laboratory before the results of the test are returned to the doctor.

5.3 Describe the difference between Type I and Type II errors.

5.4 Describe the five elements of a hypothesis test.

5.5 Describe the power of the test.

5.6 Describe the similarities between a two-tailed hypothesis test and a confidence interval.

5.7 In your own words, discuss when it is appropriate to use each of the following tests:

(a) t test.

(b) χ^2 test.

(c) F test.

5.8 State whether the following examples should be one-tailed or two-tailed test.

***(a)** Comparing a calibration baseline length against a value measured using an EDM.

(b) Comparing the variance of distances of points relative to a planar surface using two different types of laser scanners to scan the surface.

(c) Comparing the variance for check points of a DEM generated by UAS photogrammetry with the variance of check points of a DEM generated by laser scanning using the ASPRS geospatial standards.

(d) Comparing the variance in elevation from repeated observations on a control monument using GNSS against manufacturer specifications.

(e) Comparing the reference variance of a fully constrained adjustment against a minimally constrained adjustment.

(f) The reference variance of a least-squares adjustment against its a priori value of one.

5.9 Compare the variances of day 1 and day 2 in Example 5.7 at a level of significance of 0.01 ($\alpha = 0.01$).

5.10 Compare the variances of day 1 and day 4 in Example 5.7 at a level significance of 0.1 ($\alpha = 0.1$).

Note: In Problems 5.11 and 5.12, $F_{0.005,999,999} = 1.177$ and $F_{0.05,999,9} = 2.712$.

5.11 Assuming 100 measurements have been made each day instead of 10, compare the variances of day 1 and day 2 in Example 5.7 at a level of significance of 0.01 ($\alpha = 0.01$).

5.12 Assuming 1000 measurements have been made on day 1 and 10 measurements on day 4, compare the variances of day 1 and day 4 in Example 5.7 at a level significance of 0.1 ($\alpha = 0.1$).

5.13 Using the data given in Example 5.6, determine if Kathi is statistically better with the equipment than Ron at a significance level of:

***(a)** 0.05.

(b) 0.10.

5.14 Ron was not satisfied with the previous statistical test results, so he performed another set of observations. This time he completed 501 observations and achieved an improved variance of 1.19. Using these test results,

determine if Kathi is statistically better with the equipment than Ron at a significance level of:

***(a)** 0.05.

(b) 0.10.

5.15 The population reference variance from a properly weighted least-squares adjustment is 1. After running a minimally constrained adjustment having 10 degrees of freedom, the computed reference variance is found to be 2.01. Is the computed reference variance statistically equal to 1 at a

(a) 0.05 level of significance?

(b) 0.01 level of significance?

5.16 When all the control is added to the adjustment in Problem 5.15, the reference variance for the fully constrained adjustment with 12 degrees of freedom is found to be 2.98. Are the reference variances from the minimally constrained and constrained adjustments statistically equal at a

***(a)** 0.05 level of significance?

(b) 0.01 level of significance?

5.17 The calibrated length of a baseline is 300.456 m. After 10 observations with an EDM, the mean observation for the distance is 300.458 m with a standard deviation of ± 0.0031 m.

(a) Is the observed distance statistically different from the calibrated length at a 10% level of significance?

(b) Is the observed distance statistically different from the calibrated length at a 5% level of significance?

5.18 A mean length of 1051.232 m with a standard deviation of ± 0.0045 m is obtained for a distance after 15 observations. Using the technical specifications, it is found that the standard deviation for this observation is estimated to be ± 0.0041 m.

(a) Perform a statistical test to check the repeatability of the instrument at a level of significance of 0.05.

(b) Perform a statistical test to check the repeatability of the instrument at a level of significance of 0.10.

5.19 A least-squares adjustment is computed twice on a data set. When the data are minimally constrained with 24 degrees of freedom, a reference variance of 0.91 is obtained. In the second run, the fully constrained network also having 32 degrees of freedom has a reference variance of 21.84. The a priori estimate for the reference variance in both adjustments is 1; that is, $\sigma_1^2 = \sigma_2^2 = 1$.

(a) Is the minimally constrained adjustment reference variance statistically equal to 1 at a 0.05 level of significance?

(b) Is the fully constrained adjustment reference variance statistically equal to 1 at a 0.05 level of significance?

(c) Are the two variances statistically equal at a 0.05 level of significance?

(d) Is there statistical reason to be concerned about the presence of errors in either the control or the observations?

5.20 A total station with a manufacturer's specified angular accuracy of $\pm 3''$ was used the collect the data in Problem 3.17. Does the data warrant this accuracy at a 0.01 level of significance? Develop a statistical test to validate your response.

5.21 An EDM is placed on a calibration baseline and the distance between two monuments is determined to be $1500.010\,\text{m} \pm 0.0041\,\text{m}$ after 10 observations. The calibrated length between the monuments is 1500.008 m. Is the instrument measuring the length properly at a

(a) 0.05 level of significance?

(b) 0.01 level of significance?

5.22 Develop a hypothesis test to answer Problem 4.6(a).

5.23 Develop hypothesis tests to answer Problem 4.7(c).

5.24 Develop a hypothesis test to answer Problem 4.8(a).

5.25 Develop a hypothesis test to answer Problem 4.18(a).

5.26 Develop a hypothesis test to answer Problem 4.10(b).

5.27 Develop a hypothesis test to answer Problem 4.11(b).

5.28 Develop a hypothesis test to answer Problem 4.18(b).

5.29 Develop hypothesis test to answer Problem 4.23(c).

CHAPTER 6

PROPAGATION OF RANDOM ERRORS IN INDIRECTLY MEASURED QUANTITIES

6.1 BASIC ERROR PROPAGATION EQUATION



As discussed in Section 1.2, unknown values are often determined indirectly by making direct measurements on other quantities that are functionally related to the desired unknown quantities. Examples in surveying include computing station coordinates from distance and angle observations, obtaining station elevations from rod readings in differential leveling, and determining the azimuth of a line from astronomical observations. As noted in Section 1.2, since all directly observed quantities contain errors, any values computed from them will also contain errors. This intrusion, or *propagation*, of errors that occur in quantities computed from direct measurements is called *error propagation*. This topic is one of the most important topics discussed in this book.

In this chapter, it is assumed that all systematic errors and mistakes have been eliminated from a set of direct observations so that only random errors remain. To derive the basic error propagation equation, consider the simple function, $z = a_1x_1 + a_2x_2$, where x_1 and x_2 are two independently observed quantities with

standard errors σ_1 and σ_2, and a_1 and a_2 are constants. By analyzing how errors propagate in this function, a general expression can be developed for the propagation of random errors through any function.

Since x_1 and x_2 are two independently observed quantities, they each have different probability density functions. Let the errors in n determinations of x_1 be $\varepsilon_1^i, \varepsilon_1^{ii}, \ldots, \varepsilon_1^n$ and the errors in n determinations of x_2 be $\varepsilon_2^i, \varepsilon_2^{ii}, \ldots, \varepsilon_2^n$, then z_T, the true value of z for each independent observation, is

$$z_T = \begin{cases} a_1\left(x_1^i - \varepsilon_1^i\right) + a_2\left(x_2^i - \varepsilon_2^i\right) = a_1x_1^i + a_2x_2^i - \left(a_1\varepsilon_1^i + a_2\varepsilon_2^i\right) \\ a_1\left(x_1^{ii} - \varepsilon_1^{ii}\right) + a_2\left(x_2^{ii} - \varepsilon_2^{ii}\right) = a_1x_1^{ii} + a_2x_2^{ii} - \left(a_1\varepsilon_1^{ii} + a_2\varepsilon_2^{ii}\right) \\ a_1\left(x_1^{iii} - \varepsilon_1^{iii}\right) + a_2\left(x_2^{iii} - \varepsilon_2^{iii}\right) = a_1x_1^{iii} + a_2x_2^{iii} - \left(a_1\varepsilon_1^{iii} + a_2\varepsilon_2^{iii}\right) \\ \vdots \end{cases} \tag{6.1}$$

The values for z computed from the observations are

$$\begin{aligned} z^i &= a_1x_1^i + a_2x_2^i \\ z^{ii} &= a_1x_1^{ii} + a_2x_2^{ii} \\ z^{iii} &= a_1x_1^{iii} + a_2x_2^{iii} \\ &\vdots \end{aligned} \tag{6.2}$$

Substituting Equations (6.2) into Equations (6.1) and regrouping Equations (6.1) to isolate the errors for each computed value yields

$$\begin{aligned} z^i - z_T &= a_1\varepsilon_1^i + a_2\varepsilon_2^i \\ z^{ii} - z_T &= a_1\varepsilon_1^{ii} + a_2\varepsilon_2^{ii} \\ z^{iii} - z_T &= a_1\varepsilon_1^{iii} + a_2\varepsilon_2^{iii} \\ &\vdots \end{aligned} \tag{6.3}$$

From Equation (2.4) for the variance in a population, $n\sigma^2 = \sum_{i=1}^{n} \varepsilon^2$, and thus for the case under consideration, the sum of the squared errors for the value computed is

$$\sum_{i=1}^{n} \varepsilon_i^2 = \left(a_1\varepsilon_1^i + a_2\varepsilon_2^i\right)^2 + \left(a_1\varepsilon_1^{ii} + a_2\varepsilon_2^{ii}\right)^2 + \left(a_1\varepsilon_1^{iii} + a_2\varepsilon_2^{iii}\right)^2 + \cdots = n\sigma_z^2 \tag{6.4}$$

Expanding the terms in Equation (6.4) yields

$$n\sigma_z^2 = \left(a_1\varepsilon_1^i\right)^2 + 2a_1a_2\varepsilon_1^i\varepsilon_2^i + \left(a_2\varepsilon_2^i\right)^2 + \left(a_1\varepsilon_1^{ii}\right)^2 + 2a_1a_2\varepsilon_1^{ii}\varepsilon_2^{ii} + \left(a_2\varepsilon_2^{ii}\right)^2 + \cdots \tag{6.5}$$

Factoring terms in Equation (6.5) results in

$$n\sigma_z^2 = a_1^2\left(\varepsilon_1^{i^2} + \varepsilon_1^{ii^2} + \varepsilon_1^{iii^2} + \cdots\right) + a_2^2\left(\varepsilon_2^{i^2} + \varepsilon_2^{ii^2} + \varepsilon_2^{iii^2} + \cdots\right) + 2a_1a_2\left(\varepsilon_1^i\varepsilon_2^i + \varepsilon_1^{ii}\varepsilon_2^{ii} + \varepsilon_1^{iii}\varepsilon_2^{iii} + \cdots\right) \tag{6.6}$$

Inserting summation symbols for the error terms in Equation (6.6) yields

$$\sigma_z^2 = a_i^2\left(\frac{\sum_{i=1}^{n}\varepsilon_1^2}{n}\right) + 2a_1a_2\left(\frac{\sum_{i=1}^{n}\varepsilon_1\varepsilon_2}{n}\right) + a_2^2\left(\frac{\sum_{i=1}^{n}\varepsilon_2^2}{n}\right) \tag{6.7}$$

Recognizing that the terms in parentheses in Equation (6.7) are by definition: $\sigma_{x_1}^2$, $\sigma_{x_1x_2}$, and $\sigma_{x_2}^2$, respectively, Equation (6.7) can be rewritten as

$$\sigma_z^2 = a_1^2\sigma_{x_1}^2 + 2a_1a_2\sigma_{x_1x_2} + a_2^2\sigma_{x_2}^2 \tag{6.8}$$

In Equation (6.8), the middle term, $\sigma_{x_1x_2}$, is known as the *covariance*. This term shows the interdependence between the two unknown variables x_1 and x_2. As the covariance term decreases, the interdependence of the variables also decreases. When these terms are zero, the variables are said to be mathematically independent. Its importance in surveying is discussed in more detail in later chapters.

Equations (6.7) and (6.8) can be written in matrix form as

$$\Sigma_{zz} = \begin{bmatrix} a_1 & a_2 \end{bmatrix}\begin{bmatrix} \sigma_{x_1}^2 & \sigma_{x_1x_2} \\ \sigma_{x_1x_2} & \sigma_{x_2}^2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \tag{6.9}$$

where Σ_{zz} is the variance–covariance matrix for the function z. It follows logically from this derivation that, in general, if z is a function of n independently measured quantities, $x_1, x_2, \ldots, x_n$, then Σ_{zz} is

$$\Sigma_{zz} = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}\begin{bmatrix} \sigma_{x_1}^2 & \sigma_{x_1x_2} & \cdots & \sigma_{x_1x_n} \\ \sigma_{x_2x_1} & \sigma_{x_2}^2 & & \sigma_{x_2x_n} \\ \vdots & & \ddots & \\ \sigma_{x_nx_1} & \sigma_{x_nx_2} & & \sigma_{x_n}^2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \tag{6.10}$$

Further, for a set of m functions with n independently measured quantities, $x_1, x_2, \ldots, x_n$, Equation (6.10) expands to

$$\Sigma_{zz} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}\begin{bmatrix} \sigma_{x_1}^2 & \sigma_{x_1x_2} & \cdots & \sigma_{x_1x_n} \\ \sigma_{x_1x_2} & \sigma_{x_2}^2 & \cdots & \sigma_{x_2x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{x_nx_1} & \sigma_{x_2x_n} & \cdots & \sigma_{x_n}^2 \end{bmatrix}\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \cdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} \tag{6.11}$$

Similarly, if the functions are nonlinear, a first order Taylor Series expansion can be used to linearize them.[1] Thus $a_{11}, a_{12}, \ldots, a_{mn}$ are replaced by the partial derivatives of the $Z_1, Z_2, \ldots, Z_m$ with respect to the unknown parameters, $x_1, x_2, \ldots, x_n$. Thus, after linearizing a set of nonlinear equations, the matrix for the function of Z can be written in linear form as

$$\Sigma_{zz} = \begin{bmatrix} \frac{\partial Z_1}{\partial x_1} & \frac{\partial Z_1}{\partial x_2} & \cdots & \frac{\partial Z_1}{\partial x_n} \\ \frac{\partial Z_2}{\partial x_1} & \frac{\partial Z_2}{\partial x_2} & \cdots & \frac{\partial Z_2}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial Z_m}{\partial x_1} & \frac{\partial Z_m}{\partial x_2} & \cdots & \frac{\partial Z_m}{\partial x_n} \end{bmatrix} \begin{bmatrix} \sigma_{x_1}^2 & \sigma_{x_1x_2} & \cdots & \sigma_{x_1x_n} \\ \sigma_{x_1x_2} & \sigma_{x_2}^2 & \cdots & \sigma_{x_2x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{x_nx_1} & \sigma_{x_2x_n} & \cdots & \sigma_{x_n}^2 \end{bmatrix}$$
$$\times \begin{bmatrix} \frac{\partial Z_1}{\partial x_1} & \frac{\partial Z_2}{\partial x_1} & \cdots & \frac{\partial Z_m}{\partial x_1} \\ \frac{\partial Z_1}{\partial x_2} & \frac{\partial Z_2}{\partial x_2} & \cdots & \frac{\partial Z_m}{\partial x_2} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial Z_1}{\partial x_n} & \frac{\partial Z_2}{\partial x_n} & \cdots & \frac{\partial Z_m}{\partial x_n} \end{bmatrix} \tag{6.12}$$

Equations (6.11) and (6.12) are known as the *general law of propagation of variances* (GLOPOV) for linear and nonlinear equations, respectively. Both Equations (6.11) and (6.12) can be written symbolically in matrix notation as

$$\Sigma_{zz} = A\Sigma A^T \tag{6.13}$$

where Σ_{zz} is the covariance matrix for the function Z. For a nonlinear set of equations that is linearized using Taylor's Theorem, the coefficient matrix (A) is called a *Jacobian matrix* where A is replaced with a J later to indicate this difference. That is, it is a matrix of partial derivatives with respect to the unknowns, as shown in Equation (6.12).

If the observations, $x_1, x_2, \ldots, x_n$, are unrelated, that is they are statistically independent, then the covariance terms, $\sigma_{x_1x_2}, \sigma_{x_1x_3}, \ldots \sigma_{x_mx_n}$ are equal to zero, and the right side of Equations (6.10) and (6.11) can be rewritten, respectively, as

$$\Sigma_{zz} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} \sigma_{x_1}^2 & 0 & \cdots & 0 \\ 0 & \sigma_{x_2}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{x_n}^2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \cdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} \tag{6.14}$$

[1] Readers who are unfamiliar with solving nonlinear equations should refer to Appendix C of this book.

$$\Sigma_{zz} = \begin{bmatrix} \frac{\partial Z_1}{\partial x_1} & \frac{\partial Z_1}{\partial x_2} & \cdots & \frac{\partial Z_1}{\partial x_n} \\ \frac{\partial Z_2}{\partial x_1} & \frac{\partial Z_2}{\partial x_2} & \cdots & \frac{\partial Z_2}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial Z_m}{\partial x_1} & \frac{\partial Z_m}{\partial x_2} & \cdots & \frac{\partial Z_m}{\partial x_n} \end{bmatrix} \begin{bmatrix} \sigma_{x_1}^2 & 0 & \cdots & 0 \\ 0 & \sigma_{x_2}^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_{x_n}^2 \end{bmatrix} \times \begin{bmatrix} \frac{\partial Z_1}{\partial x_1} & \frac{\partial Z_2}{\partial x_1} & \cdots & \frac{\partial Z_m}{\partial x_1} \\ \frac{\partial Z_1}{\partial x_2} & \frac{\partial Z_2}{\partial x_2} & \cdots & \frac{\partial Z_m}{\partial x_2} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial Z_1}{\partial x_n} & \frac{\partial Z_2}{\partial x_n} & \cdots & \frac{\partial Z_m}{\partial x_n} \end{bmatrix} \tag{6.15}$$

If there is only one function Z, involving n unrelated quantities, $x_1, x_2, \ldots, x_n$, then Equation (6.15) can be rewritten in algebraic form as

$$\sigma_Z = \sqrt{\left(\frac{\partial Z}{\partial x_1}\sigma_{x_1}\right)^2 + \left(\frac{\partial Z}{\partial x_2}\sigma_{x_2}\right)^2 + \cdots + \left(\frac{\partial Z}{\partial x_n}\sigma_{x_n}\right)^2} \tag{6.16}$$

Equations (6.14), (6.15), and (6.16) express the *special law of propagation of variances* (SLOPOV). These equations govern the way errors from statistically independent observations (i.e., $\sigma_{x_i x_j} = 0$) propagate in a function. In these equations, individual terms represent the individual contributions to the total error that occur as the result of observational errors for each independent variable. When the size of a function's estimated error is too large, inspection of these individual terms will indicate the largest contributors to the error. The most efficient method to reduce the overall error in the function is to examine ways to reduce the largest individual error terms in Equation (6.16).

6.1.1 Generic Example

Let $A = B + C$ and assume that B and C are independently observed quantities. Note that $\partial A/\partial B = 1$ and $\partial A/\partial C = 1$. Substituting these into Equation (6.16) yields

$$\sigma_A = \sqrt{(1\sigma_B)^2 + (1\sigma_C)^2} \tag{6.17}$$

Using Equation (6.15) yields

$$\Sigma_{AA} = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_B^2 & 0 \\ 0 & \sigma_C^2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sigma_B^2 + \sigma_C^2 \end{bmatrix}$$

Equation (6.17) yields the same results as Equation (6.16) after the square root of the single element is determined. In the equations above, standard error (σ) and standard deviation (S) can be used interchangeably.

6.2 FREQUENTLY ENCOUNTERED SPECIFIC FUNCTIONS

6.2.1 Standard Deviation of a Sum

Let $A = B_1 + B_2 + \cdots + B_n$, where the B's are n independently observed quantities having standard deviations of $S_{B_1}, S_{B_2}, \ldots, S_{B_n}$, then by Equation (6.16),

$$S_A = \sqrt{S_{B_1}^2 + S_{B_2}^2 + \cdots + S_{B_n}^2} \tag{6.18}$$

6.2.2 Standard Deviation in a Series

Assume that the error for each observed value in Equation (6.18) is equal, that is $S_{B_1} = S_{B_2} = \cdots = S_{B_n} = S_B$, then Equation (6.18) simplifies to

$$S_A = S_B \sqrt{n} \tag{6.19}$$

6.2.3 Standard Deviation of the Mean

Let $\bar{y}$ be the mean obtained from n independently observed quantities $y_1, y_2, \ldots, y_n$, each of which has the same standard deviation S. As given in Equation (2.1), the mean is expressed as

$$\bar{y} = \frac{y_1 + y_2 + \cdots + y_n}{n}$$

An equation for $S_{\bar{y}}$, the standard deviation of the mean is obtained by substituting the above expression into Equation (6.16). Since the partial derivatives of with respect to the observed quantities, $y_1, y_2, \ldots, y_n$, is $\frac{\partial \bar{y}}{\partial y_1} = \frac{\partial \bar{y}}{\partial y_2} = \cdots = \frac{\partial \bar{y}}{\partial y_n} = \frac{1}{n}$, the resulting error in is

$$S_{\bar{y}} = \sqrt{\left(\frac{1}{n} S_{y_1}\right)^2 + \left(\frac{1}{n} S_{y_2}\right)^2 + \cdots + \left(\frac{1}{n} S_{y_n}\right)^2} = \sqrt{\frac{nS^2}{n^2}} = \frac{S}{\sqrt{n}} \tag{6.20}$$

Note that Equation (6.20) is the same as Equation (2.8).

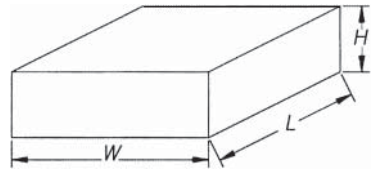


FIGURE 6.1 Rectangular tank.

6.3 NUMERICAL EXAMPLES

Example 6.1

The dimensions of a rectangular tank shown in Figure 6.1 are observed as

$$\begin{aligned} L &= 40.00 \text{ ft} & S_L &= \pm 0.05 \text{ ft} \\ W &= 20.00 \text{ ft} & S_W &= \pm 0.03 \text{ ft} \\ H &= 15.00 \text{ ft} & S_H &= \pm 0.02 \text{ ft} \end{aligned}$$

Find the tank's volume and the standard deviation in the volume using the observations above.

SOLUTION The volume of the tank is found by the formula

$$V = L\,W\,H = 40.00(20.00)(15.00) = 12{,}000 \text{ ft}^3$$

Given that the standard deviation in the computed volume is determined by using Equation (6.16), which yields

$$\begin{aligned} S_V &= \sqrt{\left(\frac{\partial V}{\partial L}S_L\right)^2 + \left(\frac{\partial V}{\partial W}S_W\right)^2 + \left(\frac{\partial V}{\partial H}S_H\right)^2} \\ &= \sqrt{(WH)^2(0.05)^2 + (LH)^2(0.03)^2 + (LW)^2(0.02)^2} \\ &= \sqrt{(300 \times 0.05)^2 + (600 \times 0.03)^2 + (800 \times 0.02)^2} \\ &= \sqrt{225 + 324 + 256} = \sqrt{805} = \pm 28 \text{ ft}^3 \end{aligned} \qquad (a)$$

In Equation (a), the second term is the largest contributor to the total error, and thus to reduce the overall error in the computed volume, it would be prudent first to try to make S_W smaller. This would yield the greatest effect in the error in the volume estimate.

Example 6.2 As shown in Figure 6.2, the altitude angle α to point B is observed at point A as $3°00'$ with S_α being $\pm 1'$. The slope distance D from A to B is observed as

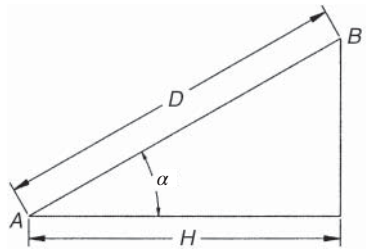


FIGURE 6.2 Horizontal distance from slope observations.

1000.00 ft with S_D being ±0.05 ft. Compute the horizontal distance and its standard deviation.

SOLUTION The horizontal distance is determined using the equation

$$H = D\ \cos(\alpha) = 1000.00\ \cos(3^\circ 00') = 998.63 \text{ ft}$$

Given $\frac{\partial H}{\partial D} = \cos\alpha$ and $\frac{\partial H}{\partial \alpha} = -D\sin\alpha$, the error in the function is determined by using Equation (6.16) as

$$S_H = \sqrt{\left(\frac{\partial H}{\partial D}S_D\right)^2 + \left(\frac{\partial H}{\partial \alpha}S_\alpha\right)^2} \tag{6.21}$$

In Equation (6.21), S_α must be converted to its equivalent radian value to achieve agreement in the units. Thus,

$$S_H = \sqrt{(\cos\alpha \times 0.05)^2 + \left(-\sin\alpha \times D \times \frac{60''}{206{,}264.8''/\text{rad}}\right)^2}$$

$$= \sqrt{(0.9986 \times 0.05)^2 + \left(\frac{-0.0523 \times 1000 \times 60''}{206264.8''}\right)^2}$$

$$= \sqrt{0.04993^2 + 0.0152^2} = \pm 0.052 \text{ ft}$$

Notice in this example that the major contributing error source (largest number under the radical) is 0.04993. This is the error associated with the distance observation, and thus if the resulting error of ±0.052 ft is too large, the logical way to improve the results (reduce the overall error) is to adopt a more precise method of measuring the distance.

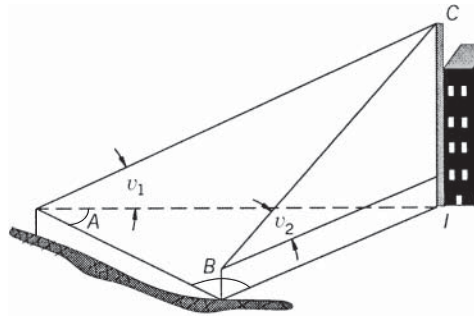


FIGURE 6.3 Elevation of chimney determined using intersecting angles.

Example 6.3 The elevation of a point C on a chimney shown in Figure 6.3 is desired. Field angles and distances are observed. Station A has an elevation of 1298.65 ± 0.006 ft and Station B has an elevation of 1301.53 ± 0.004 ft. The instrument height, hi_A, at Station A is 5.25 ± 0.005 ft and the instrument height, hi_B, at Station B is 5.18 ± 0.005 ft. The other observations and their errors are

$$AB = 136.45 \pm 0.018 \text{ ft}$$
$$A = 44°12'34'' \pm 8.6'' \qquad B = 39°26'56'' \pm 11.3''$$
$$v_1 = 8°12'47'' \pm 4.1'' \qquad v_2 = 5°50'10'' \pm 5.1''$$

What is the elevation at Point C on the chimney and the error in this computed elevation value?

SOLUTION Normally this problem is worked in several steps. The steps include computing distances AI and BI and then solving for the average elevation of C using observations obtained from both stations A and B in Figure 6.3. However, caution must be exercised when doing error analysis in a stepwise fashion since the computed values could be correlated. If so, the stepwise method might lead to an incorrect analysis of the errors. To avoid this, either use GLOPOV or a single-derived function that includes all quantities observed in the calculation of the elevation. This second method is demonstrated as follows.

From the sine law, the solution of AI and BI can be derived as

$$AI = \frac{AB \sin B}{\sin[180° - (A + B)]} = \frac{AB \sin B}{\sin(A + B)} \tag{6.22}$$

$$BI = \frac{AB \sin A}{\sin(A + B)} \tag{6.23}$$

Using Equations (6.22) and (6.23), the elevations for C from station A and B are

$$Elev_{C_A} = AI \tan v_1 + Elev_A + hi_A \tag{6.24}$$

$$Elev_{C_B} = BI \tan v_2 + Elev_B + hi_B \tag{6.25}$$

Thus, the chimney's elevation is computed as the average Equations (6.24) and (6.25) or

$$Elev_C = \frac{1}{2}(Elev_{C_A} + Elev_{C_B}) \tag{6.26}$$

Substituting Equations (6.22) through (6.25) into (6.26), a single expression for the chimney elevation can be written as

$$Elev_C = \frac{1}{2}\left[Elev_A + hi_A + \left(\frac{AB \sin B \tan v_1}{\sin(A+B)}\right) + Elev_B + hi_B + \left(\frac{AB \sin A \tan v_2}{\sin(A+B)}\right)\right] \tag{6.27}$$

From Equation (6.27), the elevation of C is 1316.49 ft. To perform the error analysis, Equation (6.16) is used. In this complex problem, it is often easier to break the problem into smaller parts. This can be done by numerically solving the necessary partial derivatives from Equation (6.16) before finally squaring and summing the results. From Equation (6.27),

$$\frac{\partial Elev_C}{\partial Elev_A} = \frac{\partial Elev_C}{\partial Elev_B} = \frac{1}{2}$$
$$\frac{\partial Elev_C}{\partial hi_A} = \frac{\partial Elev_C}{\partial hi_B} = \frac{1}{2}$$

From Equation (6.27),

$$\frac{\partial Elev_C}{\partial AB} = \frac{1}{2}\left(\frac{\sin B \tan v_1 + \sin A \tan v_2}{\sin(A+B)}\right) = 0.08199$$

$$\frac{\partial Elev_C}{\partial A} = \frac{AB}{2}\left[\frac{-\cos(A+B)(\sin B \tan v_1 + \sin A \tan v_2)}{\sin^2(A+B)} + \frac{\cos A \tan v_2}{\sin(A+B)}\right]$$
$$= 3.78596$$

$$\frac{\partial Elev_C}{\partial B} = \frac{AB}{2}\left[\frac{-\cos(A+B)(\sin B \tan v_1 + \sin A \tan v_2)}{\sin^2(A+B)} + \frac{\cos B \tan v_1}{\sin(A+B)}\right]$$
$$= 6.40739$$

$$\frac{\partial Elev_C}{\partial v_1} = \frac{AB \sin B}{2\sin(A+B)\cos^2 v_1} = 44.52499$$

$$\frac{\partial Elev_C}{\partial v_2} = \frac{AB \sin A}{2\sin(A+B)\cos^2 v_2} = 48.36511$$

Again, for compatibility of the units in this problem, all angular errors are converted to their radian equivalents by dividing each by 206,264.8″/rad. Finally, using Equation (6.16), the variance and error are

$$
\begin{aligned}
S^2_{Elev_C} &= \left(\frac{\partial Elev_C}{\partial Elev_A} S_{Elev_A}\right)^2 + \left(\frac{\partial Elev_C}{\partial Elev_B} S_{Elev_B}\right)^2 + \left(\frac{\partial Elev_C}{\partial hi_A} S_{hi_A}\right)^2 \\
&\quad + \left(\frac{\partial Elev_C}{\partial hi_B} S_{hi_B}\right)^2 + \left(\frac{\partial Elev_C}{\partial AB} S_{AB}\right)^2 + \left(\frac{\partial Elev_C}{\partial A} S_A\right)^2 \\
&\quad + \left(\frac{\partial Elev_C}{\partial B} S_B\right)^2 + \left(\frac{\partial Elev_C}{\partial v_1} S_{v_1}\right)^2 + \left(\frac{\partial Elev_C}{\partial v_2} S_{v_2}\right)^2 \\
S_{Elev_C} &= \left[\begin{array}{l} \left(\frac{0.006}{2}\right)^2 + \left(\frac{0.004}{2}\right)^2 + 2\left(\frac{1}{2}0.005\right)^2 + (0.08199 \times 0.018)^2 \\ \quad + (3.78596 \times 4.1693 \times 10^{-5})^2 + (6.40739 \times 5.4783 \times 10^{-5})^2 \\ \quad + (44.52499 \times 1.9877 \times 10^{-5})^2 + (48.36511 \times 2.4725 \times 10^{-5})^2 \end{array}\right]^{\frac{1}{2}} \\
&= \sqrt{\begin{array}{l} 0.003^2 + 0.002^2 + 2(0.0025)^2 + 0.0015^2 + 0.00016^2 \\ \quad + 0.00035^2 + 0.00089^2 + 0.0012^2 \end{array}} \\
&= \pm 0.0055 \text{ ft} = \pm 0.01 \text{ ft.}
\end{aligned}
$$

Thus, the elevation of point C is 1316.49 ± 0.01 ft. Also note that the largest errors are due to the uncertainties in the elevations of A and B and the observed heights of the instrument stations at these stations.

6.4 SOFTWARE

The computations in this chapter can be time-consuming and tedious often leading to computational errors in the results. Typically, it is more efficient to program these equations in a computational package. The programming of the examples in this chapter is demonstrated in the Mathcad® electronic book on the companion website for this book. Figure 6.4 shows a partial listing of a Mathcad® worksheet used to solve Example 6.3. Notice that a function called radian was used to convert angles from sexagesimal units to radian units. After all the variables for the problem are created, the problem is solved by creating an equation that looks very similar to those written in this book. The results of each expression are shown to the right of the equation. Oftentimes, the time that is spent learning a new programming language is more than rewarded in its efficient use to solve multiple problems. For example, the same worksheet can be used to solve problems that are similar to that in Example 6.3.

COMPUTATIONS

$$AI := \frac{AB \cdot \sin(B)}{\sin(A+B)} \qquad AI = 87.233$$

$$BI := \frac{AB \cdot \sin(A)}{\sin(A+B)} \qquad BI = 95.730$$

$$Elev_{CA} := AI \cdot \tan(v_1) + Elev_A + hi_A \qquad Elev_{CA} = 1316.491$$

$$Elev_{CB} := BI \cdot \tan(v_2) + Elev_B + hi_B \qquad Elev_{CB} = 1316.495$$

$$a := \frac{1}{2} \qquad b := \frac{1}{2}$$

$$c := \frac{1}{2}\left(\frac{\sin(B)\cdot\tan(v_1) + \sin(A)\cdot\tan(v_2)}{\sin(A+B)}\right) \qquad c = 0.081992$$

$$d := \frac{AB}{2}\left[\frac{-\cos(A+B)\cdot\left(\sin(B)\cdot\tan(v_1) + \sin(A)\cdot\tan(v_2)\right)}{\sin(A+B)^2} + \frac{\cos(A)\cdot\tan(v_2)}{\sin(A+B)}\right] \qquad d = 3.785962$$

$$e := \frac{AB}{2}\left[\frac{-\cos(A+B)\cdot\left(\sin(B)\cdot\tan(v_1) + \sin(A)\cdot\tan(v_2)\right)}{\sin(A+B)^2} + \frac{\cos(B)\cdot\tan(v_1)}{\sin(A+B)}\right] \qquad e = 6.407395$$

FIGURE 6.4 Partial listing of Example 6.3 calculated in Mathcad®.

Another efficient method of programming these types of problems is to use a spreadsheet. In some spreadsheets, cells can be named, and thus equations can be entered using variables. For example, the expression for the $\frac{\partial Elev_C}{\partial A}$, which is EC/A in Figure 6.5, was entered as AB/2*((−COS(A+B)*(SIN(B)*TAN(VA)+

	A	B	C	D	E	F	G
1	**Example 6.3**						
2	AB	136.45	±0.018			radian	
3					S	Angle	S
4	A	44°	12'	34"	±8.6	0.771600	4.1694E-05
5	B	39°	26'	56"	±11.3	0.688513	5.4784E-05
6	v_1	8°	12'	47"	±4.1	0.143345	1.9877E-05
7	v_2	5°	50'	10"	±5.1	0.101859	2.4725E-05
8							
9	$Elev_A$	1298.65	±0.006		hi_A	5.25	±0.005
10	$Elev_B$	1301.53	±0.004		hi_B	5.18	±0.005
11							
12	$Elev_C$	1316.493	±0.005				
13							
14	Partial derivatives						
15	Ec/AB	0.081710					
16	EC/A	3.785962					
17	EC/B	6.407395					
18	Ec/v1	44.524990					
19	Ec/v2	48.365113					

FIGURE 6.5 Example 6.3 performed in a spreadsheet.

SIN(A)*TAN(VB)))/(SIN(A+B))^2 + (COS(A)*TAN(VB))/SIN(A+B)) where AB referred to the cell containing the distance *AB*, A the cell containing the angle *A* in radian units, B the cell containing the angle *B* in radian units, VA the cell containing the altitude angle v_1 in radian units, and VB the cell containing the altitude angle v_2 in radian units. Similarly, the other expressions were created using the named cells. While this method is not as clear as using a program such as Mathcad®, when an expression is entered carefully the same results can be computed. Again, the resultant spreadsheet can be used to solve similar problems by simply changing the data entry values.

6.5 CONCLUSIONS

Errors associated with any indirect measurement problem can be analyzed as described earlier. Besides being able to compute the estimated error in a function, the sizes of the individual errors contributing to the functional error can also be analyzed. This identifies those observations whose errors are most critical in reducing the functional error. An alternative use of the error propagation equation involves computing the error in a function of observed values prior to fieldwork. The calculation can be based on the geometry of the problem and the observations that are included in the function. The estimated errors in each observed value can be varied to correspond with those estimated using different combinations of available equipment and field procedures. The combination that produces the desired accuracy in the final computed function can then be adopted in the field. This analysis falls under the heading of *survey planning and design*. This topic is discussed further in Chapters 7, 19, and 21.

PROBLEMS

Note: Partial answers to problems marked with an asterisk can be found in Appendix G.

***6.1** In running a line of levels, 24 instrument setups are required, with a backsight and foresight taken from each. For each rod reading, the estimated error is ± 1.5 mm. What is the error in the measured elevation difference between the origin and terminus?

6.2 The estimated error in each angle of a 8-sided traverse is $\pm 2.6''$. What is the estimated error in the angular misclosure of the traverse?

6.3 A distance is observed using both 100-ft and 200-ft tapes. The 100-ft tape observations were 100.00 ± 0.016 ft and 86.23 ± 0.013 ft. The 200-ft tape observation was 186.25 ± 0.019 ft. Which tape produced the smallest estimated error?

6.4 Determine the estimated error in length of *AE* that was observed in sections as follows:

Section	Observed length (ft)	Standard deviation (ft)
AB	416.24	±0.02
BC	1044.16	±0.05
CD	590.03	±0.03
DE	714.28	±0.04

***6.5** A slope distance is observed as 1506.843 ± 0.009 m. The zenith angle is observed as $92°37'29'' \pm 8.8''$. What is the horizontal distance and its estimated error?

6.6 A slope distance is observed as 581.98 ± 0.015 ft. The zenith angle is observed as $85°03'25'' \pm 9.5''$. What is the horizontal distance and its estimated error?

6.7 A rectangular parcel has dimensions of 538.056 ± 0.005 m by 368.459 ± 0.004 m. What is the area of the parcel and the estimated error in this area?

6.8 Same as Problem 6.7 except the dimensions are 536.792 ± 0.042 ft by 600.087 ± 0.048 ft.

***6.9** The volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$. A storage shed in the shape of a cone has a measured height of 50.0 ± 0.10 ft and radius of 50.0 ± 0.08 ft. What are the shed's volume and estimated error in this volume?

6.10 Same as Problem 6.9, but the measured height is 15.00 ± 0.008 m and radius is 15.00 ± 0.006 m.

6.11 Using an EDM instrument the rectangular dimensions of a large building 1435.67 ± 0.025 ft by 453.56 ± 0.016 ft are laid out. Assuming only errors in distance observations, what are the

(a) Area enclosed by the building and its standard deviation?

(b) Perimeter of the building and its standard deviation?

6.12 A particular total station's reading error is determined to be $\pm 3.0''$. After repeatedly pointing at a distant target with the same instrument, the observer determines an error due to both pointing and reading the circles of $\pm 4.5''$. What is the observer's pointing error?

***6.13** The dimensions of a storage shed under the rafters are 100.00 ± 0.03 ft long, 50 ± 0.02 ft wide by 14.03 ± 0.02 ft high. What is the volume of this shed and the estimated error in this volume?

6.14 Same as Problem 6.13 except the dimensions are 40.000 ± 0.006 m long, 25.000 ± 0.005 m wide by 4.000 ± 0.003 m high.

6.15 An EDM instrument manufacturer publishes its instrument's accuracy as $\pm(2\text{ mm} + 2\text{ ppm})$. [*Note*: 2 ppm means 2 parts per million. This is a scaling error and is computed as (Distance H 2/1,000,000).]

(a) What formula should be used to determine the error in a distance observed with this instrument?

(b) What is the error in a 856.78-ft distance measured with this EDM?

6.16 Same as Problem 6.15 except the observed distance is 235.781 m.



FIGURE P6.17 and P6.18

6.17 As shown in the accompanying sketch, a racetrack is measured in three simple components: a rectangle and two semicircles. Using an EDM with a manufacturer's specified accuracy of $\pm(3\text{ mm} + 3\text{ ppm})$, the rectangle's dimensions measured at the inside of the track are 5280.00 ft by 850.00 ft. Assuming only errors in the distance observations, what are the

(a) Standard deviations in each observation?

(b) Length of the perimeter of the track?

(c) Area enclosed by the track?

(d) Standard deviation in the perimeter of the track?

(e) Standard deviation in the area enclosed by the track?

6.18 Same as Problem 6.17, except the rectangle's dimensions are 1620.000 m by 250.000 m.

6.19 The elevation of point C on the chimney shown in Figure 6.3 is desired. Field angles and distances are observed. Station A has an elevation of 347.758 ± 0.008 m and Station B has an elevation of 353.816 ± 0.008 m. The instrument height, hi_A, at Station A is 1.249 ± 0.002 m and the instrument height, hi_B, at Station B is 1.155 ± 0.003 m. Zenith angles are read in the field. The other observations and their estimated errors are

$$AB = 93.505 \pm 0.005\text{ m}$$

$$A = 44°15'21'' \pm 7.8'' \qquad B = 39°26'56'' \pm 9.7''$$

$$z_1 = 81°41'06'' \pm 12.3'' \qquad z_2 = 84°10'25'' \pm 11.6''$$

What is the elevation of the stack and the standard deviation in this elevation?

6.20 Same as Problem 6.19 with the following observations.

$$Elev_A = 248.35 \pm 0.04 \text{ ft} \qquad Elev_B = 242.87 \pm 0.04 \text{ ft}$$
$$hi_A = 5.53 \pm 0.01 \text{ ft} \qquad hi_B = 5.52 \pm 0.01 \text{ ft}$$
$$A = 43°15'21'' \pm 6.8'' \qquad B = 42°02'36'' \pm 5.7''$$
$$z_1 = 85°22'30'' \pm 12.3'' \qquad z_2 = 83°00'48'' \pm 9.8''$$

6.21 Show that Equation (6.12) is equivalent to Equation (6.11) for linear equations.

6.22 Derive an expression like Equation (6.9) for the function $z = a_1x_1 + a_2x_2 + a_3x_3$.

PRACTICAL EXERCISES

6.23 With an engineer's scale, measure the radius of the circle in the accompanying sketch 10 times using different starting locations on the scale. Use a magnifying glass and interpolate the readings on the scale to a tenth of the smallest graduated reading on the scale.

(a) What are the mean radius of the circle and its standard deviation?

(b) Compute the area of the circle and its standard deviation.

(c) Calibrate a planimeter by measuring a 2-in^2. Calculate the mean constant for the planimeter (k = units/4 in^2) and based on 10 measurements, determine the standard deviation in the constant.

(d) Using the same planimeter, measure the area of the circle and determine its standard deviation.

6.24 Develop a computational worksheet that solves Problem 6.19.

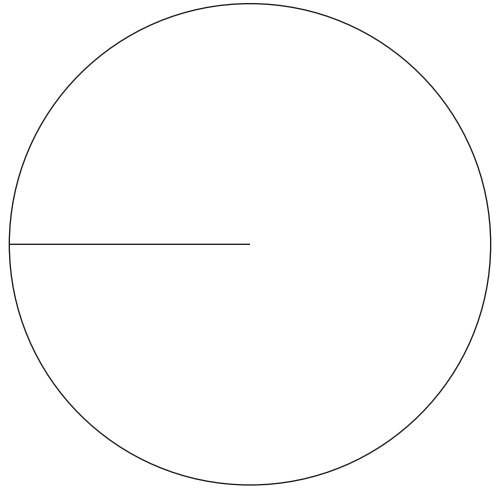


FIGURE P6.24

CHAPTER 7

ERROR PROPAGATION IN ANGLE AND DISTANCE OBSERVATIONS

7.1 INTRODUCTION

All survey observations are subject to errors from varying sources. For example, when observing an angle, the major sources of error include instrument placement and leveling, target placement, circle reading, and target pointing. Although great care may be taken in observing an angle, these error sources will nonetheless render inexact results. To fully appreciate the need for adjustments, surveyors must be able to identify the major observational error sources, know their effects on the observations, understand how they can be modeled, and understand how to keep the overall error budget within survey requirements. In this chapter, emphasis is placed on analyzing the errors in observed horizontal angles and distances. Chapter 8 discusses the way these errors propagate to produce traverse closure errors. Chapter 9 covers the propagation of errors in elevation determination. Chapter 10 then shows how these estimated errors can be used as the foundation for a valid stochastic (weighting) model in least-squares adjustments.

7.2 ERROR SOURCES IN HORIZONTAL ANGLES



Whether a transit, theodolite, or total station instrument is used in the survey, errors are present in every horizontal angle observation. Whenever an instrument's circles are read, a small error is introduced into the final angular value. This is true for both optical and digital reading instruments. Also, when pointing to a target, a small error always occurs. Other major error sources in angle observations include instrument and target setup errors as well as the instrument leveling error. Each of these sources produces random errors, which may be small or large, depending on the instrument, the operator, and the conditions at the time of the angle observation. The effects of reading, pointing, and leveling errors can be reduced, but not eliminated, by increasing the number of angle repetitions. The effects of instrument and target setup errors can only be reduced by increasing sight distances. However, increased sight distances will result in higher EDM ranging error, so these effects must be balanced.

7.3 READING ERRORS

Errors in reading conventional transits and theodolites are dependent on the quality of the instrument's optics, the size of the smallest division in the horizontal and vertical circles, and the operator's abilities. Reading errors occur with digital instruments; their size being dependent on the sensitivity of the particular electronic angular resolution system. Manufacturers usually quote the estimated combined pointing and reading precision for an individual direction measured with both faces of the instrument in terms of standard deviations. Typical values range from $\pm 1''$ for the more precise instruments to $\pm 10''$ for the less precise ones. These errors are random, and their effects on an angle are dependent on the observation method and the number of repeated observations.

7.3.1 Angles Observed by the Repetition Method

When observing a horizontal angle by the repetition method, the circle is first zeroed so that angles can be accumulated on the horizontal circle. The angle is turned several times, and finally, the cumulative angle is read and divided by the number of repetitions to determine the average angular value. In this method, a reading error exists in two positions regardless of the number of repetitions. When the circle is zeroed, the first reading error occurs. The second occurs when reading the final cumulative angle. For this procedure, the average angle is computed as

$$\alpha = \frac{\alpha_1 + \alpha_2 + \cdots + \alpha_n}{n} \qquad (a)$$

where α is the average angle, and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the n repetitions of the angle. Recognizing that readings occur only when zeroing the plates and reading the final

direction of angle α_n and applying Equation (6.18) to Equation (*a*), the standard error in reading the angle using the repetition method is

$$\sigma_{\alpha_r} = \frac{\sqrt{\sigma_0^2 + \sigma_r^2}}{n} \tag{7.1}$$

In Equation (7.1), σ_{α_r} is the error in the average angle due to reading; σ_0 is the estimated error in setting zero on the circle; σ_r is the estimated error in the final reading; and n is the number of repetitions of the angle. Note that the number of repetitions should always be an even number, with half being turned face I (direct) and half face II (reversed). This procedure compensates for systematic instrumental errors.

Assuming the observer's abilities to set zero and to read the circle are equal, Equation (7.1) is simplified to

$$\sigma_{\alpha_r} = \frac{\sigma_r\sqrt{2}}{n} \tag{7.2}$$

Example 7.1 Suppose that an angle is turned six times using the repetition method. For an observer having a personal reading error of $\pm 1.5''$, what is the error in the final angle due to circle reading?

SOLUTION From Equation (7.2), $\sigma_{\alpha_r} = \pm\frac{1.5''\sqrt{2}}{6} = \pm 0.4''$

7.3.2 Angles Observed by the Direction Method

When a horizontal angle is observed by the directional method, the horizontal circle is read in both the backsight and foresight directions. The angle is the difference between these two readings. When multiple observations of the angle are made, the circle is advanced prior to each reading to compensate for the systematic errors in the horizontal plates. The final angle is taken as the average of all the observed values. Again, an even number of repetitions are made, with half of the readings taken in face I and half in face II position. Since each repetition of the angle requires two readings, the error in the average angle due to the reading error is computed using Equation (6.18), which yields

$$\sigma_{\alpha_r} = \frac{\sqrt{(\sigma_{r_{1b}}^2 + \sigma_{r_{1f}}^2) + (\sigma_{r_{2b}}^2 + \sigma_{r_{2f}}^2) + \cdots + (\sigma_{r_{nb}}^2 + \sigma_{r_{nf}}^2)}}{n} \tag{7.3}$$

where $\sigma_{r_{ib}}$ and $\sigma_{r_{if}}$ are the estimated errors in reading the circle for both the backsight and foresight directions, respectively, and n is the number of repetitions.

Assuming one's ability to read the circle is independent of the particular direction, $\sigma_{r_{ib}} = \sigma_{r_{if}} = \sigma_r$. Hence, Equation (7.3) simplifies to

$$\sigma_{\alpha_r} = \frac{\sigma_r\sqrt{2}}{\sqrt{n}} \tag{7.4}$$

Example 7.2 Using the same parameters of six repetitions and an estimated observer reading error of $\pm 1.5''$ as given in Example 7.1, find the estimated error in the average angle due to reading when the directional method is used.

SOLUTION

$$\sigma_{\alpha_r} = \pm\frac{1.5''\sqrt{2}}{\sqrt{6}} = \pm 0.9''$$

Note that the additional readings needed in the directional method produce a larger error in the angle than that obtained when using the repetition method.

7.4 POINTING ERRORS

Accuracy in pointing to a target is dependent on several factors. These include the optical qualities of the telescope, the quality of the instrument calibration, target size, observer's personal ability to place the crosshairs on a target, and weather conditions at the time of observation. Pointing errors are random, and they will occur in every angle pointing no matter the instrument used. Since each repetition of an angle consists of two pointings, the pointing error for an angle that is the mean of n repetitions can be estimated using Equation (6.16) as

$$\sigma_{\alpha_p} = \frac{\sqrt{2\sigma_{p1}^2 + 2\sigma_{p2}^2 + \cdots + 2\sigma_{pn}^2}}{n} \tag{7.5}$$

In Equation (7.5), σ_{α_p} is the error in the angle due to pointing and $\sigma_{p_1}, \sigma_{p_2}, \ldots, \sigma_{p_n}$ are the estimated errors in pointings for the first repetition, second repetition, and so on. Again, for a given instrument and observer, the pointing error can be assumed to be the same for each repetition (i.e., $\sigma_{p_1} = \sigma_{p_2} = \cdots = \sigma_{p_n} = \sigma_p$) and Equation (7.5) simplifies to

$$\sigma_{\alpha_p} = \frac{\sigma_p\sqrt{2}}{\sqrt{n}} \tag{7.6}$$

Example 7.3 An angle is observed six times by an observer whose ability to point at a well-defined target is estimated to be $\pm 1.8''$. What is the estimated error in the average angle due to the pointing error?

SOLUTION From Equation (7.6),

$$\sigma_{\alpha_p} = \pm \frac{1.8''\sqrt{2}}{\sqrt{6}} = \pm 1.0''$$

7.5 ESTIMATED POINTING AND READING ERRORS WITH TOTAL STATIONS

With the introduction of digital theodolites and subsequently, total station instruments, new standards were developed for estimating errors in angle observations. The standards, called International Organization for Standardization (ISO) 17123-3 standard, provide values for estimated errors in the mean of two direction measurements, one each in face left and face right. Thus, in terms of a single pointing and reading error, σ_{pr}, the ISO value, σ_{ISO}, can be expressed as

$$\sigma_{ISO} = \frac{\sigma_{pr}\sqrt{2}}{2} = \frac{\sigma_{pr}}{\sqrt{2}}$$

Using this equation, the expression for the estimated error in the observation of a single direction due to pointing and reading with a digital theodolite is

$$\sigma_{pr} = \sigma_{ISO}\sqrt{2} \qquad (b)$$

Using a procedure such as that given in Equation (7.6), the estimated error in an angle measured n times and averaged due to pointing and reading is

$$\sigma_{\alpha_{pr}} = \frac{\sigma_{pr}\sqrt{2}}{\sqrt{n}} \qquad (c)$$

Substituting Equation (b) into Equation (c) yields

$$\sigma_{\alpha_{pr}} = \frac{2\sigma_{ISO}}{\sqrt{n}} \qquad (7.7)$$

Example 7.4 An angle is observed six times by an operator with a total station instrument having a published ISO 17123-3 value for the pointing and reading error of $\pm 5''$. What is the estimated error in the angle due to the pointing and reading errors?

SOLUTION From Equation (7.7), $\sigma_{\alpha_{pr}} = \frac{2(5'')}{\sqrt{6}} = \pm 4.1''$

7.6 TARGET CENTERING ERRORS

Whenever a target is set over a station, there will be some error due to faulty imperfections in the centering. It can be attributed to environmental conditions, optical plummet errors, quality of the optics, plumb bob centering error, personal abilities, and so on. When care is taken, the instrument is usually within 0.001–0.01 ft of the true station location. Although these sources produce a constant error for any individual angle, it will appear as random in the adjustment of a network involving many stations since targets and instruments will center differently over the same points. More importantly, this error will be noticed in resurveys of the same points.

An estimate of the effect of this error on an angle observation can be made by analyzing its contribution to a single direction. As shown in Figure 7.1, the angular error due to the centering error is dependent on the position of the target. If the target is online but off-center, as shown in Figure 7.1(*a*), the target centering error does not contribute to the angular error. However, as the target moves to either side of the sight line, the error size increases. As shown in Figure 7.1(*d*), the largest

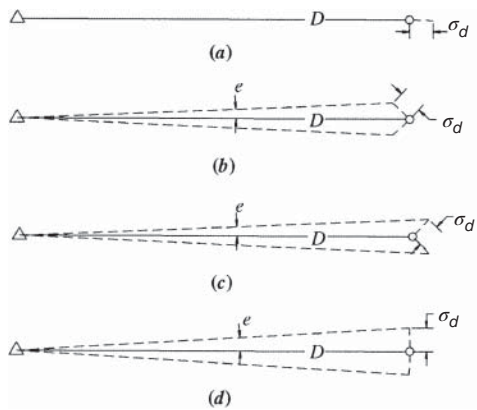


FIGURE 7.1 Possible target locations. (a) miscentering along line of sight does not affect angular error, (b) centering error closer to the instrument producing angular error, (c) centering error away from the instrument producing angular error, and (d) the largest angular error when the centering error is perpendicular to the line of sight.

error occurs when the target is offset perpendicular to the line of sight. Letting σ_d represent the distance the target is from the *true* station location, from Figure 7.1(d), the maximum error in an individual direction due to the target centering error is

$$e = \pm\frac{\sigma_d}{D} \text{ (rad)} \tag{7.8}$$

where e is the uncertainty in the direction due to the target centering error; σ_d is the amount of a centering error at the time of pointing; and D, as shown in Figure 7.2, is the distance from the instrument center to the target.

Since two directions are required for each angle observation, the contribution of the target centering error to the total angular error is

$$\sigma_{\alpha_t} = \sqrt{\left(\frac{\sigma_{d_1}}{D_1}\right)^2 + \left(\frac{\sigma_{d_2}}{D_2}\right)^2} \tag{7.9}$$

where σ_{α_t} is the angular error due to the target centering error; σ_{d_1} and σ_{d_2} are the target centering errors at stations 1 and 2, respectively; and D_1 and D_2 are the distances from the instrument to the target at stations 1 and 2, respectively. Assuming the ability to center the target over a point is independent of the direction, it can be stated that $\sigma_{d_1} = \sigma_{d_2} = \sigma_t$. Finally, the results of Equation (7.9) are in radian units. To convert the results to arc seconds, it must be multiplied by the constant ρ (206,264.8″/radian), which yields

$$\sigma''_{\alpha_t} = \pm\frac{\sqrt{D_1^2 + D_2^2}}{D_1 D_2}\sigma_t \rho \tag{7.10}$$

Notice that the same target centering error occurs on each pointing. Thus, it cannot be reduced in size by taking multiple pointings, and therefore Equation (7.10) is not divided by the number of angle repetitions. This makes the target centering error one of the more significant errors in angle observations. It also shows that

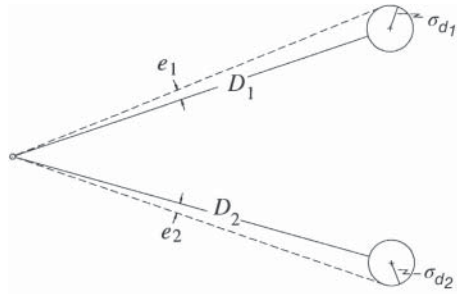


FIGURE 7.2 Error in angle due to target centering.

the most effective way to decrease the size of this error is to increase the sight distances.

Example 7.5 An observer's estimated ability at centering targets over a station is ±0.003 ft. For a particular angle observation, the backsight and foresight distances from the instrument station to the targets are approximately 250 ft and 450 ft, respectively. What is the angular error due to the error in target centering?

SOLUTION From Equation (7.10), the estimated error is

$$\sigma''_{\alpha_t} = \pm\frac{\sqrt{250^2 + 450^2}}{250(450)}0.003(206{,}264.8''/\text{rad}) = \pm 2.8''$$

If handheld range poles were used in this example with an estimated centering error of ±0.01 ft, the estimated angular error due to the target centering would be

$$\sigma''_{\alpha_t} = \pm\frac{\sqrt{250^2 + 450^2}}{250(450)}0.01(206{,}264.8''/\text{rad}) = \pm 9.4''$$

Obviously, this is a significant error source if care is not taken in target centering. Use of a bipod or tripod provides more stability and is necessary for control surveys. However, those can be impractical for points acquired for topographic features. When using hand-held range poles, multiple readings can be acquired on the target to reduce its centering error. Modern total stations can be set to average a specified number of measurements, which can reduce the error of wobbling. Some manufacturers have also developed range poles with internal tilt compensation, which can also help mitigate some of these effects, provided the tip is precisely centered.

7.7 INSTRUMENT CENTERING ERRORS

Every time an instrument is centered over a point, there is some error in its position with respect to the *true* location of the station. This error is dependent on the quality of the instrument and the state of adjustment of its optical plummet, the quality of the tripod, and the skill of the observer. As shown in Figure 7.3(*a*), this error can be compensating, or it can be at a maximum when the instrument is on the angle bisector, as shown in Figure 7.3(*b*) and (*c*). For any individual setup, this error is a constant. However, since the instrument's location is random with respect to the *true* station location, it will appear to be random in the adjustment of a network involving many stations. Like the target centering error, it will be most noticeable during a resurvey of the points. From Figure 7.3, the true angle α is

$$\alpha = (P_2 + \varepsilon_2) - (P_1 + \varepsilon_1) = (P_2 - P_1) + (\varepsilon_2 - \varepsilon_1)$$

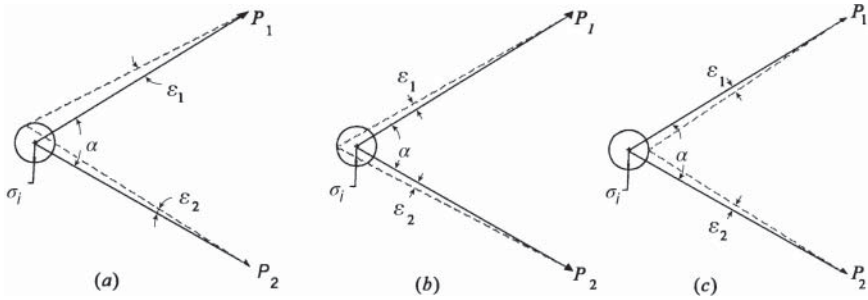


FIGURE 7.3 Error in angle due to error in instrument centering resulting in (a) compensating error, (b) maximum error when on the angle bisector away from the measured points, and (c) maximum error when on the angular bisector slightly closer to the measured points.

where P_1 and P_2 are the directions observed by the operator to the backsight and foresight stations, respectively, and ε_1 and ε_2 are errors in those directions due to faulty instrument centering. The overall error for any setup is

$$\varepsilon = \varepsilon_2 - \varepsilon_1 \tag{7.11}$$

The analysis of the error in the observed angle due to instrument centering errors is accomplished by propagating errors in a formula based on Cartesian (x, y) coordinates. In Figure 7.4, a coordinate system has been constructed with the x-axis going from the *true* station's location to the foresight station. The y-axis passes through the instrument's vertical axis and is perpendicular to the x-axis. From Figure 7.4, the following equations can be derived:

$$\begin{aligned} ih &= ip - qr \\ ih &= iq\ \cos(\alpha) - sq\ \sin(\alpha) \end{aligned} \tag{7.12}$$

Letting $sq = x$ and $iq = y$, Equation (7.12) can be rewritten as

$$ih = y\cos\alpha - x\sin\alpha \tag{7.13}$$

Furthermore, in Figure 7.4,

$$\varepsilon_1 = \frac{ih}{D_1} = \frac{y\cos\alpha - x\sin\alpha}{D_1} \tag{7.14}$$

$$\varepsilon_2 = \frac{y}{D_2} \tag{7.15}$$

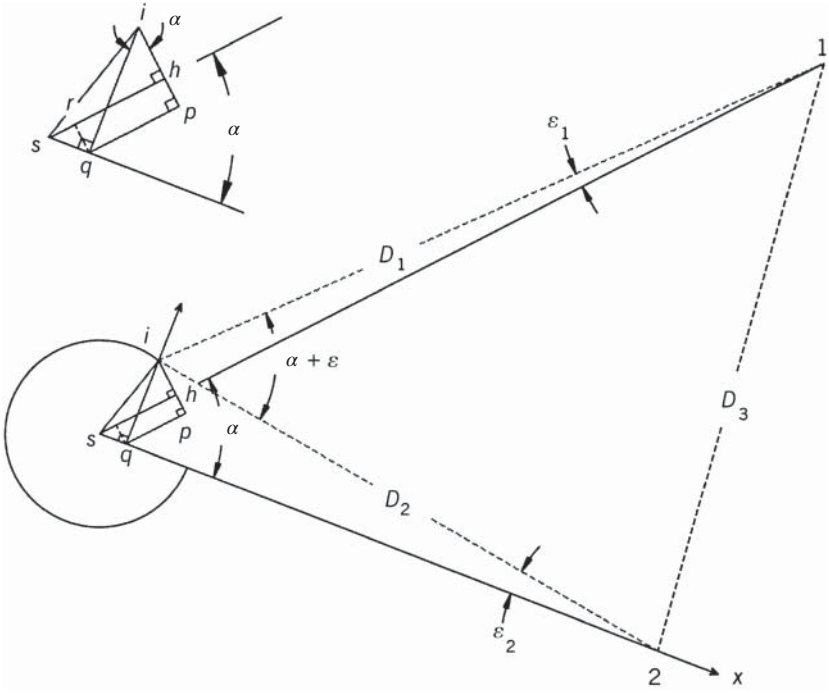


FIGURE 7.4 Analysis of instrument centering error.

By substituting Equations (7.14) and (7.15) into Equation (7.11), the error in an observed angle due to the instrument centering error is

$$\varepsilon = \frac{y}{D_2} - \frac{y\cos\alpha - x\sin\alpha}{D_1} \tag{7.16}$$

Reorganizing Equation (7.16) yields

$$\varepsilon = \frac{D_1 y + D_2 x\sin\alpha - D_2 y\cos\alpha}{D_1 D_2} \tag{7.17}$$

Now because the instrument's position is truly random, Equation (6.16) can be used to find the estimated angular error due to the instrument centering error. Taking the partial derivative of Equation (7.17) with respect to both x and y gives

$$\frac{\partial\varepsilon}{\partial x} = \frac{D_2\sin\alpha}{D_1 D_2}$$

and (7.18)

$$\frac{\partial\varepsilon}{\partial y} = \frac{D_1 - D_2\cos\alpha}{D_1 D_2}$$

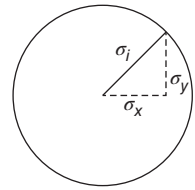


FIGURE 7.5 Centering errors at a station.

Now substituting the partial derivatives in Equation (7.18) into Equation (6.16) yields

$$\sigma_\varepsilon^2 = \left(\frac{D_2 \sin\alpha}{D_1 D_2}\right)\sigma_x^2 + \left(\frac{D_1 - D_2\cos\alpha}{D_1 D_2}\right)\sigma_y^2 \tag{7.19}$$

Because this error is a constant for any individual setup, the mean angle has the same error as a single angle, and thus it is not reduced by taking several repetitions. The estimated error in the position of a station is derived from a *bivariate* distribution,[1] where the coordinate components are independent and have equal magnitudes. Assuming that the estimated errors in the x- and y-axes are σ_x and σ_y, from Figure 7.5, it is seen that

$$\sigma_x = \sigma_y = \frac{\sigma_i}{\sqrt{2}}$$

Letting $\sigma_\varepsilon = \sigma_{\alpha_i}$, expanding the squares of Equation (7.19) and rearranging yields

$$\sigma_{\alpha_i}^2 = \frac{D_1^2 + D_2^2(\cos^2\alpha + \sin^2\alpha) - 2D_1 D_2\cos\alpha}{D_1^2 D_2^2}\frac{\sigma_i^2}{2} \tag{7.20}$$

Making the trigonometric substitutions of $\cos^2\alpha + \sin^2\alpha = 1$ and $D_1^2 + D_2^2 - 2D_1 D_2\cos\alpha = D_3^2$ in Equation (7.20), taking the square root of both sides, and multiplying by ρ (206,264.8″/radian) to convert the results to arc seconds yields

$$\sigma''_{\alpha_i} = \pm\frac{D_3}{D_1 D_2}\frac{\sigma_i}{\sqrt{2}}\rho \tag{7.21}$$

Example 7.6 An observer centers the instrument to within ±0.005 ft of a station for an angle with backsight and foresight distances of 250 ft and 450 ft, respectively. The observed angle is 50°. What is the error in the angle due to the instrument centering error?

[1]The bivariate distribution is discussed in Chapter 19.

SOLUTION Using the law of cosines, $D_3^2 = D_1^2 + D_2^2 - 2D_1D_2 \cos \angle$ and substituting in the appropriate values, D_3 is found to be

$$D_3 = \sqrt{250^2 + 450^2 - 2(250)(450)\cos 50^\circ} = 346.95 \text{ ft}$$

Substituting this value into Equation (7.21), the estimated contribution of the instrument centering error to the overall angular error is

$$\sigma''_{\alpha_i} = \pm \frac{346.95}{250(450)} \frac{0.005}{\sqrt{2}} 206,264.8''/\text{rad} = \pm 2.2''$$

7.8 EFFECTS OF LEVELING ERRORS IN ANGLE OBSERVATIONS

If an instrument is imperfectly leveled, its vertical axis is not truly vertical, and thus its horizontal circle and horizontal axis are both inclined. If horizontal angles are observed while an instrument is imperfectly leveled, the angles will be measured in some plane other than horizontal. Errors that result from this error source are most severe when the backsights or foresights are steeply inclined—for example, in making astronomical observations or traversing over mountainous terrain. If the bubble of the instrument were to remain off-center by the same amount during the entire angle-observation process, the resulting error would be systematic. However, because an operator normally carefully monitors the bubble and attempts to keep it centered while turning angles, the amount and direction by which the instrument is out of level becomes random, and hence the resulting errors tend to be random. Even if the operator does not carefully monitor the instrument's level, this error will appear to be random in a resurvey. Most modern total stations apply level compensation to each measurement and provide warnings when the instrument is out of compensation range. However, surveyors should still periodically check the instrument leveling and stability. If a tripod leg is loose or the ground is unstable, additional errors will result that cannot be reliably corrected.

In Figure 7.6, ε represents the angular error that occurs in either the backsight or foresight of a horizontal angle observation made with an instrument out of level and located at station I. The line of sight IS is elevated by the altitude angle v. In the figure, IS is shown perpendicular to the instrument's horizontal axis. The amount by which the instrument is out of level is $f_d\mu$, where f_d is the number of fractional divisions the bubble is off-center and μ is the bubble sensitivity. From the figure,

$$SP = D \tan(v) \qquad (d)$$

and

$$PP' = D\varepsilon \qquad (e)$$

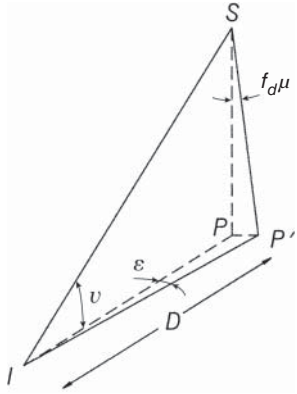


FIGURE 7.6 Effects of instrument leveling error.

where D is the horizontal component of the sighting distance, and the angular error ε is in radians. Because the amount of leveling error is small, PP' can be approximated as a circular arc, and thus

$$PP' = f_d \mu (SP) \tag{f}$$

Substituting Equation (d) into Equation (f) yields

$$PP' = f_d \mu D \tan \upsilon \tag{7.22}$$

Now substituting Equation (7.22) into Equation (e) and reducing, the error in an individual pointing due to the instrument leveling error is

$$\varepsilon = f_d \mu \tan \upsilon \tag{7.23}$$

As noted, Figure 7.6 shows the line of sight oriented perpendicular to the instrument's horizontal axis. Also, the direction a level runs is random. Thus, Equation (6.18) can be used to compute the combined angular error that results from n repetitions of an angle made with an imperfectly leveled instrument. Note that each angle observation involves both a backsight and a foresight pointing:

$$\sigma_{\alpha_l} = \pm \frac{\sqrt{(f_d \mu \tan \upsilon_b)^2 + (f_d \mu \tan \upsilon_f)^2}}{\sqrt{n}} \tag{7.24a}$$

$$\sigma_{\alpha_l} = \pm \frac{\sqrt{(f_d \mu \cot z_b)^2 + (f_d \mu \cot z_f)^2}}{\sqrt{n}} \tag{7.24b}$$

where Equation (7.24*a*) is for altitude-angle reading instruments and Equation (7.24*b*) is for zenith-angle reading instruments, where v_b and v_f are the altitude angles to the backsight and foresight targets, respectively; z_b and z_f are the zenith angles to the backsight and foresight targets, respectively; and n is the number of repetitions of the angle.

Example 7.7 A horizontal angle is observed on a hillside, where the backsight is to the top of the hill and the foresight is at the bottom. The average zenith angles to the backsight and foresight are 80° and 95°, respectively. The instrument has a level bubble with a sensitivity of 30″/div and is leveled to within 0.3 div. For the average angle obtained from six repetitions, what is the contribution of the leveling error to the overall angular error?

SOLUTION Substituting the appropriate values into Equation (7.24) yields

$$\sigma_{\alpha_l} = \pm\frac{\sqrt{[0.3\ div(30''/div)\cot 80^\circ]^2 + [0.3\ div(30''/div)\cot(-5^\circ)]^2}}{\sqrt{6}}$$

$$= \pm 0.8''$$

As shown in Example 7.7, this error is generally small for traditional surveying work when normal care in leveling the instrument is taken. Thus, it can generally be ignored for all but the most precise work. However, as noted earlier, for astronomical observations, this error can become quite large due to the steeply inclined sights to celestial objects. Thus, for astronomical observations, it is extremely important to keep the instrument precisely leveled for each observation.

7.9 NUMERICAL EXAMPLE OF COMBINED ERROR PROPAGATION IN A SINGLE HORIZONTAL ANGLE

Example 7.8 Assume that an angle is observed four times with a directional-type instrument. The observer has an estimated reading error of ±1″ and a pointing error of ±1.5″. The targets are well defined and placed on an optical plummet tribrach with an estimated centering error of ±0.003 ft. The instrument is in adjustment and centered over the station to within ±0.003 ft. The horizontal distances from the instrument to the backsight and foresight targets are approximately 251 ft and 347 ft, respectively. The average angle is 65°37′12″. What is the estimated error in the angle observation?

SOLUTION The best way to solve this type of problem is to compute estimated errors for each item in Sections 7.3–7.8 individually and then apply Equation (6.18).

Error Due to Reading: Substituting the appropriate values into Equation (7.4) yields

$$\sigma''_{\alpha_r} = \pm\frac{1''\sqrt{2}}{\sqrt{4}} = \pm 0.71''$$

Error Due to Pointing: Substituting the appropriate values into Equation (7.6) yields

$$\sigma''_{\alpha_p} = \pm\frac{1.5''\sqrt{2}}{\sqrt{4}} = \pm 1.06''$$

Error Due to Target Centering: Substituting the appropriate values into Equation (7.10) yields

$$\sigma''_{\alpha_t} = \frac{\sqrt{251^2 + 347^2}}{251(347)} 0.003 \times 206,264.8''/\text{rad} = \pm 3.04''$$

Error Due to Instrument Centering: From the cosine law, we have

$$D_3^2 = 251^2 + 347^2 - 2(251)347\cos(65°37'12'')$$
$$D_3 = 334 \text{ ft}$$

Substituting the appropriate values into Equation (7.21) yields

$$\sigma''_{\alpha_t} = \pm\frac{334}{251(347)}\frac{0.003}{\sqrt{2}} 206,264.8''/\text{rad} = \pm 1.68''$$

Combined Error: From Equation (6.18), the estimated angular error is

$$\sigma_\alpha = \sqrt{0.71^2 + 1.06^2 + 3.04^2 + 1.68^2} = \pm 3.7''$$

In Example 7.8, the largest error sources are due to the target and instrument centering errors, respectively. This is true even when the estimated error in centering the target and instrument is only ±0.003 ft. Because of these error sources, many surveyors unfortunately place more confidence in their observations than is warranted. Since these two error sources do not decrease with increased repetitions, there is a limit to what can be expected from any survey. For instance, assume that the targets were handheld reflector poles with an estimated centering error of ±0.01 ft. The error due to the target centering error becomes ±10.1″. This results in an estimated angular error of ±10.3″. If the 99% probable error were computed, a value as large as ±60″ would be possible!

7.10 USING ESTIMATED ERRORS TO CHECK ANGULAR MISCLOSURE IN A TRAVERSE



When a traverse is geometrically closed, the angles are generally checked for closure errors. By computing the errors for each angle in the traverse as described in Section 7.9 and summing the results with Equation (6.18), an estimate for the size of the angular closure error is obtained. The procedure is best demonstrated with an example.

Example 7.9 Assume each of the angles in Figure 7.7 were observed using four repetitions (2D and 2R), and their estimated errors were computed as shown in Table 7.1. Does this traverse meet acceptable angular closure at a 95% level of confidence?

SOLUTION The *actual* angular misclosure of the traverse is 30″. The *estimated* angular closure error of the traverse is found by applying Equation (6.18) with the computed errors for each angle. That is, the estimated angular closure error is

$$\sigma_{\sum\angle} = \sqrt{8.9^2 + 12.1^2 + 13.7^2 + 10.0^2 + 9.9^2} = \pm 24.7''$$

Thus, the actual angular misclosure of 30″ is greater than the value estimated (24.7″) at a 68.3% probable error level. However, since each angle was turned

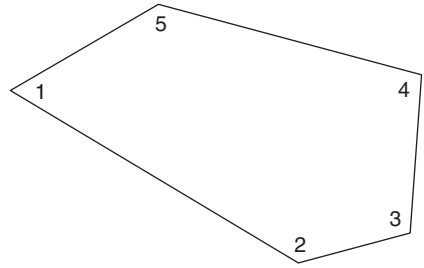


FIGURE 7.7 Closed-polygon traverse.

TABLE 7.1 Data for Example 7.9

Angle	Observed Value	Computed Angular Error
1	60°50′48″	±8.9″
2	134°09′24″	±12.1″
3	109°00′12″	±13.7″
4	100°59′54″	±10.0″
5	135°00′12″	±9.9″

only four times, a 95% probable error must be computed by using the appropriate t value from Table D.3.

This problem begs the question of what is the appropriate number of degrees of freedom for the summation of angles. Only four of the angles are required in the summation since the remaining fifth angle can be computed from the other four and thus is redundant. Since each angle is turned four times, it can be argued that there are 16 redundant observations, that is 12 angles (4×3) at the first four stations and four at the fifth station. However, this assumes that instrumental systematic errors were not present in the observational process since only the average of face I and face II reading can eliminate systematic errors in the instrument. If there are n angles and each angle is turned r times, the total number of redundant observations would be $n(r-1)+1$. In this case, it would be $5(4-1)+1=16$.

A second approach is to account for instrumental systematic errors when counting redundant observations. This method requires that an angle exists only if it is observed with both faces of the instrument. In this case, there is one redundant angle at each of the first four stations, with the fifth angle having two redundant observations for a total of six redundant observations. Using this argument, the number of redundant angles in the traverse would be $n(r/2-1)+1$. In this example, it would be $5(4/2-1)+1=6$.

A third approach would be to consider each mean angle observed at each station to be a single observation since only mean observations are being used in the computations. In this case, there would be only one redundant angle for the traverse. However, had horizon closures been observed at each station, the additional angles would add n redundant observations.

A fourth approach would be to determine the 95% probable error at each station and then use Equation (6.18) to sum up these 95% error values. In this example, each station has three redundant observations. In general, there would be $r-1$ redundant angle observations, where r represents the number of times the angle was repeated during the observation process.

The last two methods are the most conservative since they allow the most error in the sum of the angles. The fourth method is used in this text. However, the surveyor must decide which method is most appropriate for their practice. As stated in Chapter 5, the statistician must make decisions when performing any test. Using the fourth method, there are three redundant observations at each station. To finish the problem, we construct a 95% confidence interval or perform a two-tailed test to determine the range of error that is statistically equal to zero. In this case, $t_{0.025,3}=3.183$, the 95% probable error for the angular sum is

$$\sum_{95\%} = 3.183(24.7'') = \pm 78.6''$$

Thus, the traverse angles are well within the range of allowable error. We cannot reject the null hypothesis that the error in the angles is not statistically equal to zero. Thus, the survey meets the minimum level of angular closure at a 95% probable

error. However, it must be remembered that because of the possibility of Type II errors, we can only state that there is no statistical reason to believe that there might be a blunder in the angle observations.

Example 7.9 presents another question for the statistician-surveyor. That is, should a surveyor allow a field crew to have this large of an angular misclosure in the traverse? Statistically, the answer would seem to be yes, but recall that the target and instrument centering errors only affect angle observations if the instrument and targets are reset after each individual observation. Since this is never done in practice, these two errors should not be included in the summation of the angles. Instead, the allowable angular misclosure should be based solely on pointing and reading errors. For example, if the angles were observed with a total station having an ISO 17123-3 standard of $\pm 1''$, by Equation (7.7) the pointing and reading error for each angle would be estimated as

$$\sigma_{\alpha_{pr}} = \pm\frac{2 \times 1''}{\sqrt{2}} = \pm 1.4''$$

By Equation (6.19), the error in the summation of the five angles would be $\pm 1.4''\sqrt{5} = \pm 3.2''$. Using the same critical t value of 3.183, the allowable error in the angular misclosure should only be $3.183(3.2'') = \pm 10''$. If this instrument were used in Example 7.9, the field-observed angular closure of $30''$ would be unacceptable and would warrant reobservation of some or all the angles.

As stated in Sections (7.6) and (7.7), the angular misclosure of $78.6''$ computed in Example 7.9 will only be noticed when the target and instrument are reset on a survey. This most likely will happen during the resurvey. During the resurvey, the centering errors of the target and instrument from the original survey will be present in the *record* directions. Thus, record azimuths or bearings could disagree with those determined in the resurvey by this amount, if the equipment used in the resurvey is comparable or of higher quality than that used in the original survey.

7.11 ERRORS IN ASTRONOMICAL OBSERVATIONS FOR AZIMUTH

The total error in an azimuth determined from astronomical observations is dependent on errors from several sources, including those in timing, the observer's latitude and longitude, the celestial object's position at observation time, timing accuracy, instrument optics, atmospheric conditions, and others as identified in Section 7.2. The error in astronomical observations can be estimated by analyzing the hour-angle formula, which is

$$z = \tan^{-1}\left(\frac{\sin t}{\cos\phi\tan\delta - \sin\phi\cos t}\right) \tag{7.25}$$

In Equation (7.25), z is the angle used to compute the azimuth of the celestial object at the time of the observation, t the *t angle* of the *PZS* triangle at the time of observation, ϕ the observer's latitude, and δ the object's declination at the time of the observation.

The t angle is a function of the *local hour angle* (LHA) of the sun or star at the time of observation. That is, when the LHA $< 180°$, then $t = \text{LHA}$, otherwise $t = 360° - \text{LHA}$. Furthermore, the LHA is a function of the *Greenwich hour angle* (GHA) of the celestial body and the observer's longitude, that is,

$$LHA = GHA + \lambda \tag{7.26}$$

where λ is the observer's longitude, considered positive if eastern longitude and negative if western longitude. The GHA increases approximately 15° per hour of time, and thus an estimate of the error in the GHA is approximately

$$\sigma_t = 15°\sigma_T \tag{7.27}$$

where σ_T is the estimated error in time (in hours). Similarly, by using the declination at 0^h and 24^h, the amount of change in declination per second can be derived, and thus the estimated error in the hour angle is determined.

Using Equation (6.16), the error in a star's azimuth is estimated by taking the partial derivative of Equation (7.25) with respect to t, δ, ϕ, and λ. To do this, simplify Equation (7.25) by letting

$$F = \cos\phi \tan\delta - \sin\phi \cos t \tag{7.28a}$$

and

$$u = \frac{\sin t}{F} \tag{7.28b}$$

Substituting Equation (7.27), Equation (7.25) is rewritten as

$$z = \tan^{-1}\frac{\sin t}{F} = \tan^{-1} u \tag{7.29}$$

From calculus,

$$\frac{d\, tan^{-1} u}{dx} = \frac{1}{1+u^2}\frac{du}{dx}$$

Applying this fundamental relation to Equation (7.28) and letting G represent GHA yields

$$\frac{\partial z}{\partial G} = \frac{1}{1+[\sin(G-\lambda)/F]^2}\frac{du}{dG} = \frac{F^2}{F^2+\sin^2(G-\lambda)}\frac{du}{dG} \tag{7.30}$$

Now, $\frac{du}{dG}$ is

$$\frac{du}{dG} = \frac{\cos(G-\lambda)}{F} - \frac{\sin(G-\lambda)}{F^2}\sin\phi\sin(G-\lambda)$$
$$= \frac{\cos(G-\lambda)}{F} - \frac{\sin^2(G-\lambda)\sin\phi}{F^2}$$

and thus

$$\frac{du}{dG} = \frac{F\cos(G-\lambda) - \sin^2(G-\lambda)\sin\phi}{F^2} \tag{7.31}$$

Substituting Equation (7.30) into Equation (7.29) and substituting t for $G-\lambda$ yields

$$\frac{\partial z}{\partial G} = \frac{F\cos t - \sin^2 t\ \sin\phi}{F^2 + \sin^2 t} \tag{7.32}$$

In a similar fashion, the following partial derivatives are developed from Equation (7.25)

$$\frac{dz}{d\delta} = -\frac{\sin t\cos\phi}{\cos^2\delta(F^2+\sin^2 t)} \tag{7.33}$$

$$\frac{\partial z}{\partial\phi} = \frac{\sin t\cos t\cos\phi + \sin t\sin\phi\tan\delta}{F^2+\sin^2 t} \tag{7.34}$$

$$\frac{\partial z}{\partial\lambda} = \frac{\sin^2 t\ \sin\phi - F\cos t}{F^2+\sin^2 t} \tag{7.35}$$

In Equation (7.35), t is the t angle of the PZS triangle, z the celestial object's azimuth, δ the celestial object's declination, ϕ the observer's latitude, λ the observer's longitude, and $F = \cos\phi\tan\delta - \sin\phi\cos t$.

If the horizontal angle, H, is the observed angle to the right from the line to the celestial body, the equation for a line's azimuth is

$$Az = z + 360^\circ - H \tag{7.36}$$

Therefore, the error contributions from the horizontal angle observation must be included in computing the overall error in the azimuth. Since the distance to the star is considered infinite, the estimated contribution to the angular error due to the instrument centering error can be determined with a formula like that for the target centering error with one pointing. That is,

$$\sigma_{\alpha_i} = \frac{\sigma_i}{D} \tag{7.37}$$

where σ_i is the centering error in the instrument, and D the length of the azimuth line in the same units. Note that the results of Equation (7.37) are in radian units and must be multiplied by ρ to yield a value in arc seconds.

Example 7.10 Using Equation (7.25), the azimuth to Polaris was found to be 0°01′31.9″. The observation time was 1:00:00 UTC with an estimated error of $\sigma_T = \pm 0.5^{\text{s}}$. The Greenwich hour angles to the star at 0^{h} and 24^{h} UTC were 243°27′05.0″ and 244°25′50.0″, respectively. The LHA at the time of the observation was 181°27′40.4″. The declinations at 0^{h} and 24^{h} were 89°13′38.18″ and 89°13′38.16″, respectively. At the time of observation, the declination was 89°13′38.18″. The clockwise horizontal angle observed from the backsight to a target at a distance of 450.00 ft was 221°25′55.9″. The observer's latitude and longitude were scaled from a map as 40°13′54″N and 77°01′51.5″W, respectively, with estimated errors of ±1″. The altitude angle to the star was 39°27′33.1″. The observer's estimated errors in reading and pointing are ±1″ and ±1.5″, respectively, and the instrument was leveled to within 0.3 of a division with a bubble sensitivity of 25″/div. The estimated error in instrument and target centering is ±0.003 ft. What is the azimuth of the line and its estimated error? What is the error at the 95% level of confidence?

SOLUTION The azimuth of the line is $Az = 0°01'31.9'' + 360 - 221°25'55.9'' = 138°35'36''$. Using the Greenwich hour angles at 0^{h} and 24^{h}, an error of $\pm 0.5^{\text{s}}$ time will result in an estimated error in the GHA of

$$\pm \frac{360° + (244°25'50.0'' - 243°27'05.0'')}{24^h \times 3600^{s/h}} 0.05^s = \pm 7.52''$$

Since $t = 360° - \text{LHA} = 178°32'19.6''$, F in the Equations (7.29) through (7.34) is

$$\begin{aligned} F &= \cos 40°13'54'' \tan 89°13'38.18'' - \sin 40°13'54'' \cos 178°32'19.6'' \\ &= 57.249 \end{aligned}$$

The error in the observed azimuth can be estimated by computing the individual error terms, such as

(a) From Equation (7.32), the error with respect to the GHA, G, is

$$\begin{aligned} \frac{\partial z}{\partial G}\sigma_G &= \frac{57.249 \cos 178°32'19.6'' - \sin^2 178°32'19.6'' \sin 40°13'54''}{57.249^2 + \sin^2 178°32'19.6''} 7.52'' \\ &= \pm 0.13'' \end{aligned}$$

(b) By observing the change in declination, it is obvious that for this observation, the error in a time of $\pm 0.5^{\text{s}}$ is insignificant. In fact, for the entire day,

the declination changes only 0.02″. This situation is common for stars. However, the sun's declination may change from only a few seconds daily to more than 23 minutes per day, and thus, for solar observations, this error term should not be ignored.

(c) From Equation (7.34), the error with respect to latitude, ϕ, is

$$\frac{\partial z}{\partial \phi}\sigma_\phi = \pm\frac{\sin t \cos t \cos\phi + \sin t \sin\phi \tan\delta}{F^2 + \sin^2 t}\sigma_\phi = \pm 0.0004''$$

(d) From Equation (7.35), the error with respect to longitude, λ, is

$$\pm\frac{\sin^2(178^\circ 32' 19.6'')\sin(40^\circ 13' 54'') - 57.249\cos(178^\circ 32' 19.6'')}{57.249^2 + \sin^2(178^\circ 32' 19.6'')} \times 1''$$
$$= \pm 0.02''$$

From (c) and (d), we see that the errors in the latitude and longitude of the observing station are insignificant, and thus it is acceptable to scale these values from a map or obtain them from an online service such as Google Earth®.

(e) The circles are read both when pointing at the star and at the azimuth mark. Thus, from Equation (7.2), the reading contribution to the estimated error in the azimuth is

$$\sigma_{\alpha_r} = \pm\sigma_r\sqrt{2} = \pm 1''\sqrt{2} = \pm 1.41''$$

(f) Using Equation (7.6), the estimated error in the azimuth due to pointing is

$$\sigma_{\alpha_p} = \pm\sigma_p\sqrt{2} = \pm 1.5''\sqrt{2} = \pm 2.12''$$

(g) From Equation (7.8), the estimated error in the azimuth due to target centering is

$$\sigma''_{\alpha_t} = \pm\frac{d}{D} = \pm\frac{0.003}{450}206,264.8''/\text{rad} = \pm 1.37''$$

(h) Using Equation (7.37), the estimated error in the azimuth due to instrument centering is

$$\sigma''_{\alpha_i} = \pm\frac{d}{D} = \pm\frac{0.003}{450}206,264.8''/\text{rad} = \pm 1.37''$$

(i) From Equation (7.23), the estimated error in the azimuth due to the leveling error is

$$\sigma_{\alpha_b} = \pm f_d \mu \tan v = \pm 0.3(25'' \tan 39^\circ 27' 33.1'') = \pm 6.17''$$

Parts (a) through (i) are the errors for each individual error source. Using Equation (6.18), the estimated error in the azimuth observation is

$$\sigma_{AZ} = \sqrt{(0.13'')^2 + (1.32'')^2 + (0.02'')^2 + (0.0004'')^2 + (2.12'')^2 + 2(1.37'')^2 + (6.17'')^2}$$
$$= \pm 7.0''$$

Using the appropriate critical *t value* of $t_{0.025,1}$ from Table D.3, the 95% error is

$$\sigma_{Az} = \pm 12.705(7.0'') = \pm 88.9''$$

Notice that in this problem, the largest error source in the azimuth error is caused by the instrument leveling error, and that the errors caused by the observer's position are insignificant if care is taken in obtaining the position of the station. Additionally, the timing error is insignificant for stellar observations but is generally not for solar observations.

7.12 ERRORS IN ELECTRONIC DISTANCE OBSERVATIONS

All electronically measured distance observations are subject to instrumental errors that manufacturers list as a constant, a, and scalar, b, error. A typical specified accuracy is defined as $\pm(a + b$ ppm). In this expression, a is generally in the range of 1–10 mm, and b is a scalar error, which typically has the range 1–10 parts per million (ppm). Other errors involved in electronically measured distance observations stem from the target and instrument centering errors. Since these errors are random in any survey involving several stations, they should be combined using Equation (6.18). Thus, the estimated error in an EDM observed distance is

$$\sigma_D = \sqrt{\sigma_i^2 + \sigma_t^2 + a^2 + (D \times b \text{ ppm})^2} \tag{7.38}$$

where σ_D is the error in the observed distance D; σ_i is the instrument miscentering error; σ_t is the reflector miscentering error; and a and b are the instrument's specified accuracy parameters.

Example 7.11 A distance of 453.87 ft is observed using an EDM with a manufacturer's specified accuracy of $\pm(3$ mm $+ 3$ ppm). The instrument is centered over the station with an estimated error of ± 0.003 ft, and the reflector, which is mounted on a handheld prism pole, is centered with an estimated error of ± 0.01 ft. What is the error in the observed distance? What is E_{95} value?

SOLUTION Converting millimeters to feet yields

$$\frac{0.003}{0.3048} = 0.0098 \text{ ft}$$

The scalar portion of the manufacturer's estimated standard error is computed as

$$\frac{\text{distance} \times b}{1,000,000}$$

In this example, the error is 453.87(3/1,000,000) = ±0.0014 ft. Thus, according to Equation (7.38), the distance error is

$$\sigma = \sqrt{(0.003)^2 + (0.01)^2 + (0.0098)^2 + (0.0014)^2} = \pm 0.0144 \text{ ft}$$

Using the appropriate *t value* from Table 3.2, the 95% probable error is

$$E_{95} = 1.96\sigma = \pm 0.028 \text{ ft}$$

Notice in this example that the instrument's constant error and target centering errors are the largest single contributors to the overall error in the observed distance. Furthermore, since both errors are constants, their contribution to the total error is unchanged regardless of the distance. Thus, if higher accuracy is needed for this particular survey, a stable tripod or bipod should be used for the prism. Additionally, distances under 200 ft could probably be observed more accurately with a calibrated steel tape. However, this statement is dependent on the terrain, the quality of the instrument, and the skill of the surveyor in using a steel tape.

Furthermore, notice that when computing the 95% probable error, the critical *t* value is obtained from the normal distribution. This is because distance observations are seldom repeated, which means that the instrument and reflector are reset between each observation and atmospheric conditions have changed. As an example, simply pushing the distance measurement button three times does not constitute independent observations of a distance since the instrument and reflector setups are never moved and sufficient time does not pass to affect atmospheric conditions.

7.13 CENTERING ERRORS WHEN USING RANGE POLES

The estimated centering error in the EDM reflector, total station target, or GNSS receiver that is attached to the top of a range pole can be significant. From geometry, we know that the 3D displacement at the top of a pole from its point at the bottom is computed as

$$S = R\theta \tag{7.39}$$

where S is the displacement of the top of the pole relative to its bottom point; R is the length of the pole; and θ is the angular shift from vertical of the pole in radian units, which can be computed as $f_d\mu$ with variables, as previously defined for Equation (7.22). Note that although S represents a 3D displacement, for small

values of θ most of the centering error will be in the horizontal direction. Using Equation (6.16) and substituting $f_d\mu$ for θ, the centering error of a reflector or GNSS receiver is

$$\sigma_t = \sqrt{(\theta\sigma_R)^2 + (R\sigma_\theta)^2} = \sqrt{(f_d\mu\sigma_R)^2 + (R\sigma_\theta)^2} \tag{7.40}$$

where σ_t is the centering error of the reflector or GNSS receiver; θ is the angular displacement from vertical of the target/receiver in radians; and σ_R is the uncertainty in R. According to Franklin and Meyer (2016), σ_θ follows a Rayleigh distribution and can be computed as

$$\sigma_\theta = f_d\mu\sqrt{0.5(4-\pi)} \tag{7.41}$$

Assume in Equation (7.41) that a calibrated circular vial is within one-half of a division of being centered, which would be 1 mm for a 2-mm division. Then, σ_θ is computed as

$$\sigma_\theta = 0.5(0.011636)\sqrt{0.5(4-\pi)} = 0.003811 \text{ rad}$$

where 0.011636 is $40'$ converted to radian units. Further assume that the range pole is $2\text{ m} \pm 0.002\text{ m}$, the estimated centering error in the reflector is given by Equation (7.40) as

$$\begin{aligned}\sigma_t &= \sqrt{\left[\frac{0.5(40)60}{206{,}264.8}0.002\right]^2 + (2\times 0.003811)^2} \\ &= \sqrt{0.000023^2 + 0.007623^2} = \pm 0.008\text{ m}\end{aligned}$$

Note that when examining the values in the above equation for this example that the error in the height of the rod has a minimal contribution to the overall error in centering (assuming that it is a reasonably accurate value). Thus, Equation (7.41) can be simplified as

$$\sigma_t = R\sigma_\theta \tag{7.42}$$

where R and σ_θ are as previously defined.

7.14 SOFTWARE

The tedious and time-consuming computations demonstrated in this chapter often leads to mistakes. Such surveying problems involving repeated computations of a few equations with different values, can be done conveniently with a spreadsheet, worksheet, script, or program. On the companion website for this book is an electronic book prepared with Mathcad, demonstrating the programming of the

σ_{ISO} ±5"													
σ_i ±0.005													
σ_t ±0.003													
Stations			Angle				Distances						
BS	Inst	FS	°	'	"	n	D (ft)		D_3	σ_{aISO}	σ_{at}	σ_{ai}	σ_a
B	I	F	50°	00'	00"	6	250.00	450.00	346.948	±1.2"	±2.8"	±2.2"	±3.8"

FIGURE 7.8 Excel® worksheet for computing estimated errors in angles.

computational examples in this chapter. Figure 7.8 depicts a spreadsheet, *C7.xls,* also available on the companion website that can be used to compute estimated errors in an angle. It demonstrates the use of spreadsheet to solve Examples 7.4 through 7.6. The user enters the ISO 17123-3 standard for a total station instrument and the estimated instrument and target setup errors. Information about the angle includes the stations, angular value and its standard deviation, number of times the angle was turned, and the backsight and foresight distances. The spreadsheet then computes the estimated errors in the angle due to pointing, reading, and setup errors. When practicing the following problems, the reader should consider writing software to perform the computations.

PROBLEMS

Note: Partial answers to problems marked with an asterisk can be found in Appendix G.

7.1 Plot a graph of vertical angles from 0° to 60° versus the error in horizontal angle measurement due to an instrument leveling error of 10″.

7.2 For a direction with sight distances to the target of 100, 200, 300, 400, 600, 1000, and 1500 ft, construct:

(a) A table of estimated standard deviations due to target centering when $\sigma_t = \pm 0.005$ ft.

(b) A plot of distance versus the standard deviations computed in part (a).

7.3 For an angle of size 100° with equal sight distances to the targets of: 100, 200, 300, 400, 600, 1000, and 1500 ft, construct:

(a) A table of standard deviations due to instrument centering when $\sigma_i = \pm 0.005$ ft.

(b) A plot of distance versus the standard deviations computed in part (a).

***7.4** Assuming a setup errors of $\sigma_i = \pm 0.003$ ft and $\sigma_t = \pm 0.005$ ft, what is the estimated error in a distance of length 864.36 ft, using an EDM with stated accuracies of 3 mm + 3 ppm?

7.5 Similar to Problem 7.4, except the setup errors are $\sigma_i = \pm 3$ mm and $\sigma_t = \pm 5$ mm and the distance is 254.597 m.

7.6 Similar to Problem 7.4, except the length of the distance is 854.34 ft and the stated accuracies for the EDM are 5 mm + 5 ppm and $\sigma_i = \pm 0.005$ ft and $\sigma_t = \pm 0.015$ ft.

7.7 Assuming a setup errors of $\sigma_i = \pm 0.005$ ft and $\sigma_t = \pm 0.015$ ft, what is the estimated error in a distance of length 2008.36 ft using an EDM with stated accuracies of 3 mm + 3 ppm?

7.8 Similar to Problem 7.7, except that the setup errors are ± 1.5 mm and the distance has a length of 597.098 m.

***7.9** A 59°06′22″ angle having a backsight length of 125.003 m and a foresight length of 67.836 m is observed with a total station having a stated ISO 17123-3 standard of $\pm 3''$. Assuming instrument and target centering errors of $\sigma_i = \pm 0.0015$ m and $\sigma_t = 0.003$ m, what is the estimated error in the angle?

7.10 Similar to Problem 7.9, except the ISO 17123-3 standard is $\pm 2''$.

7.11 Similar to Problem 7.9, except the angle is 127°03′42″, backsight length is 155.05 ft, foresight length is 210.85 ft, the instrument and target centering errors are $\sigma_i = \pm 0.003$ ft and $\sigma_t = \pm 0.01$ ft, the angle is observed four times, and the instrument has a stated ISO 17123-3 standard of $\pm 5''$. What is the estimated error in the angle?

7.12 Similar to Problem 7.11, except the ISO 17123-3 standard is $\pm 2''$.

***7.13** For the following traverse data, compute the estimated error for each angle if ISO 17123-3 standard is $\pm 3''$, $\sigma_i = \pm 0.005$ ft, $\sigma_t = \pm 0.005$ ft, and the angles were each measured four times (two direct and two reverse). Does the traverse meet acceptable angular closures at a 95% level of confidence?

Station	Angle	Course	Distance (ft)
A	62°32′57″	*AB*	221.85
B	124°56′24″	*BC*	346.55
C	60°44′05″	*CD*	260.66
D	111°46′09″	*DA*	349.16

7.14 A total station with an ISO 17123-3 standard of $\pm 1''$ was used to turn the angles in Problem 7.13. The angles were turned four times (two direct and two reverse). Assume the same estimated errors in instrument and target centering. Does the traverse meet acceptable angular closures at a 95% level of confidence?

7.15 A total station with an ISO 17123-3 standard of $\pm 2''$ was used to turn the angles in Problem 7.13. The angles were turned two times (one direct and one reverse). Assume the same estimated errors in instrument and target centering. Does the traverse meet acceptable angular closures at a 95% level of confidence?

7.16 Assuming that the instrument and target are not moved during the angle measurement procedure:

(a) Does the traverse in Problem 7.13 meet acceptable misclosure at a 95% level?

(b) Does the traverse in Problem 7.14 meet acceptable misclosure at a 95% level?

(c) Does the traverse in Problem 7.15 meet acceptable misclosure at a 95% level?

7.17 For the following traverse data, compute the estimated error in each angle if $\sigma_{ISO} = \pm 3''$, $\sigma_i = \sigma_t = \pm 0.01$ ft, and the angles were observed four times (twice direct and twice reverse) using the repetition method. Does the traverse meet acceptable angular closures at a 95% level of confidence?

Station	Angle	Course	Distance (ft)
A	39°04′48″	*AB*	321.79
B	148°55′06″	*BC*	276.83
C	84°25′22″	*CD*	102.33
D	114°36′08″	*DE*	306.16
E	152°58′18″	*EA*	255.53

7.18 A total station with an ISO 171123-3 standard of $\pm 1''$ was used to turn the angles six times (three direct and three reverse) in Problem 7.17. Repeat the problem for this instrument.

7.19 Assuming that the instrument and target are not moved during the angle measurement procedure:

(a) Does the traverse in Problem 7.17 meet acceptable misclosure at a 95% level?

(b) Does the traverse in Problem 7.18 meet acceptable misclosure at a 95% level?

7.20 An EDM was used to measure the distances in Problem 7.13. The manufacturer's specified error for the instrument are $\pm(3\text{ mm} + 3\text{ ppm})$. Using $\sigma_i = \pm 0.005$ ft and $\sigma_t = \pm 0.005$ ft, calculate the estimated error in each distance.

7.21 Similar to Problem 7.20, but for the distances in Problem 7.17. The manufacturer's specified error for the instrument was $\pm(1\text{ mm} + 1\text{ ppm})$. Use σ_i and σ_t from Problem 7.17.

7.22 The following observations and calculations were made on a sun observation to determine the azimuth of a line.

Observation No.	UTC	Horizontal Angle	Altitude Angle	δ	LHA	z
1	16:30:00	41°02′33″	39°53′08″	−3°28′00.58″	339°54′05.5″	153°26′51.8″
2	16:35:00	42°35′28″	40°16′49″	−3°28′05.43″	341°09′06.5″	154°59′42.4″
3	16:40:00	44°09′23″	40°39′11″	−3°28′10.27″	342°24′07.5″	156°33′39.0″
4	16:45:00	45°44′25″	41°00′05″	−3°28′15.11″	343°39′08.5″	158°08′39.2″
5	16:50:00	47°20′21″	41°19′42″	−3°28′19.96″	344°54′09.5″	159°44′40.2″
6	16:55:00	48°57′24″	41°37′47″	−3°28′24.80″	346°09′10.5″	161°21′38.9″

The Greenwich hour angles for the day were 182°34′06.00″ at 0^h UT and 182°38′53.30″ at 24^h UT. The declinations were −3°12′00.80″ at 0^h and −3°35′16.30″ at 24^h. The observer's latitude and longitude were scaled from a map as 43°15′22″ and 90°13′18″, respectively, with an estimated standard error of ±1″ for both values. Stopwatch times were assumed to be correct to within an error of $\pm 0.5^s$. A Roelof's prism was used to take pointings on the center of the sun. The target was 535 ft from the observer's station. The observer's estimated reading and pointing errors were ±1.2″ and ±1.8″, respectively. The instrument was leveled to within 0.3 of a div on a level bubble with a sensitivity of 20″/div. The target was centered to within an estimated error of ±0.003 ft of the station.

(a) What is the average azimuth of the line and its standard deviation?

(b) What is the estimated error of the line at 95% level of confidence?

(c) What is the largest error contributor in the observation?

PROGRAMMING PROBLEMS

7.23 Create computational package that will compute the errors in angle observations. Use the package to compute the estimated errors for the angles in Problem 7.13.

7.24 Create a computational package that will compute the errors in EDM observed distances. Use the package to solve Problem 7.20.

CHAPTER 8

ERROR PROPAGATION IN TRAVERSE SURVEYS

8.1 INTRODUCTION

Even though the specifications for a project may allow lower accuracies, the presence of blunders in observations is never acceptable. Thus, an important question for every surveyor is, "How can I tell when blunders are present in the data?" In this chapter, we begin to address that question, and in particular, stress traverse analysis. This topic is discussed further in Chapters 21 and 25.

In Chapter 6, it was shown that the estimated error in a function of observations is dependent on the individual errors in the observations. Generally, observations in horizontal surveys (e.g., traverses) are independent. That is, the measurement of a distance observation is independent of the azimuth observation. But the latitude and departure of a line, which are computed from the distance and azimuth observations, are not independent. Figure 8.1 shows the effects of the errors in distance and azimuth observations on the computed latitude and departure. Figure 8.1 shows that there is a correlation between the latitude and departure; that is, if either distance or azimuth observation changes, corresponding changes occur in both latitude and departure.

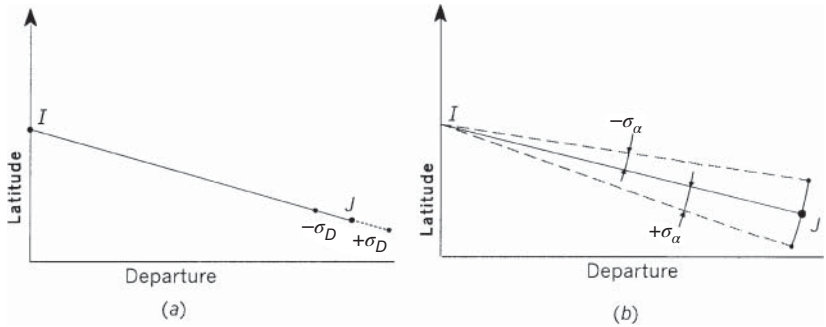


FIGURE 8.1 Latitude and departure uncertainties due to (*a*) the distance standard error (σ_D) and (*b*) the azimuth standard error (σ_α). Note that if either the distance or azimuth changes, both the latitude and departure are affected.

Because the observations from which latitudes and departures are computed are assumed to be independent with no correlation, the special law of propagation of variances (SLOPOV) approach [Equation (6.16)] can be used to determine the estimated errors in these computed values. However, for proper computation of estimated errors in functions, which use these computed values (i.e., latitudes and departures), the effects of correlation must be considered, and thus the general law of propagation of variances (GLOPOV) approach [Equation (6.13)] will be used.

8.2 DERIVATION OF ESTIMATED ERROR IN LATITUDE AND DEPARTURE



When computing the latitude and departure of a line, the following well-known equations are used:

$$Lat = D\cos Az$$
$$Dep = D\sin Az \tag{8.1}$$

where *Lat* is the latitude of a course; *Dep* is the departure; *Az* is the azimuth; and *D* is the horizontal length of the line. To derive the estimated error in the line's latitude or departure, the following partial derivatives from Equation (8.1) are required in using Equation (6.13):

$$\frac{\partial Lat}{\partial D} = \cos Az \qquad \frac{\partial Lat}{\partial Az} = -D\sin Az$$
$$\frac{\partial Dep}{\partial D} = \sin Az \qquad \frac{\partial \mathrm{Dep}}{\partial \mathrm{Az}} = D\cos Az \tag{8.2}$$

Example 8.1 A traverse course has a length of 456.87 ± 0.02 ft and an azimuth of $23°35'26'' \pm 9''$. What are the latitude and departure, and their estimated errors?

SOLUTION Using Equation (8.1), the latitude and departure of the course are

$$Lat = 456.87 \cos(23°35'26'') = 418.69 \text{ ft}$$
$$Dep = 456.87 \sin(23°35'26'') = 182.84 \text{ ft}$$

The estimated errors in these values are solved using Equation (6.16) as

$$\Sigma_{Lat,\,Dep} = \begin{bmatrix} \dfrac{\partial Lat}{\partial D} & \dfrac{\partial Lat}{\partial Az} \\ \dfrac{\partial Dep}{\partial D} & \dfrac{\partial Dep}{\partial Az} \end{bmatrix} \begin{bmatrix} \sigma_D^2 & 0 \\ 0 & \sigma_{Az}^2 \end{bmatrix} \begin{bmatrix} \dfrac{\partial Lat}{\partial D} & \dfrac{\partial Dep}{\partial D} \\ \dfrac{\partial Lat}{\partial Az} & \dfrac{\partial Dep}{\partial Az} \end{bmatrix}$$
$$= \begin{bmatrix} \sigma_{Lat}^2 & \sigma_{Lat,\,Dep} \\ \sigma_{Lat,\,Dep} & \sigma_{Dep}^2 \end{bmatrix}$$

Substituting partial derivatives into the above yields

$$\Sigma_{Lat,\,Dep} = \begin{bmatrix} \cos Az & -D \sin Az \\ \sin Az & D \cos Az \end{bmatrix} \begin{bmatrix} 0.02^2 & 0 \\ 0 & (9''/\rho)^2 \end{bmatrix} \begin{bmatrix} \cos Az & \sin Az \\ -D \sin Az & D \cos Az \end{bmatrix} \quad (8.3)$$

Notice that the angular values must be converted to radians to prevent a mixing of units. A close look at the units of the matrix multiplication shows that the final units will be ft^2. Entering the appropriate numerical values into Equation (8.3), the *covariance* matrix is

$$\Sigma_{Lat,\,Dep} = \begin{bmatrix} 0.9164 & -456.87(0.4002) \\ 0.4002 & 456.87(0.9164) \end{bmatrix} \begin{bmatrix} 0.0004 & 0 \\ 0 & \left(\dfrac{9''}{\rho}\right)^2 \end{bmatrix}$$
$$\times \begin{bmatrix} 0.9164 & 0.4002 \\ -456.87(0.4002) & 456.87(0.9164) \end{bmatrix}$$

from which

$$\Sigma_{Lat,\,Dep} = \begin{bmatrix} 0.00039958 & 0.00000096 \\ 0.00000096 & 0.00039781 \end{bmatrix} \quad (8.4)$$

In Equation (8.4), σ_{11}^2 is the variance of the latitude; σ_{22}^2 is the variance of the departure; and σ_{12} and σ_{21} are their covariances. Thus, the standard errors are: $\sigma_{Lat} = \sqrt{\sigma_{11}^2} = \sqrt{0.00039958} = \pm 0.020$ ft and $\sigma_{Dep} = \sqrt{\sigma_{22}^2} = \sqrt{0.00039781} = \pm 0.020$ ft. Note that the off-diagonal of $\Sigma_{Lat,\,Dep}$ is not equal to zero, and thus, the computed values are correlated, as illustrated in Figure 8.1.

8.3 DERIVATION OF ESTIMATED STANDARD ERRORS IN COURSE AZIMUTHS

Equation (8.1) is based on the azimuth of a course. However, in practice, traverse azimuths are normally computed from observed angles rather than measured directly. Thus, another level of error propagation exists in calculating the azimuths from angular values. In the following analysis, consider that *angles-to-the-right* are observed and that azimuths are computed in a *counterclockwise* direction successively around the traverse using the formula

$$Az_C = Az_P + 180^\circ + \theta_i \tag{8.5}$$

where Az_C is the azimuth for the current course, Az_P is the previous course azimuth, and θ_i is the appropriate interior angle to use in computing the current course azimuth. By applying Equation (6.18), the error in the current azimuth, Az_C, is

$$\sigma_{Az_C} = \sqrt{\sigma^2_{Az_P} + \sigma^2_{\theta_i}} \tag{8.6}$$

In Equation (8.6) σ_{θ_i} is the estimated error in the appropriate interior angle used in computation of the current azimuth, σ_{Az_P} is the estimated error in the azimuth of the previous course, and σ_{Az_C} is the estimated error in the course being computed. This equation is also valid for azimuth computations going clockwise around the traverse. The proof of this is left as an exercise.

8.4 COMPUTING AND ANALYZING POLYGON TRAVERSE MISCLOSURE ERRORS



From elementary surveying, it is known that the following geometric constraints exist for any closed polygon-type traverse:

$$\sum \text{interior angles} = (n-2)180^\circ, \text{and} \tag{8.7}$$

$$\sum Lat = \sum Dep = 0 \tag{8.8}$$

Deviations from these conditions, normally called *misclosures*, can be calculated from the observations in any traverse. Statistical analyses can then be performed to determine the acceptability of the misclosures and check for the presence of blunders in the observations. If blunders appear to be present, the measurements must be rejected, and the observations repeated.

However, while the initial azimuth will correctly orient the polygon traverse, its error does not affect the linear misclosure of the traverse. This can also be stated about the observed angle at the control station, since the azimuth for the first course is either given or assumed. Thus, the errors in these observations can be ignored

when checking the linear misclosure of the traverse. The error in the initial angle should be used in checking the angular misclosure in the traverse. The following example illustrates methods of making these computations for any closed polygon traverse.

Example 8.2 Compute the angular and linear misclosures for the traverse illustrated in Figure 8.2. The observations for the traverse are given in Table 8.1. Determine the estimated misclosure errors at the 95% confidence level and comment on whether the observations may contain any blunders.

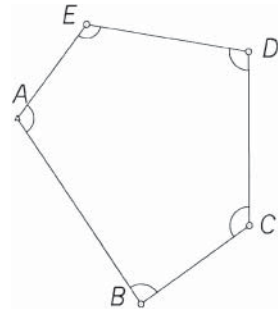


FIGURE 8.2 Link traverse example.

TABLE 8.1 Distance and Angle Observations for Figure 8.2

Occupied	Sighted	Distance (ft)	*S* (ft)	BS	Occ	FS	Angle[a]	*S*(″)
A	*B*	1,435.67	0.020	*E*	*A*	*B*	110°24′40″	±3.5
B	*C*	856.94	0.020	*A*	*B*	*C*	87°36′14″	±3.1
C	*D*	1,125.66	0.020	*B*	*C*	*D*	125°47′27″	±3.6
D	*E*	1,054.54	0.020	*C*	*D*	*E*	99°57′02″	±3.1
E	*A*	756.35	0.020	*D*	*E*	*A*	116°14′56″	±3.9
							Σ = 540°00′19″	

[a]Each angle was measured using four repetitions.

SOLUTION *Angular Check*: First, the angular misclosure is checked to see if it is within the tolerances specified. From Equation (6.18), and using the standard deviations given in Table 8.1, the angular sum should have an error within $\pm\sqrt{\sigma_{\angle 1}^2 + \sigma_{\angle 2}^2 + \cdots + \sigma_{\angle n}^2}$ 68.3% of the time. Since the angles were measured four times, each computed mean has three degrees of freedom, and the appropriate *t value* from Table D.3 (the *t* distribution) is $t_{0.025,3}$, which equals 3.183. This is a two-tailed test since we are looking for the range that is statistically equal to zero at the selected level of confidence. If this range contains the actual misclosure, then

there is no statistical reason to believe that the observations contain a blunder. In this case, the angular misclosure at a 95% confidence level is estimated as

$$\sigma_{\Sigma Angles} = 3.183\sqrt{3.5^2 + 3.1^2 + 3.6^2 + 3.1^2 + 3.9^2} = \pm 24.6'' \qquad (8.9)$$

Using the summation of the angles in Table 8.1, the actual angular misclosure in this problem is

$$540^\circ 00' 19'' - (5-2)180^\circ = 19''$$

Thus, the actual angular misclosure for the traverse ($19''$) is within its estimated range of error, and there is no reason to believe that a blunder exists in the angles.

Azimuth Computations: In this problem, no azimuth is given for the first course. However, to solve the problem, the azimuth of the first course can be assumed as $0^\circ 00' 00''$ and to be free of error. Since only geometric closure on the traverse is being checked, not the orientation of the traverse, this can be done even when the initial course azimuth is observed. For the data of Table 8.1 and using Equations (8.5) and (8.6), the values for the course azimuths and their estimated errors are computed and listed in Table 8.2.

TABLE 8.2 Estimated Errors in the Computed Azimuths of Figure 8.2

From	To	Azimuth	Estimated Error
A	*B*	$0^\circ 00' 00''$	$0''$
B	*C*	$267^\circ 36' 14''$	$\pm 3.1''$
C	*D*	$213^\circ 23' 41''$	$\sqrt{3.1^2 + 3.6^2} = \pm 4.8''$
D	*E*	$133^\circ 20' 43''$	$\sqrt{4.8^2 + 3.1^2} = \pm 5.7''$
E	*A*	$69^\circ 35' 39''$	$\sqrt{5.7^2 + 3.9^2} = \pm 6.9''$

Computation of Estimated Linear Misclosure: Equation (6.13) properly accounts for correlation in the latitude and departure when computing the linear misclosure of a traverse. Applying the partial derivatives of Equation (8.2) to the latitudes and departures, the Jacobian matrix, J, has the form

$$J = \begin{bmatrix} \cos Az_{AB} & -AB\sin Az_{AB} & 0 & 0 & \cdots & 0 & 0 \\ \sin Az_{AB} & AB\cos Az_{AB} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cos Az_{BC} & -BC\sin Az_{BC} & & 0 & 0 \\ 0 & 0 & \sin Az_{BC} & BC\cos Az_{BC} & & 0 & 0 \\ \vdots & \vdots & & & \ddots & & \\ 0 & 0 & 0 & 0 & & \cos Az_{EA} & -EA\sin Az_{EA} \\ 0 & 0 & 0 & 0 & & \sin Az_{EA} & EA\cos Az_{EA} \end{bmatrix} \qquad (8.10)$$

Because the lengths and angles were observed independently, they are uncorrelated. Thus, the appropriate covariance matrix, Σ, for solving this problem using Equation (6.13) is

$$\Sigma = \begin{bmatrix} \sigma_{AB}^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \left(\frac{\sigma_{Az_{AB}}}{\rho}\right)^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_{BC}^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \left(\frac{\sigma_{Az_{BC}}}{\rho}\right)^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_{CD}^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \left(\frac{\sigma_{Az_{CD}}}{\rho}\right)^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{DE}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \left(\frac{\sigma_{Az_{DE}}}{\rho}\right)^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{EA}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \left(\frac{\sigma_{Az_{EA}}}{\rho}\right)^2 \end{bmatrix} \tag{8.11}$$

Again, notice that the estimated errors for the computed azimuths are converted to their equivalent radian values to ensure that the results of the computations will yield estimated errors in linear units. Substituting numerical values for this problem into Equations (8.10) and (8.11), the covariance matrix, $\Sigma_{Lat,\ Dep}$, is computed for the latitudes and departures $J\Sigma J^{\mathrm{T}}$, or

$$\Sigma_{Lat,\ Dep} = \begin{bmatrix} 0.00040 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.00017 & 0.00002 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.00002 & 0.00040 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.00049 & 0.00050 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.00050 & 0.00060 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.00064 & -0.00062 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.00062 & 0.00061 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.00061 & 0.00034 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.00034 & 0.00040 \end{bmatrix} \tag{8.12}$$

By taking the square roots of the diagonal elements in the $\Sigma_{Lat,\ Dep}$ matrix [Equation (8.12)], the errors for the latitude and departure of each course are found. That is, the estimated error in the latitude for course *BC* is the square root of the "3,3" element in Equation (8.12), and the estimated error in the departure of *BC* is the square root of the "4,4" element. In a similar fashion, the estimated errors in latitude and departure can be computed for any other course.

The formula for determining the linear misclosure of a closed polygon traverse is

$$LC = \sqrt{(Lat_{AB} + Lat_{BC} + \cdots + Lat_{EA})^2 + (Dep_{AB} + Dep_{BC} + \cdots + Dep_{EA})^2} \tag{8.13}$$

where LC is the linear misclosure. To determine the estimated error in the linear misclosure, Equation (6.13) is applied to the linear misclosure formula in Equation (8.13). The necessary partial derivatives from Equation (8.13) for substitution into Equation (6.13) must first be determined. The partial derivatives with respect to the latitude and departure of course AB are

$$\frac{\partial LC}{\partial Lat_{AB}} = \frac{\sum Lats}{LC} \qquad \frac{\partial LC}{\partial Dep_{AB}} = \frac{\sum Deps}{LC} \tag{8.14}$$

Notice that these partial derivatives are independent of the course. Also, the other courses have the same partial derivatives as given by Equation (8.14), and thus, the Jacobian matrix for Equation (6.13) has the form

$$J = \left[\frac{\sum Lats}{LC} \quad \frac{\sum Deps}{LC} \quad \frac{\sum Lats}{LC} \quad \frac{\sum Deps}{LC} \quad \cdots \quad \frac{\sum Lats}{LC} \quad \frac{\sum Deps}{LC}\right] \tag{8.15}$$

As shown in Table 8.3, the sum of the latitudes is −0.083, the sum of the departures is 0.022, and $LC = 0.086$ ft. Substituting these values into Equation (8.15), which in turn is substituted into Equation (6.13), yields

$$\sum_{LC} = \begin{bmatrix}-0.9674 & 0.2531 & \cdots & -0.9674 & 0.2531\end{bmatrix} \Sigma_{Lat,\,Dep} \begin{bmatrix}-0.9674 \\ 0.2531 \\ \vdots \\ -0.9674 \\ 0.2531\end{bmatrix}$$

$$= [0.00226] \tag{8.16}$$

TABLE 8.3 Latitudes and Departures for Example 8.2

Course	Latitude	Departure
AB	1,435.670	0.000
BC	−35.827	−856.191
CD	−939.812	−619.567
DE	−723.829	766.894
EA	263.715	708.886
	$\sum = -0.083$	$\sum = 0.022$

$LC = \sqrt{(-0.083)^2 + (0.022)^2} = 0.086$ ft

In Equation (8.16), Σ_{LC} is a single-element covariance matrix; that is, the variance of the linear closure and can be called σ_{LC}^2. Also, $\Sigma_{Lat,Dep}$ is the matrix given by Equation (8.12). To compute the 95% confidence interval, a t value from Table D.3 (the t distribution) must be used with $\alpha = 0.025$ and 3 degrees of freedom.[1] The estimated misclosure for a traverse at a $(1 - \alpha)$ level of confidence is $t_{\alpha/2,3}\sigma_{LC}$. Again, we are checking to see if the traverse misclosure falls within a range of errors that are statistically equal to zero. This requires placing $\alpha/2$ into the upper and lower tails of the distribution. Thus, the estimated error in the traverse closure at a 95% level of confidence is

$$\sigma_{LC} = t_{0.025,3}\sqrt{\sigma_1^2} = 3.183\sqrt{0.00226} = \pm 0.15 \text{ ft}$$

This value is well above the actual traverse linear misclosure of 0.086 ft, and thus, there is no reason to believe that the traverse contains any blunders.

In Example 8.2, we failed to reject the null hypothesis; that is, there was no statistical reason to believe there were errors in the data. However, it is important to remember that this does necessarily indicate that the observations are free from error. There is always the possibility of a Type II error. For example, if the computations were supposed to be performed on a map projection grid, but the observations were not reduced, the traverse would still close within acceptable tolerances. However, the computed results would be incorrect since all the distances would be either too long or too short depending on where the survey existed in the map projection zone. Another example of an undetectable systematic error is an incorrectly entered instrument-reflector constant. Again, all the observed distances would be either too long or too short, but the traverse misclosure would still be within acceptable tolerances.

Surveyors must always be aware of systematic instrumental errors and follow proper field and office procedures to remove these errors. As discussed, simply passing a statistical test does not imply directly that the observations are error or mistake free. However, when the test rejects the null hypothesis, only a Type I error can occur at an α level of confidence. Dependent on the value of α, rejection of the null hypothesis is a strong indicator of problems within the data.

[1] A closed polygon traverse has $2(n - 1)$ unknown coordinates with $2n + 1$ observations, where n is the number of traverse sides. Thus the number of degrees of freedom in a simple closed traverse is always $2n + 1 - 2(n - 1) = 3$. For a five-sided traverse, there are five angle and five distance observations plus one azimuth. Also, there are four stations, each having two unknown coordinates, thus $11 - 8 = 3$ degrees of freedom.

8.5 COMPUTING AND ANALYZING LINK TRAVERSE MISCLOSURE ERRORS

As illustrated in Figure 8.3, a link traverse begins at one station and ends on a different one. Normally, they are used to establish the positions of intermediate stations, as in *A* through *D* of the figure. The coordinates at the endpoints, Stations 1 and 2 in the figure are known. Angular and linear misclosures are also computed for these types of traverses, and the resulting values used as the basis for accepting or rejecting the observations. In a link traverse, the error in the initial azimuth and angle are important in checking both the angular and linear misclosure of the traverse. Furthermore, the error in the closing azimuth is considered when checking the angular misclosure. It does not affect the linear misclosure of the traverse. Example 8.3 illustrates the computational methods.

Example 8.3 Compute the angular and linear misclosures for the traverse illustrated in Figure 8.4. The observed data for the traverse are given in Table 8.4.

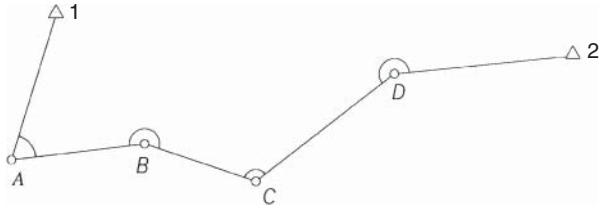


FIGURE 8.3 Closed link traverse.

TABLE 8.4 Data for Link Traverse in Example 8.3

Distance Observations

From	To	Distance (ft)	*S* (ft)
1	*A*	1,069.16	±0.021
A	*B*	933.26	±0.020
B	*C*	819.98	±0.020
C	*D*	1,223.33	±0.021
D	2	1,273.22	±0.021

Control Stations

Station	X (ft)	Y (ft)
1	1,248.00	3,979.00
2	4,873.00	3,677.00

Angle Observations

BS	Occ	FS	Angle	*S* (″)
1	*A*	*B*	66°16′35″	±4.9
A	*B*	*C*	205°16′46″	±5.5
B	*C*	*D*	123°40′19″	±5.1
C	*D*	2	212°00′55″	±4.6

Azimuth Observations

From	To	Azimuth	*S* (″)
1	*A*	197°04′47″	±4.3
D	2	264°19′13″	±4.1

Determine the estimated misclosures at the 95% confidence level, and comment on whether the observations contain any blunders.

SOLUTION *Angular Misclosure*: In a link traverse, angular misclosure is found by computing initial azimuths for each course and then subtracting the final computed azimuth from its given counterpart. The initial azimuths and their estimated errors are computed using Equations (8.5) and (8.6) and are shown in Table 8.5.

TABLE 8.5 Computed Azimuths and Their Uncertainties

Course	Azimuth	σ ($''$)
1*A*	197°04′47″	± 4.3
AB	83°21′22″	± 6.5
BC	108°38′08″	± 8.5
CD	52°18′27″	± 9.9
*D*2	84°19′22″	±11.0

The difference between the azimuth computed for course *D*2 (84°19′22″) and its actual value (264°19′13″ − 180°) is +9″. Using Equation (6.18), the estimated error in the difference is $\sqrt{11.0^2 + 4.1^2} = \pm 11.7''$, and thus there is no reason to assume that the angles contain blunders.

Linear Misclosure. First, the actual traverse misclosure is computed using Equation (8.1). From Table 8.6, the total change in latitude for the traverse is −302.128 ft, and the total change in departure is 3624.968 ft. From the control coordinates, the cumulative change in X and Y coordinate values is

$$\Delta X = X_2 - X_1 = 4873.00 - 1248.00 = 3625.00 \text{ ft}$$
$$\Delta Y = Y_2 - Y_1 = 3677.00 - 3979.00 = -302.00 \text{ ft}$$

The actual misclosures in departure and latitude are computed as

$$\Delta Dep = \sum Dep - (X_2 - X_1) = 3624.968 - 3625.00 = -0.032 \text{ ft}$$
$$\Delta Lat = \sum Lat - (Y_2 - Y_1) = -302.128 - (-302.00) = -0.128 \text{ ft} \qquad (8.17)$$

TABLE 8.6 Computed Latitudes and Departures

Course	Latitude (ft)	Departure (ft)
1A	−1,022.007	−314.014
AB	107.976	926.993
BC	−262.022	776.989
CD	747.973	968.025
D2	125.952	1,266.975
	$\sum = -302.128$	$\sum = 3{,}624.968$

In Equation (8.17), ΔDep represents the misclosure in departure and ΔLat represents the misclosure in latitude. The linear misclosure for the traverse is

$$LC = \sqrt{\Delta Lat^2 + \Delta Dep^2} = \sqrt{(-0.128)^2 + (-0.032)^2} = 0.132 \text{ ft} \tag{8.18}$$

Estimated Misclosure for the Traverse: Following procedures like those described earlier for polygon traverses, the expected misclosure in this link traverse can be estimated. The Jacobian matrix of the partial derivative for the latitude and departure with respect to distance and angle observations are

$$J = \begin{bmatrix} \cos Az_{1A} & -1A\sin Az_A & 0 & 0 & \cdots & 0 & 0 \\ \sin Az_{1A} & 1A\cos Az_{1A} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cos Az_{AB} & -AB\sin Az_{AB} & & 0 & 0 \\ 0 & 0 & \sin Az_{AB} & AB\cos Az_{AB} & & 0 & 0 \\ \vdots & \vdots & & & \ddots & & \\ 0 & 0 & 0 & 0 & & \cos Az_{D2} & -D2\sin Az_{D2} \\ 0 & 0 & 0 & 0 & & \sin Az_{D2} & D2\cos Az_{D2} \end{bmatrix} \tag{8.19}$$

Similarly, the corresponding covariance matrix Σ in Equation (6.16) has the form

$$\Sigma = \begin{bmatrix} \sigma_{1A}^2 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \left(\frac{\sigma_{Az_{1A}}}{\rho}\right)^2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \sigma_{AB}^2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \left(\frac{\sigma_{Az_{AB}}}{\rho}\right)^2 & & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & & \sigma_{D2}^2 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \left(\frac{\sigma_{Az_{D2}}}{\rho}\right)^2 \end{bmatrix} \tag{8.20}$$

Substituting the appropriate numerical values into Equations (8.19) and (8.20) and applying Equation (6.16), the covariance matrix is

$$\Sigma_{Lat,\,Dep} =$$

$$\begin{bmatrix}
0.000446 & -0.000016 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.000016 & 0.000492 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.000864 & -0.000054 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -0.000054 & 0.000406 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.00107 & 0.00023 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.00023 & 0.00048 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0.00234 & -0.00147 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -0.00147 & 0.00157 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.00453 & -0.00041 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.00041 & 0.00048
\end{bmatrix}$$

To estimate the error in the traverse misclosure, Equation (6.16) must be applied to Equation (8.18). As was the case for closed polygon traverse, the terms of the Jacobian matrix are independent of the course for which they are determined, and thus the Jacobian matrix has the form

$$J = \begin{bmatrix} \frac{\Delta Lat}{LC} & \frac{\Delta Dep}{LC} & \frac{\Delta Lat}{LC} & \frac{\Delta Dep}{LC} & \cdots & \frac{\Delta Lat}{LC} & \frac{\Delta Dep}{LC} \end{bmatrix} \tag{8.21}$$

Following procedures like those used in Example 8.2, the estimated error in the misclosure of the link traverse is

$$\sum_{LC} = J \sum_{Lat,Dep} J^{\mathrm{T}} = [0.00806]$$

From these results and using a t value from Table D.3 for three degrees of freedom, the estimated linear misclosure error for a 95% level of confidence is

$$\sigma_{95\%} = 3.183\sqrt{0.00806} = \pm 0.29 \text{ ft}$$

Since the actual misclosure of 0.13 ft is within the range of values that are statistically equal to zero at the 95% level (±0.29 ft), there is no reason to believe that the traverse observations contain any blunders. Again, this test does not remove the possibility of a Type II error occurring.

This example leads to an interesting discussion. When using traditional methods of adjusting link traverse data, such as the compass rule, the control is assumed to be perfect. However, since control coordinates are themselves derived from observations, they contain errors that are not accounted for in these computations. This fact is apparent in Equation (8.21), where the coordinate values are assumed to have no error and thus are not represented. These equations can easily be modified to consider the control errors, but this is left as an exercise for the reader.

One of the principal advantages of the least-squares adjustment method is that it allows application of varying weights to the observations, and control can be included in the adjustment with appropriate weights. A full discussion of this subject is presented in Section 21.7.

8.6 SOFTWARE

The computations in this chapter can be challenging or tedious at best to perform using a calculator. Thus, it is recommended that these computations be programmed in the software. On the companion website for this book, an Excel® spreadsheet (Chapter 8.xls) for Examples 8.2 and 8.3 is available for study. Programming allows for an easier and more reliable method of developing the required matrices. Once the initial matrices are computed, they can be manipulated in software such as MATRIX, which is on the companion website, or in the spreadsheet itself.

A spreadsheet often provides routines to manipulate the matrices. In Excel®, matrices can be multiplied using the matrix multiplication, MMULT(), function. Similarly, matrices can be transposed using the TRANSPOSE() function. These functions are demonstrated in the spreadsheet Chapter 8.xls. To simplify the selection of the matrices, the block of cells for each matrix can be "named." The help file that accompanies Excel® discusses how to name cells or groups of cells and use of the matrix functions.

For those who wish to develop a more robust program, a higher-level programming language or Mathcad can be used. ADJUST demonstrates this in its traverse computations option. The dialog box for this option is shown in Figure 8.4 where a polygon traverse with angles is selected. At the bottom of the box is the option to compute the estimated errors in the traverse. Selecting this option informs the software not to adjust the angles in the traverse, but rather to use the entered distance and angle uncertainties to compute the misclosures in both the angles and the traverse. Figure 8.5 depicts the file used in computing the estimated errors for Example 8.2. Notice that with a polygon traverse, there is no error given for the initial azimuth of 0°00′00″, and that the estimated errors in the distances and

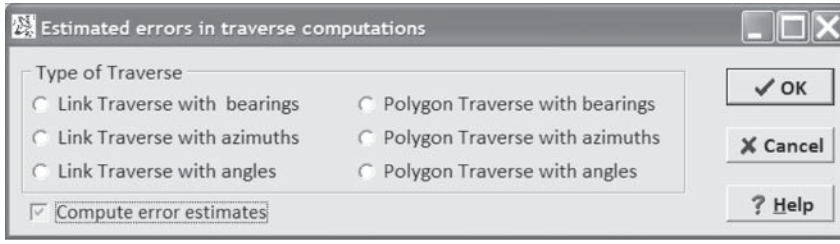


FIGURE 8.4 Traverse computations option dialog.

```
Example 8.2
5 1
0 0 0 0.0
1435.67 110 24 40
856.94 87 36 14
1125.66 125 47 27
1054.54 99 57 02
756.35 116 14 56
10000.00 5000.00
```

FIGURE 8.5 ADJUST data file for Example 8.2.

angles follow the observations for each course of the traverse. The help file that accompanies ADJUST describes the format of this file.

8.7 CONCLUSIONS

This chapter discussed the propagation of observational errors through traverse computations. Error propagation is a powerful tool for the surveyor, enabling an answer to be obtained for the question, "What is an acceptable traverse misclosure?" This is an example of surveying *engineering*. Surveyors are constantly designing measurement systems and checking their results against personal or legal standards. The subjects of error propagation and detection of observational blunders are discussed further in later chapters.

PROBLEMS

Note: Partial answers to problems marked with an asterisk can be found in Appendix G.

8.1 Which of the following are direct (independent) or indirect (dependent) observations:

***(a)** Angles

(b) Latitudes and departures

(c) Coordinates

8.2 Discuss why latitudes and departures are considered correlated observations.

***8.3** Given a course with an azimuth of 28°16′08″ with an estimated error of ±3″ and a distance of 678.24 ft with an estimated error of ±0.019 ft, what are:

(a) the latitude and departure?

(b) the estimated errors in the latitude and departure?

8.4 Same as Problem 8.3, except the azimuth is $272°14'08'' \pm 8.9''$ and the distance is 85.783 ± 0.0034 m.

8.5 Same as Problem 8.3, except the azimuth is $328°49'07'' \pm 4.3''$ and the distance is 365.977 ± 0.0065 m.

8.6 Same as Problem 8.3, except the azimuth is $44°56'22'' \pm 6.7''$ and the distance is 138.04 ± 0.018 ft.

8.7 Same as Problem 8.3, except the azimuth is $64°13'24'' \pm 9.8''$ and the distance is $156.022\text{ m} \pm 3.2$ mm.

***8.8** A polygon traverse has the following angle measurements and related standard deviations. Each angle was observed twice (1D and 1R). Do the angles meet acceptable closure limits at a 95% level of confidence?

Backsight	Occupied	Foresight	Angle	S ($''$)
A	*B*	*C*	$82°21'33''$	± 9.2
B	*C*	*D*	$101°52'27''$	± 11.3
C	*D*	*A*	$91°26'38''$	± 11.1
D	*A*	*B*	$84°19'51''$	± 10.9

8.9 Same as Problem 8.8, except the angles were measured four times (2D and 2R) with the following observations.

Backsight	Occupied	Foresight	Angle	S ($''$)
A	*B*	*C*	$142°00'52''$	± 15.9
B	*C*	*D*	$112°44'09''$	± 18.0
C	*D*	*A*	$48°54'55''$	± 13.5
D	*A*	*B*	$56°19'08''$	± 22.2

***8.10** Given an initial azimuth for course *AB* of $343°06'46''$ with an estimated error of $\pm 4.5''$, what are the azimuths and their estimated standard errors for the remaining three courses of Problem 8.8?

8.11 Using data in Problem 8.9 solve Problem 8.10, with an azimuth for course *AB* of $158°43'33''$ with an estimated error of $\pm 0.9''$.

8.12 Using the distances listed in the following table and the data from Problems 8.8 and 8.10, compute:

(a) the misclosure of the traverse.

***(b)** the estimated misclosure error.

(c) the 95% error in the traverse misclosure.

From	To	Distance (ft)	*S* (ft)
A	*B*	250.57	±0.015
B	*C*	281.62	±0.015
C	*D*	270.50	±0.015
D	*A*	340.88	±0.015

8.13 Given the traverse misclosures in Problem 8.12, does the traverse meet acceptable closure limits at a 95% level of confidence? Justify your answer statistically.

8.14 Using the distances listed in the following table and the data from Problems 8.9 and 8.11, compute:

(a) the misclosure of the traverse.

***(b)** the estimated misclosure error.

(c) the 95% error in the traverse misclosure.

(d) Given the traverse misclosures, does the traverse meet acceptable closure limits at a 95% level of confidence? Justify your answer statistically.

From	To	Distance (ft)	*S* (ft)
A	*B*	163.07	±0.015
B	*C*	178.06	±0.015
C	*D*	327.02	±0.015
D	*A*	99.62	±0.015

8.15 Using the data for the link traverse listed below, compute:

(a) the angular misclosure and its estimated error.

(b) the misclosure of the traverse.

(c) the estimated misclosure error.

(d) the 95% error in the traverse misclosure.

Distance Observations

From	To	Distance (m)	*S* (mm)
W	*X*	402.535	±4.0
X	*Y*	214.769	±4.0
Y	*Z*	270.948	±4.0

Angle Observations

BS	Occ	FS	Angle	*S* (″)
*Mk*1	*W*	*X*	155°53′58″	±3.5
W	*X*	*Y*	67°13′21″	±3.6
X	*Y*	*Z*	165°33′06″	±4.1
Y	*Z*	*Mk*2	217°01′19″	±3.9

Control Azimuths

From	To	Azimuth	S (″)
*Mk*1	*W*	353°09′26″	±2.3″
Z	*Mk*2	58°51′09″	±3.4″

Control Stations

Station	Northing (m)	Easting (m)
W	5000.000	10,000.000
Z	5,079.412	10,434.819

8.16 Does the link traverse of Problem 8.15 have acceptable traverse misclosure at a 95% level of confidence? Justify your answer statistically.

8.17 Same as Problem 8.15 using the following data.

Distance Observations

From	To	Distance (ft)	S (ft)
W	*X*	223.59	±0.012
X	*Y*	854.40	±0.013
Y	*Z*	460.95	±0.012

Angle Observations

BS	Occ	FS	Angle	S (″)
*Mk*1	*W*	*X*	171°52′06″	±5.5
W	*X*	*Y*	137°07′03″	±5.9
X	*Y*	*Z*	290°02′35″	±3.4
Y	*Z*	*Mk*2	77°44′55″	±4.8

Control Azimuths

From	To	Azimuth	S (″)
Mk1	*W*	251°33′54″	±5.0″
Z	*Mk2*	28°21′00″	±3.0″

Control Stations

Station	Northing (ft)	Easting (ft)
W	1000.00	1000.00
Z	1850.00	1600.00

8.18 Does the link traverse of Problem 8.17 have acceptable traverse misclosure at a 95% level of confidence? Justify your answer statistically.

8.19 Repeat Problem 8.15 with data and results from Problems 7.13 and 7.20. Assume the azimuth of *AB* is 24°02′16″ ± 6″ and the *xy* coordinates at station *A* are (10,000.00, 5,000.00) in units of feet.

8.20 A survey produces the following set of data. The angles were obtained from the average of four measurements (two face left and two face right) made with a total station. The estimated uncertainties in the observations are:

$$\sigma_{\text{ISO}} = \pm 2'', \sigma_t = \pm 0.010 \text{ ft}, \sigma_i = \pm 0.005 \text{ ft}$$

The EDM instrument has a specified accuracy of ± (3 mm + 3 ppm).

Distance Observations

From	To	Distance (ft)
1	2	999.99
2	3	801.55
3	4	1,680.03
4	5	1,264.92
5	1	1,878.82

Angle Observations

BS	Occ	FS	Angle
5	1	2	191°40′12″
1	2	3	56°42′22″
2	3	4	122°57′10″
3	4	5	125°02′11″
4	5	1	43°38′10″

Control Azimuths

From	To	Azimuth	σ (″)
1	2	216°52′11″	±3

Control Stations

Station	Easting (ft)	Northing (ft)
1	1,000.00	1,000.00

Compute:

(a) the estimated errors in angles and distances.

(b) the angular misclosure and its 95% probable error.

(c) the misclosure of the traverse.

***(d)** the estimated misclosure error and its 95% value.

(e) Did the traverse meet acceptable closures? Justify your response statistically.

8.21 Develop new matrices for the link traverse of Example 8.3 that considers the errors in the control.

8.22 Show that Equation (8.6) is valid for azimuth computations going clockwise around a traverse.

PROGRAMMING PROBLEMS

8.23 Develop a computational package that will compute the course azimuths and their estimated errors given an initial azimuth and measured angles. Use this package to answer Problem 8.9.

8.24 Develop a computational package will compute estimated traverse misclosure error, given course azimuths, distances, and their estimated errors. Use this package to answer Problem 8.10.

CHAPTER 9

ERROR PROPAGATION IN ELEVATION DETERMINATION

9.1 INTRODUCTION

Differential and trigonometric leveling are the two most commonly employed methods for determining elevation differences between stations. Both methods are subject to systematic and random errors. The primary systematic errors include Earth curvature, atmospheric refraction, and instrument maladjustment. The effects of these systematic errors can be minimized by following proper field procedures, or they can be modeled and corrected computationally. The random errors in differential and trigonometric leveling occur in instrument leveling, distance observations, and reading graduated scales. These must be treated according to the random error theory.

9.2 SYSTEMATIC ERRORS IN DIFFERENTIAL LEVELING



During the differential leveling process, sight distances are held short, and equal, to minimize the effects of systematic errors. Still, it should always be assumed that these errors are present in differential leveling observations, and

thus corrective field procedures, described in this section, should be followed to minimize their effects.

9.2.1 Collimation Error

Collimation error occurs when the line of sight of an instrument is not truly horizontal and is minimized by keeping sight distances short and balanced. Figure 9.1 shows the effects of a collimation error. For an individual setup, the resulting error in the elevation difference due to collimation is

$$e_C = D_1\alpha - D_2\alpha \tag{9.1}$$

where e_C is the error in elevation due to the presence of a collimation error, D_1 and D_2 the distances to the backsight and foresight rods, respectively, and α the amount of collimation error present at the time of the observation expressed in radian units. Applying Equation (9.1), the collimation error for a line of levels can be expressed as:

$$e_C = \alpha[(D_1 - D_2) + (D_3 - D_4) + \cdots + (D_{n-1} - D_n)] \tag{9.2}$$

where $D_1, D_3, \ldots, D_{n-1}$ are the backsight distances and $D_2, D_4, \ldots, D_n$ the foresight distances. If the backsight and foresight distances are grouped in Equation (9.2), then

$$e_C = \alpha\left(\sum D_{BS} - \sum D_{FS}\right) \tag{9.3}$$

The collimation error determined from Equation (9.3) is treated as a correction and is thus subtracted from the observed elevation difference to obtain the corrected value.

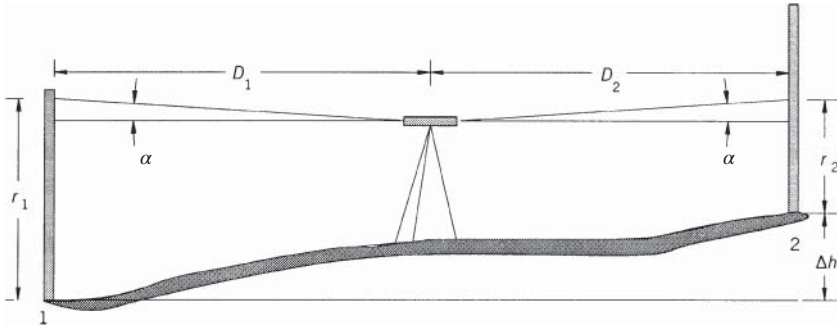


FIGURE 9.1 Collimation error in differential leveling.

Example 9.1 A level having a collimation error of 0.04 mm/m is used on a level line where the backsight distances sum to 863 m and the foresight distances sum to 932 m. If the observed elevation difference for the line is 22.865 m, what is the corrected elevation difference?

SOLUTION Using Equation (9.3), the error due to collimation is

$$e_C = 0.00004(863 - 932)\,\text{m} = -0.00276\,\text{m}$$

Thus, the corrected elevation difference is

$$22.865 - (-0.00276) = 22.868\,\text{m}$$

9.2.2 Earth Curvature and Refraction

As the line of sight extends from an instrument, the level surface curves down and away. This condition always causes rod readings to be too high. Similarly, as the line of sight extends from the instrument, refraction bends it toward the surface, causing readings to be too low. The combined effect of Earth curvature and refraction on an individual's sight always causes a rod reading to be too high by an amount approximated as:

$$h_{CR} = CR\left(\frac{D}{1000}\right)^2 \tag{9.4}$$

where h_{CR} is the error in the rod reading (in feet or meters), CR is 0.0675 when the units of D are meters or 0.0206 when the units of D are feet, and D is the individual sight distance.

The effect of this error on a single elevation difference is minimized by keeping backsight and foresight distances short and equal. For unequal sight distances, the resulting error is expressed as:

$$e_{CR} = CR\left(\frac{D_1}{1000}\right)^2 - CR\left(\frac{D_2}{1000}\right)^2 \tag{9.5}$$

where e_{CR} is the error due to Earth curvature and refraction on a single elevation difference. Factoring common terms in Equation (9.5) yields

$$e_{CR} = \frac{CR}{1000^2}\left(D_1^2 - D_2^2\right) \tag{9.6}$$

The curvature and refraction error computed from Equation (9.6) is treated as a correction and is thus subtracted from the observed elevation difference to get the

corrected value. It is important to note that D_1 and D_2 in Equation (9.6) are the individual backsight and foresight distances, respectively, that occur in a line of levels. Since these terms are squared, each individual sight distance must be treated separately. That is, substituting the sum of the backsight and foresight distances into Equation (9.6) will yield incorrect results. These distances must be squared before they are summed.

Example 9.2 An elevation difference between two stations on a hillside is determined to be 1.256 m. What would be the error in the elevation difference and the corrected elevation difference if the backsight distance were 100 m and the foresight distance 20 m only?

SOLUTION Substituting the distances into Equation (9.6) and using $CR = 0.0675$

$$e_{CR} = \frac{0.0675}{1000^2}(100^2 - 20^2) = 0.0006 \text{ m}$$

From this, the corrected elevation difference is

$$\Delta h = 1.256 - 0.0006 = 1.255 \text{ m}$$

For a line of differential leveling, the combined effect of this error is

$$e_{CR} = \frac{CR}{1000^2}(D_1^2 - D_2^2 + D_3^2 - D_4^2 + \cdots) \tag{9.7}$$

Regrouping backsight and foresight distances, Equation (9.7) becomes

$$e_{CR} = \frac{CR}{1000^2}\left(\sum D_{BS}^2 - \sum D_{FS}^2\right) \tag{9.8}$$

The error due to refraction caused by the vertical gradient of temperature can be large when sight lines are allowed to pass through the lower layers of the atmosphere. Since observing the temperature gradient along the sight line would be cost prohibitive, a field procedure is generally adopted that requires all sight lines to be at least 0.5 m above the ground. This requirement eliminates the lower layers of the atmosphere where refraction is difficult to model.

9.2.3 Combined Effects of Systematic Errors on Elevation Differences

With reference to Figure 9.1, and by combining Equations (9.1) and (9.5), a corrected elevation difference, Δh, for one instrument setup is

$$\Delta h = (r_1 - r_2) - (D_1\alpha - D_2\alpha) - \frac{CR}{1000^2}(D_1^2 - D_2^2) \tag{9.9a}$$

where r_1 is the backsight rod reading, r_2 is the foresight rod reading, and the other terms are as defined previously. For a line of levels, the corrected elevation difference is

$$\Delta h = \sum r_{BS} - \sum r_{FS} - \alpha \left(\sum D_{BS} - \sum D_{FS} \right) - \frac{CR}{1000^2} \left(\sum D_{BS}^2 - \sum D_{FS}^2 \right) \tag{9.9b}$$

9.3 RANDOM ERRORS IN DIFFERENTIAL LEVELING

Differential leveling is subject to several sources of random errors. Included are errors in leveling the instrument and in reading the rod. The sizes of these errors are affected by atmospheric conditions, the quality of the optics of the telescope, the sensitivity of the level bubble or compensator, and the graduation scale on the rods. These errors are discussed below.

9.3.1 Reading Errors

The estimated error in rod readings is usually expressed as a ratio of the estimated standard error in the rod reading per unit sight distance length. For example, if an observer's ability to read a rod is within ±0.005 ft per 100 ft, then $\sigma_{r/D}$ is ±0.005/100 = ±0.00005 ft/ft. Using this, rod reading errors for any individual sight distance D can be estimated as:

$$\sigma_r = D\sigma_{r/D} \tag{9.10}$$

where $\sigma_{r/D}$ is the estimated error in the rod reading per unit length of sight distance and D is the length of the sight distance.

9.3.2 Instrument Leveling Errors

The estimated error in leveling for an automatic compensator or level vial is generally given in the technical data for each instrument. For precise levels, this information is usually listed in arc seconds or as an estimated elevation error for a given distance. As an example, the estimated error may be listed as ±1.5 mm/km, which corresponds to $\pm(1.5/1{,}000{,}000)\rho = \pm 0.3''$. A precise level has a compensator accuracy or setting accuracy between $\pm 0.1''$ and $\pm 0.2''$ usually, while for a less precise level, the value may be as high as $\pm 10''$.

9.3.3 Rod Plumbing Error

While a level rod that is held nonvertical always causes the reading to be too high, this error will appear random in a leveling network due to its presence in

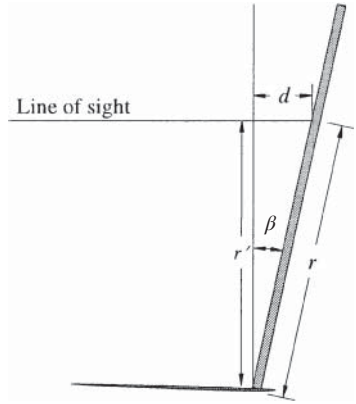


FIGURE 9.2 Nonvertical level rod.

all backsight and foresight distances of the network. Thus, the rod plumbing error should be modeled when computing the standard error in an elevation difference. With reference to Figure 9.2 for any rod reading, the rod plumbing error is approximated as:

$$e_{LS} = r - r' = \frac{d^2}{2r} \tag{9.11}$$

where d is the linear amount that the rod is out of plumb at the location of the rod reading, r. The size of d is dependent on the rod level bubble centering error and the reading location. If the rod bubble is out of level by β, then d is

$$d = r \sin \beta \tag{9.12}$$

Substituting Equation (9.12) into (9.11) yields

$$e_{LS} = \frac{r}{2} \sin^2 \beta \tag{9.13}$$

Example 9.3 Assume that a rod level bubble is within $\pm 5'$ of level and the rod reading is at 4 m. What is the estimated error in the rod reading?

SOLUTION

$$e_{LS} = \frac{4}{2} \sin^2 5' = 0.004 \text{ mm}$$

Since the rod plumbing error occurs on every sighting, backsight errors will tend to cancel foresight errors. With precise leveling techniques, the combined effect of

this error can be written as:

$$e = \left(\frac{r_1 \sin^2\beta}{2} - \frac{r_2 \sin^2\beta}{2}\right) + \left(\frac{r_3 \sin^2\beta}{2} - \frac{r_4 \sin^2\beta}{2}\right) + \cdots \tag{9.14}$$

Grouping like terms in Equation (9.14) yields

$$e = \frac{1}{2}\sin^2\beta(r_1 - r_2 + r_3 - r_4 + \cdots) \tag{9.15}$$

Recognizing that the quantity in parentheses in Equation (9.15) is the elevation difference for the leveling line yields

$$e_{LS} = \frac{\Delta\text{Elev}}{2} sin^2\beta \tag{9.16}$$

Example 9.4 If a level rod is maintained to within $\pm 5'$ of level, and the elevation difference is 22.865 m, the estimated error in the final elevation is

$$e_{LS}\frac{22.865}{2}\sin^2 5' = 0.02 \text{ mm}$$

The rod plumbing error can be practically eliminated by carefully centering the bubble of a well-adjusted rod level. It is generally small, as the example illustrates, and thus will be ignored in subsequent computations.

9.3.4 Estimated Errors in Differential Leveling

From the preceding discussion, the major random error sources in differential leveling are caused by random errors in rod readings and instrument leveling. Furthermore, in Equation (9.9*a*), the collimation error is systematic and is effectively negated by balancing the backsight and foresight distances. However, no matter what method is used to observe the lengths of the sight distances, some random error in these lengths will be present. This causes random errors in the elevation differences due to the effects of Earth curvature, refraction, and instrumental collimation errors. Equation (6.16) can be applied to Equation (9.9*a*) to model the effects of the random errors in rod readings, leveling, and sighting lengths. The following partial derivatives are needed:

$$\frac{\partial \Delta h}{\partial r_1} = \frac{\partial \Delta h}{\partial r_2} = 1 \quad \frac{\partial \Delta h}{\partial \alpha_1} = -D_1 \quad \frac{\partial \Delta h}{\partial \alpha_2} = -D_2$$
$$\frac{\partial \Delta h}{\partial D_1} = -\left(\alpha + \frac{CR(D_1)}{500,000}\right) \qquad \frac{\partial \Delta h}{\partial D_2} = -\left(\alpha + \frac{CR(D_2)}{500,000}\right) \tag{9.17}$$

By substituting Equations (9.17) into (6.16) with their corresponding estimated standard errors and dropping the negative signs, the standard error in a single elevation difference can be estimated as:

$$\sigma_{\Delta h} = \sqrt{\begin{array}{l}(D_1\sigma_{r/D})^2 + (D_2\sigma_{r/D})^2 + (D_1\sigma_{\alpha 1})^2 + (D_2\sigma_{\alpha 2})^2 \\ +\left[-\left(\alpha + \dfrac{CR(D_1)}{500,000}\right)\sigma_{D1}\right]^2 + \left[-\left(\alpha + \dfrac{CR(D_2)}{500,000}\right)\sigma_{D2}\right]^2\end{array}} \tag{9.18}$$

where $\sigma_{r/D}$ is the estimated error in a rod reading; $\sigma_{\alpha 1}$ and $\sigma_{\alpha 2}$ are the estimated collimation errors in the backsight and foresight, respectively; and σ_{D1} and σ_{D2} are the estimated errors in the sight lengths, D_1, and D_2, respectively.

In normal differential leveling procedures, $D_1 = D_2 = D$. Also, the estimated standard errors in the sight distances are equal, $\sigma_{D1} = \sigma_{D2} = \sigma_D$. Furthermore, the estimated collimation error for the backsight and foresight can be assumed equal, $\sigma_{\alpha 1} = \sigma_{\alpha 2} = \sigma_\alpha$. Thus, Equation (9.18) simplifies to

$$\sigma_{\Delta h} = \sqrt{2D^2\left(\sigma_{r/D}^2 + \sigma_\alpha^2\right) + 2\sigma_D^2\left(\alpha + \frac{CR(D)}{500.000}\right)^2} \tag{9.19}$$

Equation (9.19) is appropriate for a single elevation difference when the sight distances are approximately equal. In general, if sight distances are kept equal for N instrument setups, the total estimated error in an elevation difference is

$$\sigma_{\Delta h} = \sqrt{2ND^2\left(\sigma_{r/D}^2 + \sigma_\alpha^2\right) + 2N\sigma_D^2\left(\alpha + \frac{CR(D)}{500,000}\right)^2} \tag{9.20}$$

Since the estimated error in the elevation difference due to Earth curvature and refraction, and collimation error, α, are small, the last term can be ignored typically, which will be demonstrated in Example 9.5. Thus, the final equation for the estimated standard error in differential leveling is

$$\sigma_{\Delta h} = D\sqrt{2N\left(\sigma_{r/D}^2 + \sigma_\alpha^2\right)} \tag{9.21}$$

Example 9.5 A level line runs from benchmark A to benchmark B. The estimated standard error in rod readings is ± 0.01 mm/m. The instrument is within $\pm 2.0''$ of level. A collimation test shows that the instrument is within 4 mm per 100 m. Fifty-meter sight distances are within ± 2 m. The total line length from A to B is 1000 m. What is the estimated error in the elevation difference between A and B? If A had an elevation of 212.345 ± 0.005 m, what is the estimated error in the computed elevation of B?

SOLUTION The total number of setups in this problem is 1000/[(2)(50)] = 10 setups. Substituting the appropriate values into Equation (8.20) yields

$$\sigma_{\Delta h} = \sqrt{2(10)50^2\left[\left(\frac{0.01}{1000}\right)^2 + \left(\frac{2.0''}{\rho}\right)^2\right] + 2(10)2^2\left(\frac{0.004}{100} + \frac{0.0675(50)}{500,000}\right)^2}$$

$$= \sqrt{0.0031^2 + 0.0004^2} = 0.0031 \text{ m} = \pm 3.1 \text{ mm}$$

From an analysis of the individual error components in the previous equation, it is seen that the error caused by the error in the sight distances is negligible for all but the most precise leveling. Thus, like the error due to rod bubble centering, this error can be ignored in all but the most precise work. Thus, the simpler Equation (9.21) is used to solve the problem:

$$\sigma_{\Delta h} = 50\sqrt{2(10)\left[\left(\frac{0.01}{1000}\right)^2 + \left(\frac{2.0''}{\rho}\right)^2\right]}$$

$$= \sqrt{0.0031^2} = \pm 0.0031 \text{ m} = \pm 3.1 \text{ mm}$$

The estimated error in the elevation of B is found by applying Equation (6.18) as

$$\sigma_{Elev_B} = \sqrt{\sigma^2_{Elev_A} + \sigma^2_{\Delta Elev}} = \sqrt{5^2 + 3.1^2} = \pm 5.9 \text{ mm}$$

9.4 ERROR PROPAGATION IN TRIGONOMETRIC LEVELING



With the introduction of total station instruments, it is becoming increasingly convenient to observe elevation differences using trigonometric methods. However, in this procedure, because sight distances are not balanced typically, it is important that the systematic effects of Earth curvature and refraction, and inclination in the instrument's line of sight (collimation error) be removed. From Figure 9.3, the corrected elevation difference, Δh, between two points is

$$\Delta h = hi + S\sin\alpha + h_{CR} - hr \tag{9.22}$$

Equation (9.22) for zenith angle reading instruments is

$$\Delta h = hi + S\cos z + h_{CR} - hr \tag{9.23}$$

where hi is the instrument height above ground; S is the slope distance between the two points; α is the altitude angle between the instrument and the prism; z is the

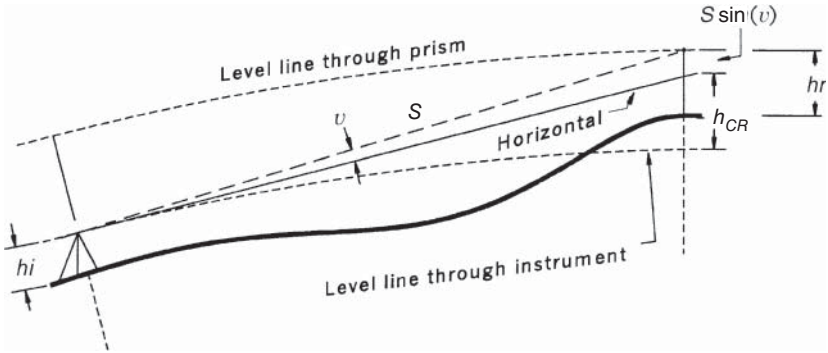


FIGURE 9.3 Determination of elevation difference by trigonometric leveling.

zenith angle between the instrument and the prism; h_{CR} is the Earth curvature and refraction correction given in Equation (9.4); and *hr* is the rod reading. Substituting the curvature and refraction formula into Equation (9.23) yields

$$\Delta h = hi + S\cos z + CR\left(\frac{S\sin z}{1000}\right)^2 - hr \tag{9.24}$$

In developing an error propagation formula for Equation (9.24), not only must errors relating to the height of the instrument and prism be considered, but errors in leveling, pointing, reading, and slope distances, as discussed in Chapter 7, must also be included. Applying Equation (6.16) to (9.24), the following partial derivatives apply:

$$\frac{\partial \Delta h}{\partial hi} = 1$$

$$\frac{\partial \Delta h}{\partial hr} = -1$$

$$\frac{\partial \Delta h}{\partial S} = \cos z + \frac{CR(S)\sin^2 z}{500,000}$$

$$\frac{\partial \Delta h}{\partial z} = \frac{CR(S^2)\sin z\cos z}{500,000} - S\sin z$$

Entering the partial derivatives and the standard errors of the observations into Equation (6.16), the total error in trigonometric leveling is

$$\sigma_{\Delta h} = \sqrt{\begin{array}{l}\sigma_{hi}^2 + \sigma_{hr}^2 + \left[\left(\cos z + \dfrac{CR(S)\sin^2 z}{500,000}\right)\sigma_S\right]^2 \\ \quad + \left[\left(\dfrac{CR(S^2)\sin z\cos z}{500,000} - S\sin z\right)\dfrac{\sigma_z}{\rho}\right]^2\end{array}} \tag{9.25}$$

where z is the zenith angle; CR is 0.0675 if units of meters are used or 0.0206 if units of feet are used; S is the slope distance; and ρ is the seconds to radian conversion of 206,264.8″/rad.

In Equation (9.25), errors from several sources make up the estimated error in the zenith angle. These include the operator's ability to point and read the instrument, accuracy of the vertical compensator, or operator's ability to center the vertical circle bubble, and sensitivity of the compensator or vertical circle bubble. For best results, zenith angles should be observed using both faces of the instrument and an average taken. Using Equations (7.4) and (7.6), the estimated error in a zenith angle that is observed in both positions (face left and face right) with a total station is

$$\sigma_z = \sqrt{\frac{2\sigma_r^2 + 2\sigma_p^2 + 2\sigma_B^2}{N}} \tag{9.26a}$$

where σ_r is the error in reading the circle; σ_p is the error in pointing; σ_B is the error in the vertical compensator or in leveling the vertical circle bubble; and n is the number of face-left and face-right observations of the zenith angle. For total stations, the appropriate formula is

$$\sigma_z = \sqrt{\frac{2\sigma_{ISO}^2 + 2\sigma_B^2}{n}} \tag{9.26b}$$

where σ_{ISO} is the ISO 17123-3 standard for the instrument and all other values are as above.

Notice that if only a single zenith angle observation is made (i.e., it is observed only in the face left), its estimated error is simply

$$\sigma_z = \sqrt{\sigma_r^2 + \sigma_p^2 + \sigma_B^2} \tag{9.27a}$$

For the total station, the estimated error for a single observation is

$$\sigma_z = \sqrt{2\sigma_{ISO}^2 + \sigma_B^2} \tag{9.27b}$$

Recall, the estimated error in the slope distance, S, is computed using Equation (7.38).

Example 9.6 A total station instrument has a vertical compensator accurate to within ±0.3″, a digital reading accuracy of ±5″, and a distance accuracy of ±(5 mm + 5 ppm). The observed slope distance is 1256.78 ft. It is estimated that the instrument is set to within ±0.005 ft of the station, and the target is set to within ±0.01 ft. The height of the instrument is 5.12 ± 0.01 ft, and the prism height is 6.72 ± 0.01 ft. The zenith angle is observed in only one position and

recorded as $88°13'15''$. What is the corrected elevation difference and its estimated error?

SOLUTION Using Equation (9.24), the corrected elevation difference is

$$\Delta h = 5.12 + 1256.78\cos(88°13'15'') + 0.0206\left(\frac{1256.78\sin(88°13'15'')}{1000}\right)^2 - 6.72$$
$$= 37.45 \text{ ft}$$

With Equation (9.27b), the zenith angle error is estimated as $\sigma_z = \sqrt{2(5)^2 + 0.3^2} = \pm 7.1''$. From Equation (7.38) and converting 5 mm to 0.0164 ft, the estimated error in the distance is

$$\sigma_S = \sqrt{0.005^2 + 0.01^2 + 0.0164^2 + \left(\frac{5}{1,000,000}1256.78\right)^2} = \pm 0.021 \text{ ft}$$

Substituting these values into Equation (9.25), the estimated error in the elevation difference is

$$\sigma_{\Delta h} = \sqrt{0.01^2 + 0.01^2 + (0.031 \times 0.021)^2 + \left(-1256.172\frac{7.1''}{\rho}\right)^2}$$
$$= \sqrt{0.01^2 + 0.01^2 + 0.00065^2 + 0.043^2} = \pm 0.045 \text{ ft}$$

Note in this example that the estimated error in the elevation difference caused by the distance error is negligible (± 0.00065 ft), whereas the error in the zenith angle is the largest (± 0.043 ft). Furthermore, since the vertical angle is not observed with both faces of the instrument, it is possible that uncompensated systematic errors are present in the final computed value. For example, assume that a $10''$ indexing error existed on the vertical circle. If the observations are taken using both faces of the instrument, the effects of this error are removed. However, by making only a face-left observation, the systematic error due to the vertical indexing error is

$$1256.78\sin(10'') = 0.061 \text{ ft}$$

The uncompensated systematic error in the final value is considerably larger than the estimated error for the observation. One should always account for systematic errors by using proper field procedures. Failure to do so can only lead to poor results. In trigonometric leveling, a minimum of a face-left and face-right reading should always be taken. However, a prudent surveyor will observe the elevation difference several times to provide redundant observations as a check.

PROBLEMS

Note: Partial answers to problems marked with an asterisk can be found in Appendix G.

***9.1** A level with a collimation error of 5.5 mm/1 km is used in leveling a line that has a sum of 525 ft for the backsight distances and only 1532 ft for the foresight distances. If the observed elevation difference is 8.07 ft, what is the elevation difference corrected for the collimation error?

9.2 Similar to Problem 9.1, except that the collimation error is 0.02 mm/km, the sum of the backsight distances is 423 m, the sum of the foresight distances is 1148 m, and the observed elevation difference is −10.486 m.

***9.3** To expedite going down a hill, the backsight distances were consistently held to 80 ft while the foresight distances were held to 155 ft. There were 46 setups. If the observed elevation difference was −157.89 ft, what is the elevation difference corrected for Earth curvature and refraction?

9.4 Similar to Problem 9.3, except that the backsight distances are 15 m, the foresight distances are 40 m, there were 55 setups, and the elevation difference is −84.387 m.

***9.5** How far from vertical must the top of a 16-ft level rod have to be to create an error of 0.005 ft with a reading at 13.00 ft?

9.6 Similar to Problem 9.5, except for a reading at 16.00 ft on a 25-ft rod.

9.7 Similar to Problem 9.5, except for a 1.0-mm error with a reading at 2.890 m on a 3-m rod.

***9.8** A line of three-wire differential levels goes from benchmark Rock to benchmark Train. The length of the line was determined to be 2400 m. The instrument had a stated compensator accuracy of $\pm 0.3''$. The instrument-rod combination had an estimated reading error of ± 0.3 mm at 30 m. The instrument is found to have an inclined collimation error of 0.06 mm per 30 m, so the sight distances were kept to approximately 30 ± 3 m. If the observed difference in elevation is −58.679 m, what is the estimated error in the final elevation difference?

9.9 Same as Problem 9.8, except the adjusted instrument's collimation error is zero.

9.10 If in Problem 9.8, benchmark Rock had an elevation of 261.978 m and Train had an elevation of 203.304 m, did the job meet acceptable closure limits at a 95% level of confidence? Justify your answer statistically.

9.11 If in Problem 9.9, benchmark Rock had an elevation of 163.807 m and Train had an elevation of 105.135 m, did the job meet acceptable closure limits at a 95% level of confidence? Justify your answer statistically.

9.12 Similar to Problem 9.9, except that the compensator accuracy is ±1.0 mm/km and the estimated error in reading the rod is 0.2 mm per 30 m.

***9.13** An elevation must be established on a benchmark on an island that is 834.75 ft from the nearest benchmark on the shore of the lake. The surveyor decides to use a total station that has a stated distance measuring accuracy of ±(3 mm + 3 ppm), and a vertical compensator accurate to within ±0.4″. The height of the instrument was 5.26 ft with an estimated error of ±0.02 ft. The prism height was 6.00 ft with an estimated error of ±0.02 ft. The single zenith angle is read as 96°07′34″. The estimated errors in instrument and target centering are ±0.005 ft. If the elevation of the occupied benchmark is 85.57 ft, what is the correct benchmark elevation on the island? (Assume that the instrument does not correct for Earth curvature and refraction.)

9.14 In Problem 9.13, what is the estimated error in the computed benchmark elevation if the instrument has an ISO standard of ±2″?

9.15 After completing the job in Problem 9.13, the surveyor discovered that the instrument had a vertical indexing error that caused the sight line to be inclined by 10″.

(a) How much error would be created in the elevation of the island benchmark if the indexing error were ignored?

(b) What is the corrected elevation?

9.16 A construction level is used to run a set of levels to a construction project from the benchmark DAM that had an elevation of 153.608 m. The line is run along a road that goes up a steep incline. To expedite the job, backsight distances are kept to 40 m ± 3 m while foresight distances are 20 m ± 3 m. The total length of the differential levels is 2460 m. The observed elevation difference is 52.609 m. What is the corrected project elevation if the instrument has a sight line that inclines at the rate of 1.5 mm per 40 m?

9.17 In Problem 9.16, the instrument's compensator is state to be within ±0.5″ for each sight, and the rod is read to ±3.0 mm per 46 m.

(a) What is the estimated error in the elevation difference?

(b) What is the 95% error in the final established benchmark?

9.18 An elevation must be established on a benchmark on a hill from the nearest benchmark in the valley. The surveyor decides to use a total station that has a stated distance measuring accuracy of ±(1 mm + 1 ppm), and a vertical compensator accurate to within ±0.3″. The slope distance to the benchmark is 2594.034 m. The height of the instrument is 1.68 m with an estimated error of ±5 mm. The prism height was 2.000 m with an estimated error of ±1 mm. The single zenith angle is read as 85°26′06″. The estimated errors

in instrument and target centering are ±2 mm. If the elevation of the occupied benchmark is 106.579 m, what is the correct benchmark elevation on the hill? (Assume that the instrument does not correct for Earth curvature and refraction.)

9.19 In Problem 9.18, what is the estimated error in the computed benchmark elevation if the instrument has an ISO standard of ±1″?

9.20 After completing the job in Problem 9.18, the surveyor discovered that the instrument had a vertical indexing error that caused the sight line to be inclined by 5″. What is the estimated error in the computed benchmark elevation?

9.21 Which method of leveling presented in this chapter offers the most precision? Defend your answer statistically.

PROGRAMMING PROBLEMS

9.22 Create a computation package that will compute a corrected elevation difference and its estimated error using the method of differential leveling. Use this package to solve Problem 9.16.

9.23 Create a computation package that will compute a corrected elevation difference and its estimated error using the method of trigonometric leveling. Use this package to solve Problem 9.14.

CHAPTER 10

WEIGHTS OF OBSERVATIONS

10.1 INTRODUCTION



When surveying data are collected, they must conform to a given set of geometric conditions, and when they do not, the observations are adjusted to force that geometric closure. For a set of uncorrelated observations, a measurement with high precision, as indicated by a small variance, implies a good observation, and in the adjustment, it should receive a relatively small portion of the overall correction. Conversely, a measurement with a lower precision, as indicated by a larger variance, implies an observation with a larger error, and thus it should receive a larger portion of the correction.

The weight of an observation is a measure of an observation's relative worth compared to other observations. *Weights* are used to control the size of corrections applied to observations in an adjustment. The more precise an observation is, the higher its weight should be; in other words, the smaller the variance, the higher the weight. From this analysis, it can be stated intuitively that *weights are inversely proportional to variances*. Thus, it also follows that *correction sizes should be inversely proportional to weights*.

In situations where observations are correlated, weights are related to the inverse of the covariance matrix, Σ. As discussed in Chapter 6, the elements of this matrix are variances and covariances. Since weights are relative, variances and covariances are often replaced by cofactors. A *cofactor* is related to its covariance by the equation

$$q_{ij} = \frac{\sigma_{ij}}{\sigma_0^2} \tag{10.1}$$

where q_{ij} is the cofactor of the ijth observation; σ_{ij} is the covariance of the ijth observation; and the *reference variance*, a value that can be used for scaling. Equation (10.1) can be expressed in matrix notation as:

$$Q = \frac{1}{\sigma_0^2}\Sigma \tag{10.2}$$

where Q is defined as the *cofactor matrix*. The structure and individual elements of the Σ matrix are

$$\Sigma = \begin{bmatrix} \sigma_{x_1}^2 & \sigma_{x_1x_2} & \cdots & \sigma_{x_1x_n} \\ \sigma_{x_2x_1} & \sigma_{x_2}^2 & & \sigma_{x_2x_n} \\ \vdots & \vdots & \ddots & \\ \sigma_{x_nx_1} & \sigma_{x_nx_2} & & \sigma_{x_n}^2 \end{bmatrix}$$

From the discussion above, the weight matrix W is

$$W = Q^{-1} = {\sigma_0}^2\,\Sigma^{-1} \tag{10.3}$$

For uncorrelated observations, the covariances are equal to zero (i.e., all $\sigma_{x_ix_j} = 0$) and the matrix Σ is diagonal. Thus, Q is also a diagonal matrix with elements equal to $\sigma_{x_i}^2/\sigma_0^2$. The inverse of a diagonal matrix is also a diagonal matrix, with its elements being the reciprocals of the original diagonals, and thus Equation (10.3) becomes

$$W = \begin{bmatrix} \dfrac{\sigma_0^2}{\sigma_{x_1}^2} & 0 & \cdots & 0 \\ 0 & \dfrac{\sigma_0^2}{\sigma_{x_2}^2} & & 0 \\ 0 & & \ddots & \\ 0 & 0 & & \dfrac{\sigma_0^2}{\sigma_{x_n}^2} \end{bmatrix} = \sigma_0^2\Sigma^{-1} \tag{10.4}$$

From Equation (10.4), any independent observation with variance equal to σ_i^2 has a weight of

$$w_i = \frac{\sigma_0^2}{\sigma_i^2} \tag{10.5}$$

If the ith observation has a weight, $w_i = 1$, then $\sigma_0^2 = \sigma_i^2$ or $\sigma_0^2 = 1$. Thus, σ_0^2 is often called the *variance of an observation of unit weight* or shortened to the *variance of unit weight* or simply *unit variance*. Its square root is called the *standard deviation of unit weight*. If σ_0^2 is set equal to 1 in Equation (10.5), then

$$w_i = \frac{1}{\sigma_i^2} \tag{10.6}$$

Note, as earlier stated, in Equation (10.6) the weight of an observation is inversely proportional to its variance.

With correlated observations, it is possible to have a covariance matrix, Σ, and a cofactor matrix, Q, but not a weight matrix. This occurs when the cofactor matrix is singular, and thus an inverse for Q does not exist. Most situations in surveying involve uncorrelated observations. For the remainder of this chapter, only the uncorrelated case with a variance of unit weight will be considered.

10.2 WEIGHTED MEAN

If two observations are taken of a quantity and the first is twice as good as the second, their relative worth can be expressed by giving the first observation a weight of 2 and the second a weight of 1. A simple adjustment involving these two observations would be to compute the mean value. In this calculation, the observation having the weight of 2 could be added twice, and the observation of weight 1 could be added once. As an illustration, suppose a distance is measured with a tape to be 151.9 ft, and the same distance is measured with an EDM instrument as 152.5 ft. Assume that previous experience indicates that the electronically measured distance is twice as good as the taped distance, and accordingly, the taped distance is given a weight of 1, and the electronically measured distance is given a weight of 2. Then, one method of computing the mean from these observations is

$$\overline{M} = \frac{151.9 + 152.5 + 152.5}{3} = 152.3$$

As an alternative, the calculation can be rewritten as

$$\overline{M} = \frac{1(151.9) + 2(152.5)}{1 + 2} = 152.3$$

Note that the weights of 1 and 2 were entered directly into the second computation and that the result of this calculation is the same as the first. Also, note that the computed mean tends to be closer to the measured value having the higher weight (i.e., 152.3 is closer to 152.5 than to 151.9). A mean value computed from weighted observations is called the *weighted mean*.

To develop a general expression for computing the weighted mean, suppose that we have m independent, uncorrelated observations ($z_1, z_2, \ldots, z_m$) for a quantity z, and each of these observations has standard deviation σ. Then the mean of the observations is

$$\bar{z} = \frac{\sum_{i=1}^{m} z_i}{m} \tag{10.7}$$

If the m observations were now separated into two sets, one of size m_a and the other m_b, such that $m_a + m_b = m$, the means for these two sets would be:

$$\bar{z}_a = \frac{\sum_{i=1}^{m_a} z_i}{m_a} \tag{10.8}$$

$$\bar{z}_b = \frac{\sum_{i=m_a+1}^{m} z_i}{m_b} \tag{10.9}$$

The mean $\bar{z}$ is found by combining the means of these two sets as:

$$\bar{z} = \frac{\sum_{i=1}^{m_a} z_i + \sum_{i=m_a+1}^{m} z_i}{m} = \frac{\sum_{i=1}^{m_a} z_i + \sum_{i=m_a+1}^{m} z_i}{m_a + m_b} \tag{10.10}$$

But from Equations (10.8) and (10.9),

$$\bar{z}_a m_a = \sum_{i=1}^{m_a} z_i \quad \text{and} \quad \bar{z}_b m_b = \sum_{i=m_a+1}^{m} z_i \tag{10.11}$$

Thus,

$$\bar{z} = \frac{\bar{z}_a m_a + \bar{z}_b m_b}{m_a + m_b} \tag{10.12}$$

Note that the correspondence between Equation (10.12) and the second equation is used to compute the weighted mean in our simple illustration given earlier. By intuitive comparison, it should be clear that m_a and m_b correspond to weights that could be symbolized as w_a and w_b, respectively. Thus, Equation (10.12) can be written as

$$\bar{z} = \frac{w_a \bar{z}_a + w_b \bar{z}_b}{w_a + w_b} = \frac{\sum wz}{\sum w} \tag{10.13}$$

Equation (10.13) is used in calculating the weighted mean for a group of uncorrelated observations having unequal weights. In Chapter 11, it is shown that the weighted mean is the most probable value for a set of weighted observations.

Example 10.1 As a simple example of computing a weighted mean using Equation (10.13), suppose that a distance d is measured three times with the following results: 92.61 with weight 3, 92.60 with weight 2, and 92.62 with weight 1. Calculate the weighted mean.

$$\bar{d} = \frac{3(92.61) + 2(92.60) + 1(92.62)}{3 + 2 + 1} = 92.608$$

Note that if weight had been neglected, the simple mean would have been 92.610.

10.3 RELATIONSHIP BETWEEN WEIGHTS AND STANDARD ERRORS

By applying the special law of propagation of variances [Equation (6.16)] to Equation (10.8), the variance $\bar{z}_a$ in Equation (10.8) is

$$\sigma^2_{\bar{z}_a} = \left(\frac{\partial \bar{z}_a}{\partial z_1}\right)\sigma^2 + \left(\frac{\partial \bar{z}_a}{\partial z_2}\right)\sigma^2 + \cdots + \left(\frac{\partial \bar{z}_a}{\partial z_{m_a}}\right)\sigma^2 \tag{10.14}$$

Substituting partial derivatives with respect to the observations into Equation (10.14) yields

$$\sigma^2_{\bar{z}_a} = \left(\frac{1}{m_a}\right)^2\sigma^2 + \left(\frac{1}{m_a}\right)^2\sigma^2 + \cdots + \left(\frac{1}{m_a}\right)^2\sigma^2$$

Thus,

$$\sigma^2_{\bar{z}_a} = m_a\left(\frac{1}{m_a}\right)^2\sigma^2 = \frac{1}{m_a}\sigma^2 \tag{10.15}$$

Using a similar procedure, the variance of $\bar{z}_b$ is

$$\sigma^2_{\bar{z}_b} = \frac{1}{m_b}\sigma^2 \tag{10.16}$$

In Equations (10.15) and (10.16), σ is constant, and the weights of $\bar{z}_a$ and $\bar{z}_b$ were established as m_a and m_b, respectively, from Equation (10.13). Since the weights are relative, from Equations (10.15) and (10.16),

$$w_a = \frac{1}{\sigma^2_{\bar{z}_a}} \text{ and } w_b = \frac{1}{\sigma^2_{\bar{z}_b}} \tag{10.17}$$

Conclusion: With uncorrelated observations, the weights of the observations are inversely proportional to their variances.

10.4 STATISTICS OF WEIGHTED OBSERVATIONS

10.4.1 Standard Deviation



By definition, an observation is said to have a weight w when its precision is equal to that of the mean of w observations of unit weight. Let σ_0 be the standard error of an observation of weight 1, or *unit weight*. If $y_1, y_2, \ldots, y_n$ are observations having standard errors $\sigma_1, \sigma_2, \ldots, \sigma_n$ and weights $w_1, w_2, \ldots, w_n$, then by Equation (10.5)

$$\sigma_1 = \frac{\sigma_0}{\sqrt{w_1}}, \sigma_2 = \frac{\sigma_0}{\sqrt{w_2}}, \ldots, \sigma_n = \frac{\sigma_0}{\sqrt{w_n}} \tag{10.18}$$

In Section 2.7, the standard error for a group of observations of equal weight was defined as

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n} \varepsilon_i^2}{n}}$$

Now, in the case where the observations are not equal in weight, the equation above becomes

$$\sigma = \sqrt{\frac{w_1\varepsilon_1^2 + w_2\varepsilon_2^2 + \cdots w_n\varepsilon_n^2}{n}} = \sqrt{\frac{\sum_{i=1}^{n} w_i\varepsilon_i^2}{n}} \tag{10.19}$$

When modified for the standard deviation in Equation (2.7), it is

$$S = \sqrt{\frac{w_1v_1^2 + w_2v_2^2 + \cdots + w_nv_n^2}{n-1}} = \sqrt{\frac{\sum_{i=1}^{n} w_iv_i^2}{n-1}} \tag{10.20}$$

10.4.2 Standard Error of Weight w and Standard Error of the Weighted Mean

The relationship between the standard error and the standard error of weight w is given in Equation (10.18). Combining this with Equation (10.19) and dropping the summation limits, equations for standard errors of weight w are obtained in terms of σ_0 as follows:

$$\sigma_1 = \frac{\sigma_0}{\sqrt{w_1}} = \sqrt{\frac{\sum w\varepsilon^2}{n}}\frac{1}{\sqrt{w_1}} = \sqrt{\frac{\sum w\varepsilon^2}{nw_1}}$$

$$\sigma_2 = \frac{\sigma_0}{\sqrt{w_2}} = \sqrt{\frac{\sum w\varepsilon^2}{n}}\frac{1}{\sqrt{w_2}} = \sqrt{\frac{\sum w\varepsilon^2}{nw_2}}$$

$$\vdots$$

$$\sigma_n = \frac{\sigma_0}{\sqrt{w_n}} = \sqrt{\frac{\sum w\varepsilon^2}{n}}\frac{1}{\sqrt{w_n}} = \sqrt{\frac{\sum w\varepsilon^2}{nw_n}} \tag{10.21}$$

Similarly, the standard deviations of weight w can be expressed as:

$$S_1 = \sqrt{\frac{\sum wv^2}{w_1(n-1)}},\quad S_2 = \sqrt{\frac{\sum wv^2}{w_2(n-1)}},\ \ldots\ , S_n = \sqrt{\frac{\sum wv^2}{w_n(n-1)}} \tag{10.22}$$

Finally, the standard error of the weighted mean is calculated as:

$$\sigma_{\overline{M}} = \sqrt{\frac{\sum w\varepsilon^2}{n\sum w}} \tag{10.23}$$

and the standard deviation of the weighted mean is

$$\sigma_{\overline{M}} = \sqrt{\frac{\sum wv^2}{(n-1)\sum w}} \tag{10.24}$$

10.5 WEIGHTS IN ANGLE OBSERVATIONS

Suppose that the three angles α_1, α_2, and α_3 in a plane triangle are observed n_1, n_2, and n_3 times, respectively, with the same instrument under the same conditions. What are the relative weights of the angles?

To analyze the relationship between weights and the number of times an angle is turned, let S be the standard deviation of a single angle observation. The means of the three angles are

$$\alpha_1 = \frac{\sum \alpha_1}{n_1},\quad \alpha_2 = \frac{\sum \alpha_2}{n_2},\quad \alpha_3 = \frac{\sum \alpha_3}{n_3}$$

The variances of the means, as obtained by applying Equation (6.16), are

$$S_{\alpha_1}^2 = \frac{1}{n_1}S^2,\quad S_{\alpha_2}^2 = \frac{1}{n_2}S^2,\quad S_{\alpha_3}^2 = \frac{1}{n_3}S^2$$

Again, since the weights of the observations are inversely proportional to the variances and relative, the weights of the three angles are

$$w_1 = \frac{1}{S_{\alpha_1}^2} = \frac{n_1}{S^2}, \quad w_2 = \frac{1}{S_{\alpha_2}^2} = \frac{n_2}{S^2}, \quad w_3 = \frac{1}{S_{\alpha_3}^2} = \frac{n_3}{S^2}$$

In the above expressions, S is a constant term in each of the weights, and because weights are relative, it can be dropped. Thus, the weights of the angles are $w_1 = n_1$, $w_2 = n_2$, and $w_3 = n_3$.

In summary, it has been shown that when all conditions in angle observation are equal except for the number of turnings, *angle weights are proportional to the number of times the angles are turned*.

10.6 WEIGHTS IN DIFFERENTIAL LEVELING

Suppose that for the leveling network shown in Figure 10.1, the lengths of the lines 1, 2, and 3 are 2, 3, and 4 miles, respectively. For these varying lengths of lines, it can be expected that the errors in their elevation differences will vary, and thus, the weights assigned to the elevation differences should also be varied. What relative weights should be used for these lines?

To analyze the relationship of weights and level lines lengths, recall from Equation (9.21) that the variance in Δh is

$$\sigma_{\Delta h}^2 = D^2 \left[2N \left(\sigma_{r/D}^2 + \sigma_\alpha^2 \right) \right] \qquad (a)$$

where D is the length of the individual sights; N is the number of setups; $\sigma_{r/D}$ is the estimated error in a rod reading; and σ_α is the estimated collimation error for each sight. Let l_i be the length of the ith course between benchmarks; then

$$N = \frac{l_i}{2D} \qquad (b)$$

Substituting Equation (b) into Equation (a) yields

$$\sigma_{\Delta h}^2 = l_i D \left(\sigma_{r/D}^2 + \sigma_\alpha^2 \right) \qquad (c)$$

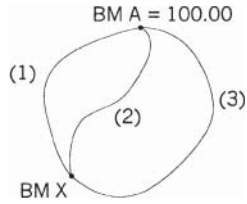


FIGURE 10.1 Differential leveling network.

However, D, $\sigma_{r/D}$, and σ_α are constants, and thus by letting $k = D(\sigma_{r/D}^2 + \sigma_\alpha^2)$, Equation ($c$) becomes

$$\sigma_{\Delta h}^2 = l_i k \qquad (d)$$

For this example, it can be said that the weights are

$$w_1 = \frac{1}{l_1 k}, \quad w_2 = \frac{1}{l_2 k}, \quad w_3 = \frac{1}{l_3 k} \qquad (e)$$

Now, since k is a constant and weights are relative, Equation (e) can be simplified to

$$w_1 = \frac{1}{l_1}, \quad w_2 = \frac{1}{l_2}, \quad w_3 = \frac{1}{l_3}$$

In summary, it has been shown that the *weights of differential leveling lines are inversely proportional to their lengths*, and since any course length is proportional to its number of setups, the *weights are also inversely proportional to the number of setups*.

10.7 PRACTICAL EXAMPLES

Example 10.2 Suppose the angles in an equilateral triangle ABC were observed by the same operator using the same instrument, but the number of repetitions for each angle varied. The results were: $A = 45°15'25''$, $n = 4$; $B = 83°37'22''$, $n = 8$; and $51°07'39''$, $n = 6$. Adjust the angles.

SOLUTION Weights are proportional to the number of repetitions, and corrections are made in inverse proportion to those weights. The sum of the three angles is $180°00'26''$, and thus the misclosure that must be adjusted is $26''$. The correction process is demonstrated in Table 10.1.

TABLE 10.1 Adjustment of Example 10.2

Angle	n (Weight)	Correction Factor	Correction	Corrected Angle
A	4	(1/4)24 = 6	(6/13)26 = 12″	45°15′13″
B	8	(1/8)24 = 3	(3/13)26 = 06″	83°37′16″
C	6	(1/6)24 = 4	(4/13)26 = 08″	51°07′31″
		$\sum = 13$	$\sum = 26''$	$\sum = 180°00'00''$

Note that a multiplier of 24 was used for convenience to avoid fractions in computing correction factors. Because weights are relative, this did not alter the adjustment. Note, also, that two computational checks are secured in the above solution; the sum of the individual corrections totaled $26''$, and the sum of the corrected angles totaled $180°00'00''$.

Example 10.3 In the leveling network of Figure 10.1, recall that the lengths of lines 1, 2, and 3 were 2, 3, and 4 miles, respectively. If the observed elevation differences in lines 1, 2, and 3 were +21.20 ft, +21.23 ft, and +21.29 ft, respectively, find the weighted mean for the elevation difference and the adjusted elevation of BMX. (Note: All level lines were run from BMA to BMX.)

SOLUTION The weights of lines 1, 2, and 3 are 1/2, 1/3, and 1/4, respectively. Again, since weights are relative, these weights can arbitrarily be multiplied by 12 to obtain weights of 6, 4, and 3, respectively. Applying Equation (10.13), the weighted mean is

$$\text{mean } \Delta Elev = \frac{6(21.20) + 4(21.23) + 3(21.29)}{6 + 4 + 3} = +21.23$$

Thus, the elevation of BMX = 100.00 + 21.23 = 123.23 ft. Note that if the weights had been neglected, the simple average would have given a mean elevation difference of +21.24.

Example 10.4 A distance is observed as 625.79 ft using a cloth tape and given a weight of 1; it is measured again as 625.71 ft using a steel tape and assigned a weight of 2; and finally, it is measured a third time as 625.69 ft with an EDM instrument and given a weight of 4. Calculate the most probable value of the length (*weighted mean*) and find the standard deviation of the weighted mean.

SOLUTION By Equation (10.13), the weighted mean is

$$\overline{M} = \frac{1(625.79) + 2(623.71) + 4(625.69)}{1 + 2 + 4} = 625.71 \text{ ft}$$

By Equation (10.24), the standard deviation of the weighted mean is

$$S_{\overline{M}} = \sqrt{\frac{\sum wv^2}{(n-1)\sum w}} = \sqrt{\frac{0.0080}{(2)7}} = \pm 0.024 \text{ ft}$$

where

$$\begin{aligned} v_1 &= 625.71 - 625.79 = -0.08 & w_1 v_1{}^2 &= 1(-0.08)^2 = 0.0064 \\ v_2 &= 625.71 - 625.71 = 0.00 & w_2 v_2^2 &= 2(0.00)^2 \phantom{{}= 0.0000} = 0.0000 \\ v_3 &= 625.71 - 625.69 = +0.02 & w_3 v_3^2 &= 4(+0.02)^2 = 0.0016 \\ & & \textstyle\sum wv^2 &= 0.0080 \end{aligned}$$

Example 10.5 In leveling from benchmark A to benchmark B, four different routes of varying length are taken. The data in Table 10.2 are obtained. (Note that the weights were computed for computational convenience only.)

TABLE 10.2 Route Data for Example 10.5

Route	Length (miles)	$\Delta Elev$	W
1	1	+25.35	18
2	2	+25.41	9
3	3	+25.38	6
4	6	+25.30	3

Calculate the most probable elevation difference (weighted mean), the standard deviation of unit weight, the standard deviation of the weighted mean, and the standard deviations of the weighted observations.

SOLUTION By Equation (10.13), the weighted mean for elevation difference is

$$\overline{M} = \frac{18(25.35) + 9(25.41) + 6(25.38) + 3(25.30)}{18 + 9 + 6 + 3} = +25.366 \text{ ft}$$

Note that the arithmetic mean for this set of observations is 25.335, but the weighted mean is 25.366. To find the standard deviations for the weighted observations, the data in Table 10.3 are first created.

TABLE 10.3 Data for Standard Deviations in Example 10.5

Route	w	v	v^2	wv^2
1	18	+0.016	0.0002	0.0045
2	9	−0.044	0.002	0.0171
3	6	−0.014	0.0002	0.0012
4	3	+0.066	0.0043	0.0130
				$\sum wv^2 = 0.0361$

By Equation (10.20), the standard deviation of unit weight is

$$S_0 = \sqrt{\frac{0.0361}{3}} = \pm 0.11 \text{ ft}$$

By Equation (10.24), the standard deviation of the weighted mean is

$$S_{\overline{M}} = \sqrt{\frac{0.0361}{36(3)}} = \pm 0.018 \text{ ft}$$

By Equation (10.22), the standard deviations for the weighted observations are

$$S_1 = \sqrt{\frac{0.0361}{18(3)}} = \pm 0.026 \text{ ft} \qquad S_2 = \sqrt{\frac{0.0361}{9(3)}} = \pm 0.037 \text{ ft}$$

$$S_3 = \sqrt{\frac{0.0361}{6(3)}} = \pm 0.045 \text{ ft} \qquad S_4 = \sqrt{\frac{0.0361}{3(3)}} = \pm 0.063 \text{ ft}$$

PROBLEMS

Note: Partial answers to problems marked with an asterisk are found in Appendix G.

***10.1** An angle was measured as 56°15′00″ using an engineer's transit and had a standard deviation of ±30″. It was measured again using a repeating optical theodolite as 56°14′44″ with a standard deviation of ±10″. This angle was measured a third time with a directional theodolite 56°14′48″ with a standard deviation of ±5″. Calculate the weighted mean of the angle and its standard deviation.

***10.2** What is the standard deviation in the weighted mean for Problem 10.1?

10.3 A zenith angle was measured six times with both faces of a total station. The average direct reading is 96°28′43″ with a standard deviation of ±6.8″. With the reverse face, it was observed as 263°31′14″ with a standard deviation of ±7.2″. What is the most probable value for the zenith angle in the direct face?

10.4 Similar to Problem 10.3, except the angle was measured four times in both faces, and the average of the direct readings was 78°56′59″ ± 5.8″, and the average of the reverse readings was 281°03′08″ ± 5.6″.

10.5 An angle was measured on four different days with the following results:

Day	Angle	S (″)
1	68°25′34″	±3.2
2	68°25′36″	±4.7
3	68°25′28″	±2.8
4	68°25′26″	±3.6

What is the most probable value for the angle and the standard deviation in the mean?

***10.6** A distance was measured by pacing as 155 ft with a standard deviation of ±5 ft. It was then observed as 153.69 ft with a steel tape, having a standard deviation of ±0.05 ft. Finally, it was measured as 153.79 ft with an EDM instrument with a standard deviation of ±0.02 ft. What is the most probable value for the distance and its standard deviation?

10.7 A distance was measured by pacing as 268 ft with a standard deviation of ±10 ft. It was then measured as 268.32 ft with a steel tape and had a standard deviation of ±0.06 ft. Finally, it was measured as 268.39 ft with an EDM. The EDM instrument and reflector setup standard deviations were ±0.005 ft, and the manufacturer's estimated standard deviation for the EDM instrument is ±(3 mm + 3 ppm). What is the most probable value for the distance and the standard deviation of the weighted mean?

10.8 What is the computed standard deviation for each weighted observation of Problem 10.5?

10.9 During a construction project, the difference in elevation from the deck to the surface of the underpass is observed four different times using differential leveling. The observations and the lengths of the lines are

Route	$\Delta Elev$ (m)	Length (m)
1	13.087	49.708
2	13.095	52.035
3	13.078	48.694
4	13.084	55.842

What is

***(a)** The weighted mean for the elevation difference?

(b) Its standard deviation?

(c) Standard deviation of the weighted mean?

(d) Standard deviation for each weighted observation?

10.10 An angle is observed on three different days with the following results.

Day	Observation	S (″)
1	28°26′54″	±6.5
2	28°26′48″	±4.3
3	28°26′52″	±6.9

What is the:

(a) Weighted mean of the angle?

(b) Standard deviation in the weighted mean?

10.11 Three crews level to a benchmark following three different routes. The lengths of the routes and the observed differences in elevation are

Route	$\Delta Elev$ (ft)	Length (ft)
1	36.48	2200
2	36.56	3800
3	36.52	2900

What is the

(a) Weighted mean for the difference in elevation?

(b) Standard deviation for the weighted elevation difference?

(c) Standard deviation for the elevation difference?

10.12 Find the standard deviations for the weighted observations in Problem 10.11.

CHAPTER 11

PRINCIPLES OF LEAST SQUARES

11.1 INTRODUCTION



In surveying, observations must often satisfy established numerical relationships known as *geometric constraints*. As examples, in a closed-polygon traverse, horizontal angle and distance observations should conform to the geometric constraints given in Section 8.4, and in a differential leveling loop, the elevation differences should sum to a given quantity. However, because the geometric constraints rarely meet perfectly, an adjustment of the data is performed.

As discussed in earlier chapters, errors in observations conform to the laws of probability; that is, they follow normal distribution theory. Thus, they should be adjusted in a manner that follows these mathematical laws. While the arithmetic mean has been used extensively throughout history, the earliest works on least squares started in the late eighteenth century. Its earliest application was primarily for adjusting celestial observations. Laplace first investigated the subject and laid its foundation in 1774. The first published article on the subject, titled "Méthode des Moindres Quarrés" (Method of Least Squares), was written in 1805 by Legendre. However, it is well known that although Gauss did not publish until 1809, he developed and used the method extensively as a student at the University of Göttingen

beginning in 1794 and thus is given credit for the development of the subject. In this chapter, equations for performing least-squares adjustments are developed, and their uses are illustrated with several examples.

11.2 FUNDAMENTAL PRINCIPLE OF LEAST SQUARES

To develop the principle of least squares, a specific case will be considered. Suppose there are n independent equally weighted measurements, $z_1, z_2, \ldots, z_n$, of the same quantity z that has a *most probable value* denoted by M. By definition

$$\begin{aligned} M - z_1 &= v_1 \\ M - z_2 &= v_2 \\ &\vdots \\ M - z_n &= v_n \end{aligned} \tag{11.1}$$

In Equation (11.1), the v's are the residual errors. Note that residuals behave in a manner like errors, and thus they can be used interchangeably in the normal distribution function given by Equation (3.2). By substituting v for x, there result

$$f_x(v) = y = \frac{1}{\sigma\sqrt{2\pi}} e^{-v^2/2\sigma^2} = Ke^{-h^2v^2} \tag{11.2}$$

where $h = 1/\sigma\sqrt{2}$ and $K = h/\sqrt{\pi}$.

As discussed in Chapter 3, probabilities are represented by areas under the normal distribution curve. Thus, the individual probabilities for the occurrence of residuals $v_1, v_2, \ldots, v_n$ are obtained by multiplying their respective ordinates $y_1, y_2, \ldots, y_n$ by some infinitesimally small increment of v, which is denoted as Δv. The following probability statements result

$$\begin{aligned} P_1 &= y_1 \Delta v = Ke^{-h^2 v_1^2} \Delta v \\ P_2 &= y_2 \Delta v = Ke^{-h^2 v_2^2} \Delta v \\ &\vdots \\ P_n &= y_n \Delta v = Ke^{-h^2 v_n^2} \Delta v \end{aligned} \tag{11.3}$$

From Equation (3.1), the probability of the simultaneous occurrence of all the residuals v_1 through v_n is the product of the individual probabilities, and thus

$$P = \left(Ke^{-h^2 v_1^2} \Delta v\right)\left(Ke^{-h^2 v_2^2} \Delta v\right) \cdots \left(Ke^{-h^2 v_n^2} \Delta v\right) \tag{11.4}$$

Simplifying Equation (11.4) yields

$$P = K^n(\Delta v)^n e^{-h^2\left(v_1^2+v_2^2+\cdots+v_n^2\right)} \tag{11.5}$$

M is a quantity that is to be selected in such a way that it gives the greatest probability of occurrence; stated differently, the value of M that maximizes the value of P. In Equation (11.5), the values of K, h, and Δv are all constant values, and thus only the residuals can be modified by selecting different values for M. Figure 11.1 shows a plot of the e^{-x} versus x. From this plot, it is readily seen that e^{-x} is maximized by minimizing x, and thus, in relation to Equation (11.5), the probability P is maximized when the quantity $\left(v_1^2 + v_2^2 + \cdots + v_n^2\right)$ is minimized. In other words, *to maximize P, the sum of the squared residuals must be minimized*. Equation (11.6) expresses the *principle of least squares*:

$$\sum v^2 = \left(v_1^2 + v_2^2 + \cdots + v_n^2\right) = \text{minimum} \tag{11.6}$$

This condition states: "The most probable value (MPV) for a quantity obtained from repeated observations of equal weight is the value that renders the sum of the squared residuals a minimum." From calculus, the minimum value of a function can be found by taking its first derivative and equating the resulting function with zero. That is, the condition stated in Equation (11.6) is enforced by taking the first derivative of the function with respect to the unknown variable M and setting the results equal to zero. Substituting Equation (11.1) into Equation (11.6) yields

$$\sum v^2 = (M - z_1)^2 + (M - z_2)^2 + \cdots + (M - z_n)^2 \tag{11.7}$$

Taking the first derivative of Equation (11.7) with respect to M and setting the resulting equation equal to zero yields

$$\frac{d\left(\sum v^2\right)}{dM} = 2(M - z_1)(1) + 2(M - z_2)(1) + \cdots + 2(M - z_n)(1) = 0 \tag{11.8}$$

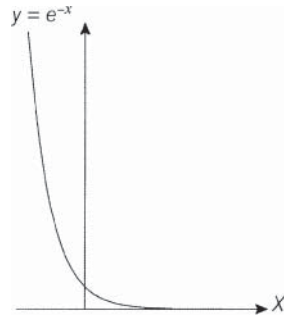


FIGURE 11.1 Plot of e^{-x}.

Now dividing Equation (11.8) by 2 and simplifying yields

$$M - z_1 + M - z_2 + \cdots + M - z_n = 0$$
$$nM = z_1 + z_2 + \cdots z_n$$
$$M = \frac{z_1 + z_2 + \cdots + z_n}{n} \tag{11.9}$$

In Equation (11.9) the quantity $(z_1 + z_2 + \cdots z_n)/n$ is the mean of the observed values. This is proof that when a quantity has been observed independently several times, the MPV is the arithmetic mean.

11.3 THE FUNDAMENTAL PRINCIPLE OF WEIGHTED LEAST SQUARES

In Section 11.2, the fundamental principle of a least-squares adjustment was developed for observations having equal or unit weights. The more general case of least-squares adjustment assumes that the observations have varying degrees of precision and thus varying weights.

Consider a set of measurements $z_1, z_2, \ldots, z_n$ having relative weights $w_1, w_2, \ldots, w_n$ and residuals $v_1, v_2, \ldots, v_n$. Denote the weighted MPV as M. As in Section 11.2, the residuals are related to the observations through Equations (11.1), and the total probability of their simultaneous occurrence is given by Equation (11.5). However, notice in Equation (11.2) that $h^2 = 1/2\sigma^2$, and since weights are inversely proportional to variances, they are directly proportional to h^2. Thus, Equation (11.5) can be rewritten as

$$P = K^n(\Delta v)^n e^{-(w_1 v_1^2 + w_2 v_2^2 + \cdots + w_n v_n^2)} \tag{11.10}$$

To maximize P in Equation (11.10), the negative exponent must be minimized. To achieve this, the sum of the products of the weights times their respective squared residuals must be minimized. This is the condition imposed in a weighted least-squares adjustment. The condition of weighted least-squares adjustment in equation form is

$$w_1 v_1^2 + w_2 v_2^2 + \cdots + w_n v_n^2 = \sum wv^2 \rightarrow \text{minimum} \tag{11.11}$$

Substituting the values for the residuals given in Equation (11.1) into Equation (11.11) yields

$$w_1(M - z_1)^2 + w_2(M - z_2)^2 + \cdots + w_n(M - z_n)^2 \rightarrow \text{minimum} \tag{11.12}$$

The condition for a weighted least-square adjustment is: The most probable value for a quantity obtained from repeated observations having various weights is

that value that renders the sum of the weight times their respective squared residual a minimum.

The minimum condition is imposed by differentiating Equation (11.12) with respect to M and setting the resultant equation equal to zero. This yields

$$2w_1(M - z_1)(1) + 2w_2(M - z_2)(1) + \cdots + 2w_n(M - z_n)(1) = 0 \tag{11.13}$$

Dividing Equation (11.13) by 2 and rearranging results in

$$w_1(M - z_1) + w_2(M - z_2) + \cdots + w_n(M - z_n) = 0 \tag{11.14a}$$

Rearranging Equation (11.14*a*) yields

$$w_1z_1 + w_2z_2 + \cdots + w_nz_n = w_1M + w_2M + \cdots w_nM \tag{11.14b}$$

Equation (11.14*b*) can be written as $\sum wz = \sum wM$. Thus,

$$M = \frac{\sum wz}{\sum w} \tag{11.15}$$

Notice that Equation (11.15) is the same as Equation (10.13), which is the formula for computing the weighted mean.

11.4 THE STOCHASTIC MODEL

The determination of variances, and subsequently the weights of the observations, is known as the *stochastic model* in a least-squares adjustment. In Section 11.3, the inclusion of weights in the adjustment was discussed. It is crucial to the adjustment to select a proper stochastic (weighting) model since, as was discussed in Section 10.1, the weight of an observation controls the amount of correction it receives during the adjustment. However, the development of the stochastic model is important to more than the weighted adjustments. When performing an unweighted adjustment, all observations are assumed to be of equal weight, and thus the stochastic model is created implicitly. The foundations for selecting a proper stochastic model in surveying were established in Chapters 7 through 10. It will be shown in Chapter 21 that failure to select the stochastic model properly will also affect one's ability to isolate blunders in observational sets.

11.5 FUNCTIONAL MODEL

A functional model in adjustment computations is an equation or set of equations/functions that represent or define an adjustment condition. It must either be known or assumed. If the functional model represents the physical situation

adequately, the observational errors can be expected to conform to the normal distribution curve. For example, a well-known functional model states that the sum of angles in a triangle is 180°. This model is adequate if the survey is limited to a small region. However, when the survey covers very large areas, this model does not account for the systematic errors caused by the curvature of the Earth. In this case, the functional model is inadequate and needs to be modified to include corrections for spherical excess. In traversing, the functional model of plane computations is suitable for smaller surveys, but if the extent of the survey becomes too large, again the model must be changed to account for the systematic errors caused by the curvature of the Earth. This can be accomplished by transforming the observations into a plane map projection system, such as the state plane coordinate system, or by using geodetic observation equations. If the model does not fit the physical situation, an incorrect adjustment will result. Chapter 23 discusses a three-dimensional geodetic model and the systematic errors, which must be considered in a three-dimensional geodetic network adjustment.

There are two basic forms for functional models: *conditional* and *parametric* adjustments. In the conditional adjustment, geometric conditions are enforced upon the observations and their residuals. Examples of conditional adjustment are: (1) the sum of the angles in a closed polygon is $(n - 2)180°$, where n is the number of sides in the polygon; (2) the latitudes and departures of a polygon traverse sum to zero; and (3) the sum of the angles in the horizon equal 360°. A least-squares adjustment example using condition equations is given in Section 11.13.

When performing a parametric adjustment, observations are expressed in terms of unknown parameters that were never observed directly. For example, the well-known coordinate equations are used to model the observed angles, directions, and distances in a horizontal plane survey. The adjustment yields the most probable values for the coordinates (parameters), which, in turn, provide the most probable values for the adjusted observations.

The choice of the functional model will determine which quantities or parameters are adjusted. The primary purpose of an adjustment is to ensure that all observations are used to find the most probable values for the unknowns in the model. In least-squares adjustments, no matter if conditional or parametric, the geometric checks at the end of the adjustment are satisfied, and the same adjusted observations are obtained. In complicated networks, it is often difficult and time-consuming to write equations to express all conditions that must be met for a conditional adjustment. Thus, this book will focus on the parametric adjustment, which generally leads to larger systems of equations but is straightforward in its development and solution and, as a result, is well suited to computers.

The *mathematical model* for an adjustment is the combination of the stochastic model and functional model. Both the stochastic model and functional model must be correct if the adjustment is to yield the most probable values. That is, it is just as important to use a correct stochastic model as it is to use a correct functional model. Improper weighting of observations will result in the unknown parameters being determined incorrectly.

11.6 OBSERVATION EQUATIONS



Equations that relate observed quantities to both observational residuals and independent, unknown parameters are called *observation equations*. One equation is written for each observation and for a unique set of unknowns. For a unique solution of unknowns, the number of equations must equal the number of unknowns. Usually, there are more observations (and hence equations) than unknowns, and this permits the determination of the most probable values for the unknowns based on the principle of least squares.

11.6.1 Elementary Example of Observation Equation Adjustment

As an example of a least-squares adjustment by the observation equation method, consider the following three equations:

$$\begin{aligned} &(1)\quad x + y = 3.0 \\ &(2)\quad 2x - y = 1.5 \\ &(3)\quad x - y = 0.2 \end{aligned} \tag{11.16}$$

Equations (11.16) relate the two unknowns, x and y, to the observed quantities 3.0, 1.5, and 0.2. One equation is redundant since the values for x and y can be obtained from any two of the three equations. For example, if Equations (1) and (2) are solved then x would equal 1.5 and y would equal 1.5, but if Equations (2) and (3) are solved then x would equal 1.3 and y would equal 1.1, and if Equations (1) and (3) are solved then x would equal 1.6 and y would equal 1.4. Based on the inconsistency of these equations, the observations must contain errors. Therefore, new expressions, called *observation equations*, can be rewritten that include residuals. The resulting set of equations is

$$\begin{aligned} &(4)\quad v_1 = x + y - 3.0 \\ &(5)\quad v_2 = 2x - y - 1.5 \\ &(6)\quad v_3 = x - y - 0.2 \end{aligned} \tag{11.17}$$

Equations (11.17) relate the unknown parameters to the observations and their errors. Equations of this form are known as *residual equations*. Obviously, it is possible to select values of v_1, v_2, and v_3 that will yield the same values for x and y, no matter which pair of equations is used. For example, to obtain consistency through all the equations, arbitrarily let $v_1 = 0$, $v_2 = 0$, and $v_3 = -0.2$. In this arbitrary solution, x would equal 1.5 and y would equal 1.5, no matter which pair of equations is solved. This is a consistent solution; however, there are other values for the residuals (v's) that will produce a smaller sum of squares, and thus the most probable values for the unknowns x and y.

To find the least-squares solution for x and y, the residual equations are squared, and these squared expressions are added to give a function, $f(x,y)$, that equals the $\sum v^2$. Doing this for Equations (11.17) yields

$$f(x,y) = \sum v^2 = (x + y - 3.0)^2 + (2x - y - 1.5)^2 + (x - y - 0.2)^2 \qquad (11.18)$$

As discussed previously, to minimize a function, its derivatives must be set equal to zero. Thus, in Equation (11.18), the partial derivatives of Equation (11.18) with respect to each unknown must be taken and set equal to zero. This leads to two equations:

$$\frac{\partial f(x,y)}{\partial x} = 2(x + y - 3.0) + 2(2x - y - 1.5)(2) + 2(x - y - 0.2) = 0$$
$$\frac{\partial f(x,y)}{\partial y} = 2(x + y - 3.0) + 2(2x - y - 1.5)(-1) + 2(x - y - 0.2)(-1) = 0 \qquad (11.19)$$

Equations (11.19) are called *normal equations*. Simplifying those gives reduced normal equations of

$$6x - 2y - 6.2 = 0$$
$$-2x + 3y - 1.3 = 0 \qquad (11.20)$$

The simultaneous solution of Equations (11.20) yields x equal to 1.514 and y equal to 1.442. Substituting these adjusted values into the residual equations [Equations (11.17)] results in the numerical values for the three residuals. Table 11.1 provides a comparison of the arbitrary solution to the least-squares solution. The tabulated summations of residuals squared show that the least-squares solution yields the smaller total and thus the better solution. In fact, it is the most probable solution for the unknown x and y based on the observations.

TABLE 11.1 Comparison of an Arbitrary and Least-Squares Solution

Arbitrary Solution		Least-Squares Solution	
$v_1 = 0.00$	$v_1^2 = 0.00$	$v_1 = -0.044$	$v_1^2 = 0.002$
$v_2 = 0.00$	$v_2^2 = 0.00$	$v_2 = 0.085$	$v_2^2 = 0.007$
$v_3 = -0.02$	$v_3^2 = 0.04$	$v_3 = -0.128$	$v_3^2 = 0.016$
	$\sum v^2 = 0.04$		$\sum v^2 = 0.025$

11.7 SYSTEMATIC FORMULATION OF THE NORMAL EQUATIONS

11.7.1 Equal-Weight Case



In large systems of observation equations, it would be helpful to use systematic procedures to formulate the normal equations. In developing these procedures, consider the following generalized system of linear observation equations having variables of $(A, B, C, \ldots, N)$.

$$\begin{aligned} a_1A + b_1B + c_1C + \cdots + n_1N &= l_1 + v_1 \\ a_2A + b_2B + c_2C + \cdots + n_2N &= l_2 + v_2 \\ &\vdots \\ a_mA + b_mB + c_mC + \cdots + n_mN &= l_m + v_m \end{aligned} \tag{11.21}$$

The squares of the residuals for Equations (11.21) are

$$\begin{aligned} v_1^2 &= (a_1A + b_1B + c_1C + \cdots + n_1N - l_1)^2 \\ v_2^2 &= (a_2A + b_2B + c_2C + \cdots + n_2N - l_2)^2 \\ &\vdots \\ v_m^2 &= (a_mA + b_mB + c_mC + \cdots + n_mN - l_m)^2 \end{aligned} \tag{11.22}$$

Summing Equations (11.22), the function $f(A, B, C, \ldots, N) = \sum v^2$ is obtained. This expresses the equal-weight least-squares condition as

$$\begin{aligned} \sum v^2 = {} & (a_1A + b_1B + c_1C + \cdots + n_1N - l_1)^2 \\ & + (a_2A + b_2B + c_2C + \cdots + n_2N - l_2)^2 + \cdots \\ & + (a_mA + b_mB + c_mC + \cdots + n_mN - l_m)^2 \end{aligned} \tag{11.23}$$

According to the theory of least squares, the minimum for Equation (11.23) is found by setting the partial derivatives of the function with respect to each unknown equal to zero. This results in the following normal equations

$$\begin{aligned} \frac{\partial \sum v^2}{\partial A} = {} & 2(a_1A + b_1B + c_1C + \cdots + n_1N - l_1)a_1 \\ & + 2(a_2A + b_2B + c_2C + \cdots + n_2N - l_2)a_2 + \cdots \\ & + 2(a_mA + b_mB + c_mC + \cdots + n_mN - l_m)a_m = 0 \end{aligned}$$

$$\begin{aligned}
\frac{\partial \sum v^2}{\partial B} &= 2(a_1A + b_1B + c_1C + \cdots + n_1N - l_1)b_1 \\
&\quad + 2(a_2A + b_2B + c_2C + \cdots + n_2N - l_2)b_2 + \cdots \\
&\quad + 2(a_mA + b_mB + c_mC + \cdots + n_mN - l_m)b_m = 0 \\
&\vdots \\
\frac{\partial \sum v^2}{\partial N} &= 2(a_1A + b_1B + c_1C + \cdots + n_1N - l_1)n_1 \\
&\quad + 2(a_2A + b_2B + c_2C + \cdots + n_2N - l_2)n_2 + \cdots \\
&\quad + 2(a_mA + b_mB + c_mC + \cdots + n_mN - l_m)n_m = 0
\end{aligned} \tag{11.24}$$

Dividing each expression by 2 and regrouping the remaining terms in Equation (11.24) results in Equation (11.25):

$$\begin{aligned}
&(a_1^2 + a_2^2 + \cdots + a_m^2)A + (a_1b_1 + a_2b_2 + \cdots + a_mb_m)B + (a_1c_1 + a_2c_2 + \cdots + a_mc_m)C \\
&\quad + \cdots + (a_1n_1 + a_2n_2 + \cdots + a_mn_m)N - (a_1l_1 + a_2l_2 + \cdots + a_ml_m) = 0 \\
&(b_1a_1 + b_2a_2 + \cdots + b_ma_m)A + \left(b_1^2 + b_2^2 + \cdots + b_m^2\right)B + (b_1c_1 + b_2c_2 + \cdots + b_mc_m)C \\
&\quad + \cdots + (b_1n_1 + b_2n_2 + \cdots + b_mn_m)N - (b_1l_1 + b_2l_2 + \cdots + b_ml_m) = 0 \\
&\qquad\qquad \vdots \\
&(n_1a_1 + n_2a_2 + \cdots + n_ma_m)A + (n_1b_1 + n_2b_2 + \cdots + b_ma_m)B + (n_1c_1 + n_2c_2 + \cdots + n_mc_m)C \\
&\quad + \cdots + \left(n_1^2 + n_2^2 + \cdots + n_m^2\right)N - (n_1l_1 + n_2l_2 + \cdots + n_ml_m) = 0
\end{aligned} \tag{11.25}$$

Generalized equations expressing normal Equations (11.25) are now written as

$$\begin{aligned}
\left(\sum a^2\right)A + \left(\sum ab\right)B + \left(\sum ac\right)C + \cdots + \left(\sum an\right)N &= \sum al \\
\left(\sum ba\right)A + \left(\sum b^2\right)B + \left(\sum bc\right)C + \cdots + \left(\sum bn\right)N &= \sum bl \\
&\vdots \\
\left(\sum na\right)A + \left(\sum nb\right)B + \left(\sum nc\right)C + \cdots + \left(\sum n^2\right)N &= \sum nl
\end{aligned} \tag{11.26}$$

In Equation (11.26), the a's, b's, c's, … , n's are the coefficients for the unknowns A, B, C, … , N; the l values are the observations; and $\sum$ signifies the summation from $i = 1$ to m.

11.7.2 Weighted Case

In a manner like that of Section 11.7.1, it can be shown that normal equations can be systematically formed for weighted observation equations in the following manner:

$$\begin{aligned}
\left(\sum wa^2\right)A + \left(\sum wab\right)B + \left(\sum wac\right)C + \cdots + \left(\sum wan\right)N &= \sum wal \\
\left(\sum wba\right)A + \left(\sum wb^2\right)B + \left(\sum wbc\right)C + \cdots + \left(\sum wbn\right)N &= \sum wbl \\
&\vdots \\
\left(\sum wna\right)A + \left(\sum wnb\right)B + \left(\sum wnc\right)C + \cdots + \left(\sum wn^2\right)N &= \sum wnl
\end{aligned} \tag{11.27}$$

In Equation (11.27), w are the weights of the observations, l; a's, b's, c's, … , n's are the coefficients for the unknowns A, B, C, … , N; l values are the observations; and $\sum$ signifies the summation from $i = 1$ to m.

Notice that the terms in Equations (11.27) are the same as those in Equations (11.26) except for the addition of the w's, which are the relative weights of the observations. In fact, Equations (11.27) can be thought of as the general set of equations in forming the normal equations, since if the weights are equal, they can all be given a value of 1. In this case, they will cancel out of Equations (11.27) to produce the special case given by Equations (11.26).

11.7.3 Advantages of the Systematic Approach

Using the systematic methods just demonstrated, the normal equations can be formed for a set of linear equations without writing the residual equations, compiling their summation equation, or taking partial derivatives with respect to the unknowns. Rather, for any set of linear equations, the normal equations for the least-squares solution can be written directly.

11.8 TABULAR FORMATION OF THE NORMAL EQUATIONS

Formulation of normal equations from observation equations can be simplified further by handling Equations (11.26) and (11.27) in a table. In this way, a large quantity of numbers can easily be manipulated. Tabular formulation of the normal equations for the example in Section 11.6.1 is illustrated below. First, Equations (11.17) are made compatible with the generalized form of Equations (11.21). These equations, the so-called *observation equations*, are shown in Equations (11.28):

$$\begin{aligned}
&(7) \quad x + y = 3.0 + v_1 \\
&(8) \quad 2x - y = 1.5 + v_2 \\
&(9) \quad x - y = 0.2 + v_3
\end{aligned} \tag{11.28}$$

TABLE 11.2 Tabular Formation of Normal Equations

Eqn.	a	b	L	a^2	ab	b^2	al	bl
7	1	1	3.0	1	1	1	3.0	3.0
8	2	−1	1.5	4	−2	1	3.0	−1.5
9	1	−1	0.2	1	−1	1	0.2	−0.2
				$\sum a^2 = 6$	$\sum ab = -2$	$\sum b^2 = 3$	$\sum al = 6.2$	$\sum bl = 1.3$

In Equations (11.28), there are two unknowns, x and y, with different coefficients for each equation. Placing the coefficients and the observations, l's, for each expression of Equation (11.28) into a table, the normal equations are formed systematically. Table 11.2 shows the coefficients, appropriate products, and summations in accordance with Equations (11.26).

After substituting the appropriate values for $\sum a^2$, $\sum ab$, $\sum b^2$, $\sum al$, and $\sum bl$ from Table 11.2 into Equations (11.26), the following normal equations are

$$\begin{aligned} 6x - 2y &= 6.2 \\ -2x + 3y &= 1.3 \end{aligned} \tag{11.29}$$

Notice that Equations (11.29) are the same as those obtained in Section 11.6 using the theoretical least-squares method. That is, Equations (11.29) match Equations (11.20). In Chapter 28, this summation method is developed to avoid matrix operations, which will be discussed next.

11.9 USING MATRICES TO FORM THE NORMAL EQUATIONS



Note that the number of normal equations in a parametric least-squares adjustment is always equal to the number of unknown variables. Often, the system of normal equations becomes quite large. But even when dealing with three unknowns, their solution by hand is time-consuming. Consequently, computers and matrix methods, as described in Appendixes A through C, are almost always used today. In the following subsections, we present the matrix methods used in performing least-squares adjustments.

11.9.1 Equal-Weight Case

To develop the matrix expressions for performing least-squares adjustments, an analogy will be made with the systematic procedures demonstrated in Section 11.7. For this development, let a system of observation equations be represented by the matrix notation

$$AX = L + V \tag{11.30}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad L = \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_m \end{bmatrix} \qquad V = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_m \end{bmatrix}$$

Note that the above system of observation Equations (11.30) is identical to Equations (11.21) except that the unknowns are $x_1, x_2, \ldots, x_n$ instead of A, $B, \ldots, N$, and the coefficients of the unknowns are $a_{11}, a_{12}, \ldots, a_{1n}$ instead of a_1, $b_1, \ldots, n_1$.

Subjecting the above matrices to the manipulations given in the following expression, Equation (11.31), produces the normal equations [i.e., matrix Equation (11.31a) is exactly equivalent to Equations (11.26)]:

$$A^{\mathrm{T}} A X = A^{\mathrm{T}} L \tag{11.31a}$$

Equation (11.31a) can also be expressed as

$$NX = A^{\mathrm{T}} L \tag{11.31b}$$

where N represents the *normal matrix*. The correspondence between Equations (11.31) and Equations (11.26) becomes clear if the matrices are multiplied and analyzed as follows:

$$A^{\mathrm{T}} A = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} n_{11} & n_{12} & \cdots & n_{1n} \\ n_{21} & n_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = N$$

$$A^{\mathrm{T}} L = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} l_1 \\ l_2 \\ \vdots \\ l_m \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{m} a_{i1} l_i \\ \sum_{i=1}^{m} a_{i2} l_i \\ \vdots \\ \sum_{i=1}^{m} a_{in} l_i \end{bmatrix}$$

The individual elements of the normal matrix can be expressed in the following summation forms:

$$\begin{array}{llll} n_{11} = \sum_{i=1}^{m} a_{i1}^2 & n_{12} = \sum_{i=1}^{m} a_{i1}a_{i2} & \cdots & n_{1n} = \sum_{i=1}^{m} a_{i1}a_{in} \\ n_{21} = \sum_{i=1}^{m} a_{i2}a_{i1} & n_{22} = \sum_{i=1}^{m} a_{i2}^2 & \cdots & n_{2n} = \sum_{i=1}^{m} a_{i2}a_{in} \\ \vdots & \vdots & \ddots & \vdots \\ n_{n1} = \sum_{i=1}^{m} a_{in}a_{i1} & n_{n2} = \sum_{i=1}^{m} a_{in}a_{i2} & \cdots & n_{nn} = \sum_{i=1}^{m} a_{in}^2 \end{array}$$

By comparing the summations above with those obtained in Equations (11.26), it should be clear that they are the same. Therefore, it is demonstrated that Equations (11.31a) and (11.31b) produce the normal equations for a least-squares adjustment. By inspection, it can also be seen that the normal matrix is always symmetric (i.e., $n_{ij} = n_{ji}$). In Chapter 28, methods are presented to take advantage of this fact.

By employing matrix algebra, the solution of normal equations like Equation (11.31*a*) is

$$X = (A^{\mathrm{T}}A)^{-1}A^{\mathrm{T}}L = N^{-1}A^{\mathrm{T}}L \tag{11.32}$$

Example 11.1 To demonstrate this procedure, the problem of Section 11.6 will be solved. Equations (11.28) can be expressed in matrix form as

$$AX = \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3.0 \\ 1.5 \\ 0.2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = L + V \tag{a}$$

Applying Equation (11.31) to Equation (*a*) yields

$$A^{\mathrm{T}}AX = NX = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \tag{b}$$

and

$$A^{\mathrm{T}}L = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 3.0 \\ 1.5 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 6.2 \\ 1.3 \end{bmatrix} \tag{c}$$

Finally, the adjusted unknowns, X matrix, are obtained using the matrix methods of Equation (11.32). These yield

$$X = N^{-1}A^{\mathrm{T}}L = \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 6.2 \\ 1.3 \end{bmatrix} = \begin{bmatrix} 1.514 \\ 1.442 \end{bmatrix} \qquad (d)$$

Notice that the normal equations and the solution in this method are the same as those obtained in Section 11.6.

11.9.2 Weighted Case

A system of weighted linear observation equations can be expressed in matrix notation as

$$WAX = WL + WV \qquad (11.33)$$

Using the same methods as demonstrated in Section 11.9.1, it is possible to show that the normal equations for this weighted system are

$$A^{\mathrm{T}}WAX = A^{\mathrm{T}}WL \qquad (11.34a)$$

Equation (11.34a) can also be expressed as

$$NX = A^{\mathrm{T}}WL \qquad (11.34b)$$

where $N = (A^{\mathrm{T}}WA)$ is the so-called normal matrix.

By employing matrix algebra, the least-squares solution of these weighted normal equations is

$$X = (A^{\mathrm{T}}WA)^{-1}A^{\mathrm{T}}WL = N^{-1}A^{\mathrm{T}}WL \qquad (11.35)$$

In Equation (11.35), W is the weight matrix as defined in Chapter 10.

11.10 LEAST-SQUARES SOLUTION OF NONLINEAR SYSTEMS



In Appendix C, a method is presented to solve a nonlinear system of equations using a first-order Taylor series approximation of the nonlinear equation. Following this procedure, the least-squares solution for a system of nonlinear equations can be found as follows:

Step 1: Write the first-order Taylor series approximation for each equation.

Step 2: Determine initial approximations for the unknowns in the equations of Step 1.

Step 3: Use matrix methods like those discussed in Section 11.9 to find the least-squares solution for the equations in Step 1 (these are corrections to the initial approximations).

Step 4: Apply the corrections to the initial approximations.

Step 5: Repeat Steps 3 through 4 until the corrections become sufficiently small.

A system of nonlinear equations that are linearized by a Taylor series approximation can be written as

$$JX = K + V \tag{11.36}$$

where the *Jacobian* matrix J contains the coefficients of the linearized observation equations. The individual matrices in Equation (11.36) are

$$J = \begin{bmatrix} \dfrac{\partial F_1}{\partial x_1} & \dfrac{\partial F_1}{\partial x_2} & \cdots & \dfrac{\partial F_1}{\partial x_n} \\ \dfrac{\partial F_2}{\partial x_1} & \dfrac{\partial F_2}{\partial x_2} & \cdots & \dfrac{\partial F_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial F_m}{\partial x_1} & \dfrac{\partial F_m}{\partial x_2} & \cdots & \dfrac{\partial F_m}{\partial x_n} \end{bmatrix}$$

$$X = \begin{bmatrix} dx_1 \\ dx_2 \\ \vdots \\ dx_n \end{bmatrix} \quad K = \begin{bmatrix} l_1 - F_1(x_1, x_2, \cdots x_n) \\ l_2 - F_2(x_1, x_2, \cdots x_n) \\ \vdots \\ l_m - F_m(x_1, x_2, \cdots x_n) \end{bmatrix} \quad V = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

The vector of least-squares corrections in the equally weighted system of Equation (11.36) is given by

$$X = (J^{\mathrm{T}}J)^{-1}J^{\mathrm{T}}K = N^{-1}J^{\mathrm{T}}K \tag{11.37}$$

Similarly, the system of weighted equations is

$$WJX = WK \tag{11.38}$$

and its solution is

$$X = (J^{\mathrm{T}}WJ)^{-1}J^{\mathrm{T}}WK = N^{-1}J^{\mathrm{T}}WK \tag{11.39}$$

where W is the weight matrix as defined in Chapter 10. Notice that the least-squares solution of a nonlinear system of equations is like the linear case. In fact, the only difference is the use of the Jacobian matrix J rather than the coefficient matrix A and the use of the K matrix rather than the observation matrix, L. Many authors use

the same nomenclature for both the linear and nonlinear cases. In these cases, the differences in the two systems of equations are stated implicitly.

Example 11.2 Find the least-squares solution for the following system of nonlinear equations:

$$\begin{aligned} F&: \; x + y - 2y^2 = -4 \\ G&: \; x^2 + y^2 \quad\;\; = 8 \\ H&: \; 3x^2 - y^2 \quad\;\; = 7.7 \end{aligned} \tag{e}$$

SOLUTION

Step 1: Determine the elements of the J matrix by taking partial derivatives of Equation (e) with respect to the unknowns x and y. Then write the first-order Taylor series equations:

$$\frac{\partial F}{\partial x} = 1 \qquad \frac{\partial G}{\partial x} = 2x \qquad \frac{\partial H}{\partial x} = 6x$$

$$\frac{\partial F}{\partial y} = 1 - 4y \qquad \frac{\partial G}{\partial y} = 2y \qquad \frac{\partial H}{\partial y} = -2y$$

$$JX = \begin{bmatrix} 1 & 1-4y_0 \\ 2x_0 & 2y_0 \\ 6x_0 & -2y_0 \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} -4 - F(x_0, y_0) \\ 8 - G(x_0, y_0) \\ 7.7 - H(x_0, y_0) \end{bmatrix} = K \tag{f}$$

Step 2: Determine initial approximations for the solution of the equations. Initial approximations can be derived by solving any two equations for x and y. This was done in Section C.3 for the equations for (F) and (G), and their solution was $x_0 = 2$ and $y_0 = 2$. Using these values, the evaluation of the equations yields

$$F(x_0, y_0) = -4 \quad G(x_0, y_0) = 8 \quad H(x_0, y_0) = 8 \tag{g}$$

Substituting Equations (g) into the K matrix of Equation (f), the K matrix becomes

$$K = \begin{bmatrix} -4 - (-4) \\ 8 - 8 \\ 7.7 - 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -0.3 \end{bmatrix}$$

It should not be surprising that the first two rows of the K matrix are zero since the initial approximations were determined using these two equations. In successive iterations, these values will change, and all terms will become nonzero.

Step 3: Solve the system using Equation (11.37):

$$N = J^{\mathrm{T}}J = \begin{bmatrix} 1 & 4 & 12 \\ -7 & 4 & -4 \end{bmatrix} \begin{bmatrix} 1 & -7 \\ 4 & 4 \\ 12 & -4 \end{bmatrix} = \begin{bmatrix} 161 & -39 \\ -39 & 81 \end{bmatrix} \qquad (h)$$

$$J^{\mathrm{T}}K = \begin{bmatrix} 1 & 4 & 12 \\ -7 & 4 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -0.3 \end{bmatrix} = \begin{bmatrix} -3.6 \\ 1.2 \end{bmatrix}$$

Substituting the matrices of Equation (*h*) into Equation (11.37), the solution for the first iteration is[1]

$$X = N^{-1}J^{\mathrm{T}}K = \begin{bmatrix} -0.02125 \\ 0.00458 \end{bmatrix}$$

Step 4: Apply the corrections to the initial approximations for the first iteration:

$$x_0 = 2.00 - 0.02125 = 1.97875 \qquad y_0 = 2.00 + 0.00458 = 2.00458$$

Step 5: Repeating steps 2 through 4 results in

$$X = N^{-1}J^{\mathrm{T}}K = \begin{bmatrix} 157.61806 & -38.75082 \\ -38.75082 & 81.40354 \end{bmatrix}^{-1} \begin{bmatrix} -0.017225 \\ -0.003307 \end{bmatrix} = \begin{bmatrix} -0.00011 \\ -0.00001 \end{bmatrix}$$

$$x = 1.97875 - 0.00011 = 1.97864$$

$$y = 2.00458 - 0.00001 = 2.00457$$

Iterating a third time yields extremely small corrections, and thus the final solution, rounded to the hundredths place, is $x = 1.98$ and $y = 2.00$. Notice that N changed by a relatively small amount from the first iteration to the second iteration. If the initial approximations are close to their final values, this can be expected. Thus, when doing these computations by hand, it is common to use the initial N for each iteration, making it necessary only to recompute $J^{\mathrm{T}}K$ between iterations. However, this procedure should be used with caution since if the initial approximations are poor, it will result in an incorrect solution. One should always perform complete computations when doing the solution with the aid of a computer.

[1]Note, although the solution represents more significant figures than can be warranted by the observations, it is important to carry more digits than are desired for the final solution. Failure to carry enough digits can result in a system that will never converge; rather it may *bounce* above and below the solution, or it may take more iterations due to these rounding errors. Many beginning students make this mistake. The answer should be rounded only after solving the problem.

11.11 LEAST-SQUARES FIT OF POINTS TO A LINE OR CURVE

Frequently in engineering work, it is desirable or necessary to fit a straight line or curve to a set of points with known coordinates. In solving this type of problem, it is necessary to decide on the appropriate functional model for the data. The decision on whether to use a straight line, parabola, or some other higher-order curve can generally be made after plotting the data and studying their form or by checking the size of the residuals after the least-squares solution with the first selected line or curve.

11.11.1 Fitting Data to a Straight Line

Consider the data illustrated in Figure 11.2. The straight line shown in the figure can be represented by this equation:

$$y = mx + b \tag{11.40}$$

In Equation (11.40), x and y are the coordinates of a point, m is the slope of a line, and b is the y intercept at $x = 0$. If the points were truly linear and there were no observational or experimental errors, all coordinates would lie on a straight line. However, this is rarely the case, as seen in Figure 11.2, and thus it is possible that (1) the points contain errors, (2) the functional model is a higher-order curve, or both. If a line is selected as the model for the data, the equation of the best-fitting straight line is found by adding residuals to Equation (11.40). This accounts for the errors shown in the figure. Observation equations for the four data points A, B, C, and D of Figure 11.2 are rewritten as

$$\begin{aligned} mx_A + b &= y_A + v_{y_A} \\ mx_B + b &= y_B + v_{y_B} \\ mx_C + b &= y_C + v_{y_C} \\ mx_D + b &= y_D + v_{y_D} \end{aligned} \tag{11.41}$$

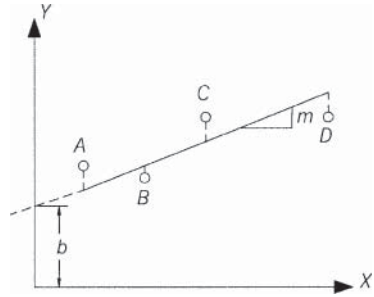


FIGURE 11.2 Fitting points on a line.

Equations (11.41) contain two unknowns, m and b, with four observations. Their matrix representation is

$$AX = L + V \tag{11.42}$$

where

$$A = \begin{bmatrix} x_A & 1 \\ x_B & 1 \\ x_C & 1 \\ x_D & 1 \end{bmatrix} \qquad X = \begin{bmatrix} m \\ b \end{bmatrix} \qquad L = \begin{bmatrix} y_A \\ y_B \\ y_C \\ y_D \end{bmatrix} \qquad V = \begin{bmatrix} v_{y_A} \\ v_{y_B} \\ v_{y_C} \\ v_{y_D} \end{bmatrix}$$

Equation (11.42) is solved by the least-squares method using Equation (11.32). If some data were more reliable than others, relative weights could be introduced and a weighted least-squares solution could be obtained using Equation (11.35).

Example 11.3 Find the best-fit straight line for the following points, whose x and y coordinates are given in parentheses.

A: (3.00, 4.50) B: (4.25, 4.25) C: (5.50, 5.50) D: (8.00, 5.50)

SOLUTION Following Equations (11.41), the four observation equations for the coordinate pairs are

$$\begin{aligned} 3.00m + b &= 4.50 + v_A \\ 4.25m + b &= 4.25 + v_B \\ 5.50m + b &= 5.50 + v_C \\ 8.00m + b &= 5.50 + v_D \end{aligned} \tag{i}$$

Rewriting Equations (i) into matrix form yields

$$\begin{bmatrix} 3.00 & 1 \\ 4.25 & 1 \\ 5.50 & 1 \\ 8.00 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 4.50 \\ 4.25 \\ 5.50 \\ 5.50 \end{bmatrix} + \begin{bmatrix} v_A \\ v_B \\ v_C \\ v_D \end{bmatrix} \tag{j}$$

To form the normal equations, premultiply matrices A and L of Equation (j) by A^{T} and get

$$\begin{bmatrix} 121.3125 & 20.7500 \\ 20.7500 & 4.0000 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 105.8125 \\ 19.7500 \end{bmatrix} \tag{k}$$

The solution of Equation (k) is

$$X = \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 121.3125 & 20.7500 \\ 20.7500 & 4.0000 \end{bmatrix}^{-1} \begin{bmatrix} 105.8125 \\ 19.7500 \end{bmatrix} = \begin{bmatrix} 0.246 \\ 3.663 \end{bmatrix}$$

Thus, the most probable values for m and b to the nearest hundredth are 0.25 and 3.66, respectively. To obtain the residuals, Equation (11.30) is rearranged into residual equations and solved as

$$V = AX - L = \begin{bmatrix} 3.00 & 1 \\ 4.25 & 1 \\ 5.50 & 1 \\ 8.00 & 1 \end{bmatrix} \begin{bmatrix} 0.246 \\ 3.663 \end{bmatrix} - \begin{bmatrix} 4.50 \\ 4.25 \\ 5.50 \\ 5.50 \end{bmatrix} = \begin{bmatrix} -0.10 \\ 0.46 \\ -0.48 \\ 0.13 \end{bmatrix}$$

11.11.2 Fitting Data to a Parabola

For certain data sets or in special situations, a parabola will fit the situation best. An example would be fitting a vertical curve to an existing roadbed. The general equation of a parabola is

$$Ax^2 + Bx + C = y \tag{11.43}$$

Again, since the data rarely fit the equation exactly, residuals are introduced. For the data shown in Figure 11.3, the following observation equations can be written:

$$\begin{aligned} Ax_a^2 + Bx_a + C &= y_a + v_a \\ Ax_b^2 + Bx_b + C &= y_b + v_b \\ Ax_c^2 + Bx_c + C &= y_c + v_c \\ Ax_d^2 + Bx_d + C &= y_d + v_d \\ Ax_e^2 + Bx_e + C &= y_e + v_e \end{aligned} \tag{11.44}$$

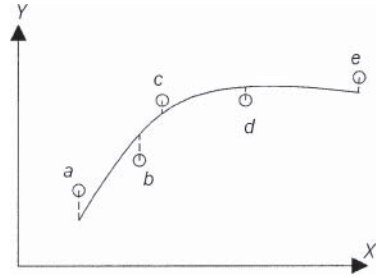


FIGURE 11.3 Fitting points on a parabolic curve.

Equations (11.44) contain three unknowns, A, B, and C, with five equations. Thus, this represents a redundant system that can be solved using least squares. In terms of the unknown coefficients, Equations (11.44) are linear and can be represented in matrix form as

$$AX = L + V \tag{11.45}$$

Since this is a linear system, it is solved using Equation (11.32). If weights were introduced, Equation (11.35) would be used. The steps taken would be similar to those used in Section 11.11.1.

11.12 CALIBRATION OF AN EDM INSTRUMENT

Calibration of an EDM is necessary to ensure confidence in the distances it measures. In calibrating these devices, if they internally make corrections and reductions for atmospheric conditions, Earth curvature, and slope, it is first necessary to determine if these corrections are made properly. Once these corrections are applied properly, the instruments with their reflectors must be checked to determine their constant and scaling corrections. This is often accomplished using a calibration baseline. The observation equation for an electronically observed distance on a calibration baseline is

$$SD_A + C = D_H - D_A + V_{DH} \tag{11.46}$$

In Equation (11.46), S is a scaling factor for the EDM; C is an instrument–reflector constant; D_H is the observed horizontal distance with all atmospheric and slope corrections applied; D_A is the published horizontal calibrated distance for the baseline; and V_{DH} is the residual error for each observation. This is a linear equation with two unknowns, S and C. Systems of these equations can be solved using Equation (11.31).

Example 11.4 A surveyor wishes to use an instrument–reflector combination that has an unknown constant value. Calibration baseline observations were carefully made, and following the manufacturer's recommendations, the necessary corrections were applied for the atmospheric conditions, Earth curvature, and slope. Use these corrected distances and their published values listed in Table 11.3

TABLE 11.3 EDM instrument–Reflector Calibration Data

Distance	D_A (meters)	D_H (meters)	Distance	D_A (meters)	D_H (meters)
0–150	149.9975	150.0175	150–0	149.9975	150.0174
0–430	430.0101	430.0302	430–0	430.0101	430.0304
0–1400	1400.0030	1400.0223	1400–0	1400.0030	1400.0221
150–430	280.0126	280.0327	430–150	280.0126	280.0331
150–1400	1250.0055	1250.0248	1400–150	1250.0055	1250.0257
430–1400	969.9929	970.0119	430–1400	969.9929	970.0125

to determine the instrument–reflector constant (C) and scaling factor (S) for the system.

SOLUTION Following Equation (11.46), the matrix equation for this problem is

$$\begin{bmatrix} 149.9975 & 1 \\ 149.9975 & 1 \\ 430.0101 & 1 \\ 430.0101 & 1 \\ 1400.0030 & 1 \\ 1400.0030 & 1 \\ 280.0126 & 1 \\ 280.0126 & 1 \\ 1250.0055 & 1 \\ 1250.0055 & 1 \\ 969.9929 & 1 \\ 969.9929 & 1 \end{bmatrix} \begin{bmatrix} S \\ C \end{bmatrix} = \begin{bmatrix} 150.0175 - 149.9975 \\ 150.0174 - 149.9975 \\ 430.0302 - 430.0101 \\ 430.0304 - 430.0101 \\ 1400.0223 - 1400.0030 \\ 1400.0221 - 1400.0030 \\ 280.0327 - 280.0126 \\ 280.0331 - 280.0126 \\ 1250.0248 - 1250.0055 \\ 1250.0257 - 1250.0055 \\ 970.0119 - 969.9929 \\ 970.0125 - 969.9929 \end{bmatrix} + V$$

Using Equation (11.32), the solution is $S = -0.0000007$ (-0.7 ppm) and $C = 0.0203$. Thus, the constant value for the instrument–reflector pair is approximately 0.020 m or 20 mm.

11.13 LEAST-SQUARES ADJUSTMENT USING CONDITIONAL EQUATIONS

As stated in Section 11.5, observations can also be adjusted using conditional equations. In this section, this form of adjustment is demonstrated by using the condition that the sum of the angles in the horizon at a single station must equal 360°.

Example 11.5 While observing angles at a station, the horizon was closed. The following observations and their standard deviations were obtained.

No.	Angle	S (″)
a_1	134°38′56″	±6.7
a_2	83°17′35″	±9.9
a_3	142°03′14″	±4.3

What are the most probable values for the observations above?

SOLUTION In a conditional adjustment, the most probable set of residuals is found to satisfy a given functional condition. In this case, the condition is that the

sum of the three angles is equal to 360°. Since the three observed angles sum to 359°59′45″, the angular misclosure is 15″. Thus, errors are present. The following residual equation is written for the observations listed above.

$$v_1 + v_2 + v_1 = 360^\circ - (a_1 + a_2 + a_3) = 15'' \qquad (l)$$

In Equation (*l*), the a_1, a_2, and a_3 represent the observations and v_1, v_2, and v_3 their residuals.

Applying the fundamental condition for a weighted least-squares adjustment, the following equation must be minimized:

$$F = w_1 v_1^2 + w_2 v_2^2 + w_3 v_3^2 \qquad (m)$$

where the w_1, w_2, and w_3 are weights, which are the inverses of the observations' variances.

Equation (*l*) can be rearranged such that v_3 is expressed as a function of the other two residuals:

$$v_3 = 15'' - (v_1 + v_2) \qquad (n)$$

Substituting Equation (*n*) into Equation (*m*) yields

$$F = w_1 v_1^2 + w_2 v_2^2 + w_3 [15'' - (v_1 + v_2)]^2 \qquad (o)$$

Taking the partial derivatives of F with respect to both v_1 and v_2, respectively, in Equation (*o*) results in the following two equations:

$$\begin{aligned} \frac{\partial F}{\partial v_1} &= 2w_1 v_1 + 2w_3 [15'' - (v_1 + v_2)](-1) = 0 \\ \frac{\partial F}{\partial v_2} &= 2w_2 v_2 + 2w_3 [15'' - (v_1 + v_2)](-1) = 0 \end{aligned} \qquad (p)$$

Rearranging Equations (*p*) and substituting in the appropriate weights yields the following normal equations.

$$\begin{aligned} \left(\frac{1}{6.7^2} + \frac{1}{4.3^2}\right) v_1 + \frac{1}{4.3^2} v_2 &= 15 \left(\frac{1}{4.3^2}\right) \\ \frac{1}{4.3^2} v_1 + \left(\frac{1}{9.9^2} + \frac{1}{4.3^2}\right) v_2 &= 15 \left(\frac{1}{4.3^2}\right) \end{aligned} \qquad (q)$$

Solving Equations (*q*) for v_1 and v_2 yields

$$v_1 = 4.2''$$

$$v_2 = 9.1''$$

By substituting these residual values into Equation (*n*), residual v_3 is computed as

$$v_3 = 15'' - (4.2'' + 9.1'') = 1.7''$$

Finally, the adjusted observations are obtained by adding the computed residuals to the observations.

No.	Observed Angle	v ($''$)	Adjusted Angle
a_1	134°38′56″	4.2	134°39′00.2″
a_2	83°17′35″	9.1	83°17′44.1″
a_3	142°03′14″	1.7	142°03′15.7″
			$\sum = 360°00'00.0''$

Note that geometric closure has been enforced in the adjusted angles to make their sum exactly 360°. Also, notice that the angle having the smallest standard deviation received the smallest correction (i.e., its residual is smallest).

11.14 THE PREVIOUS EXAMPLE USING OBSERVATION EQUATIONS

Example 11.5 can also be done using observation equations. In this case, the three observations are related to their adjusted values and their residuals by writing observation equations:

$$\begin{aligned} a_1 &= 134°38'56'' + v_1 \\ a_2 &= 83°17'35'' + v_2 \\ a_3 &= 142°03'14'' + v_3 \end{aligned} \qquad (r)$$

While these equations relate the adjusted observations to their observed values, they cannot be solved in this form. What is needed is the *geometric constraint*[2] that states that the sum of the three angles equals 360°. This can be represented in equation form as

$$a_1 + a_2 + a_3 = 360° \qquad (s)$$

Rearranging Equation (*s*) to solve for a_3 yields

$$a_3 = 360° - (a_1 + a_2) \qquad (t)$$

[2]Chapter 20 covers the use of constraint equations in a least squares adjustment. However, the fixing of control is also an example of a constraint. Constraints are typically required for an adjustment. In Chapter 20, the adjustment of control is discussed.

Substituting Equation (t) into Equations (r) produces

$$\begin{aligned} a_1 &= 134^\circ 38' 56'' + v_1 \\ a_2 &= 83^\circ 17' 35'' + v_2 \\ 360^\circ - (a_1 + a_2) &= 142^\circ 03' 14'' + v_3 \end{aligned} \qquad (u)$$

This is a linear problem, with two unknowns, a_1 and a_2. The weighted observation equation solution is obtained by solving Equation (11.35). The appropriate matrices for this problem are

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} \quad W = \begin{bmatrix} \frac{1}{6.7^2} & 0 & 0 \\ 0 & \frac{1}{9.9^2} & 0 \\ 0 & 0 & \frac{1}{4.3^2} \end{bmatrix} \quad L = \begin{bmatrix} 134^\circ 38' 56'' \\ 83^\circ 17' 35'' \\ 142^\circ 03' 14'' - 360^\circ \end{bmatrix}$$

Performing the matrix manipulations, the matrices for the normal equations are

$$A^{\mathrm{T}}WA = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{6.7^2} & 0 & 0 \\ 0 & \frac{1}{9.9^2} & 0 \\ 0 & 0 & \frac{1}{4.3^2} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 0.07636 & 0.05408 \\ 0.050408 & 0.06429 \end{bmatrix}$$

$$A^{\mathrm{T}}WL = \begin{bmatrix} 14.7867721 \\ 12.6370848 \end{bmatrix}$$

Finally, X is computed as

$$X = (A^{\mathrm{T}}WA)^{-1}A^{\mathrm{T}}WL = \begin{bmatrix} 134^\circ 39' 00.2'' \\ 83^\circ 17' 44.1'' \end{bmatrix}$$

Using Equation (t), it can now be determined that a_3 is $360^\circ - 134^\circ 39' 00.2'' - 83^\circ 17' 44.1'' = 142^\circ 03' 15.7''$. The same results are obtained as in Section 11.13. It is important to note that no matter what method of least-squares adjustment is used, if the procedures are performed properly, the same solution will always be obtained. This example involved a *constraint equation* (t). This topic is covered in more detail in Chapter 20.

11.15 SOFTWARE

As was initially stated, the method of least squares was not commonly used due to its computational intensiveness. Today, software has eliminated this hindrance. On the companion website (http://www.wiley.com/college/ghilani), the spreadsheet in the file *Chapter 11.xls* demonstrates the least-squares solution to the example problems in this chapter. It uses several of the techniques discussed in Section 8.6 to manipulate the matrices. Additionally, on the companion website is the Mathcad® worksheet *C11.xmcd*, which demonstrates the programming of all the examples in this chapter. These examples should be studied by the reader. Similar programming can be used to solve the problems at the end of this chapter. For those who wish to create a more robust program, a higher-level programming language can be used.

Both spreadsheet and Mathcad® software are available for the reader in many of the remaining chapters. The spreadsheet software files are named as *Chapter##.xls*, where ## represents the chapter number. The Mathcad® worksheets are similarly named after their representative chapter. These files demonstrate some of the programming techniques that are used to solve the example problems in this book. Readers are encouraged to investigate these files while studying the subject material in this book.

PROBLEMS

Note: Partial answers to problems marked with an asterisk can be found in Appendix G.

***11.1** Calculate the most probable values for A and B in the equations below by the method of least squares to three decimal places. Consider the observations to be of equal weight. (Use the tabular method to form normal equations.)

(a) $4A + 5B = 18.2 + v_1$

(b) $-3A + 2B = -3.3 + v_2$

(c) $5A - 2B = 8.1 + v_3$

11.2 If observations a, b, and c in Problem 11.1 have weights of 4, 4, and 1, respectively, solve the equations for the most probable values of A and B using weighted least squares. (Use the tabular method to form normal equations.)

11.3 Repeat Problem 11.1 using matrices.

11.4 Repeat Problem 11.2 using matrices.

11.5 Calculate the most probable values of X and Y for the following system of equations using:

(a) Tabular method.

(b) Matrix method.

$$3X + 4Y = 20.0 + v_1$$

$$4X - 2Y = 4.0 + v_2$$

$$2X - 3Y = 4.0 + v_3$$

***11.6** What are the residuals for the system of equations in Problem 11.5?

***11.7** Solve the following nonlinear equations using the least-squares method. Use initial approximations of $x_0 = 2.1$ and $y_0 = 0.45$.

$$3x^2 + 3xy - y^2 = 7.0 + v_1$$
$$7x^3 - 3y^2 = 55.2 + v_2$$
$$2x^2 - 6xy + 3y^2 = -1.2 + v_3$$

11.8 What are the residuals for the system of equations in Problem 11.7?

11.9 Repeat Problem 11.7 for the following system of nonlinear equations using initial approximations of $x_0 = 3.5$ and $y_0 = -2.3$:

$$1: x^2 + 2y = 7.5 + v_1$$
$$2: -4x + 6y^2 = 17.5 + v_2$$
$$3: 3x^2 + 4xy + 5y^2 = 31 + v_3$$

11.10 If observations 1, 2, and 3 in Problem 11.9 have weights of 2, 4, and 6, respectively, solve the equations for:

(a) the most probable values of x and y.

(b) the residuals.

11.11 The following coordinates of points on a line were computed for a block. What are the slope and y-intercept of the line? What is the azimuth of the line?

Point	X (ft)	Y (ft)
1	10,320.54	14,820.50
2	10,411.64	14,911.66
3	10,525.68	15,025.68
4	10,634.28	15,134.27

***11.12** Using the conditional equations, what are the most probable values for the three angles observed to close the horizon at station Red. The observed values and their standard deviations are

Angle	Value	S (″)
1	114°23′05″	±6.5
2	138°17′59″	±3.5
3	107°19′03″	±4.9

11.13 Do Problem 11.12 using the observation equation method.

11.14 Using the conditional equation method, what are the most probable values for the three interior angles of a triangle that were measured as:

Angle	Value	S (″)
1	58°26′48″	±5.1
2	67°06′56″	±4.3
3	54°26′24″	±2.6

11.15 What are the most probable values for the three angles observed to close the horizon at station Red? The observed values and their standard deviations are:

Angle	Value	S (″)
1	123°32′56″	2.5
2	110°07′28″	1.5
3	126°19′44″	4.9

11.16 A block of a street is to be reconstructed. The coordinates of stations *A* through *I* are listed below. How much deviation from a straight line is found?

Station	X (ft)	Y (ft)
A	300.00	953.16
B	349.99	945.35
C	400.00	937.55
D	449.98	929.73
E	500.01	921.95
F	550.00	914.13
G	599.98	906.31
H	650.00	898.52
I	700.01	890.72

11.17 Use the ADJUST software to do Problem 11.16.

11.18 The property corners on a single block with an alley are shown as a straight line with a Due East bearing on a recorded plat. During a recent survey, all the lot corners were found, and measurements from Station A to each were obtained. The surveyor wants to determine the possibility of disturbance of the corners by checking their fit to a straight line. A sketch of the situation is shown in Figure P11.18, and the results of the survey are given below. Assuming Station A has coordinates of $X = 5000.00$ and $Y = 5000.00$, and that the coordinates of the backsight station are $X = 5000.10$ and $Y = 5200.00$, determine the best-fitting line for the corners. Give the Y intercept and the azimuth of the best-fit line.

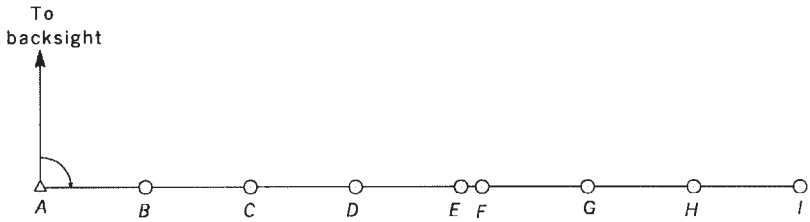


FIGURE P11.18

Course	Distance (ft)	Angle at A
AB	100.02	90°00′16″
AC	200.12	90°00′08″
AD	300.08	89°59′48″
AE	399.96	90°01′02″
AF	419.94	89°59′48″
AG	519.99	90°00′20″
AH	620.04	89°59′36″
AI	720.08	90°00′06″

11.19 Use the ADJUST software to solve Problem 11.18.

11.20 Discuss the importance of selecting the stochastic model when adjusting data.

11.21 The values for three angles in a triangle, observed using a total station, are

Angle	Number of Repetitions	Value
A	2	14°25′20″
B	4	58°16′00″
C	4	107°19′10″

The observed lengths of the course are

$$AB = 971.25 \text{ ft}; BC = 253.25 \text{ ft}; CA = 865.28 \text{ ft}$$

The following estimated errors are assumed for each measurement:

$$\sigma_i = \pm 0.005 \text{ ft}; \sigma_t = \pm 0.005 \text{ ft}; \sigma_{ISO} = \pm 2''$$

What are the most probable values for the angles? Use the condition equation method.

11.22 Do Problem 11.21 using observation equations and a constraint, as presented in Section 11.13.

11.23 The following data were collected on a calibration baseline. Atmospheric refraction and Earth curvature corrections were made to the measured distances, which are in units of meters. Determine the instrument/reflector constant and any scaling factor.

Distance	D_A	D_H	Distance	D_A	D_H
0–150	149.9104	149.9447	150–0	149.9104	149.9435
0–430	430.0010	430.0334	430–0	430.0010	430.0340
0–1400	1399.9313	1399.9777	1400–0	1399.9313	1399.9519
150–430	280.0906	208.1238	430–150	280.0906	280.1230
150–1400	1250.0209	1250.0795	1400–150	1250.0209	1250.0664
430–1400	969.9303	969.9546	1400–430	969.9303	969.9630

11.24 A survey of the centerline of a horizontal curve is done to determine the as-built curve specifications. The coordinates for the points along the curve are:

Point	X (ft)	Y (ft)
1	9,821.68	9,775.84
2	9,876.40	9,842.74
3	9,975.42	9,955.42
4	10,079.50	10,063.40
5	10,151.60	10,132.68

(a) Using Equation (C.10), compute the most probable values for the radius and center of the circle.

(b) If two points located on the tangents have coordinates of (9761.90, 9700.66) and (10,277.88, 10,245.84), what are the coordinates of the *PC* and *PT* of the curve?

CHAPTER 12

ADJUSTMENT OF LEVEL NETS

12.1 INTRODUCTION

Differential leveling observations are used to determine differences in elevations between stations. As with all observations, these measurements are subject to random errors that can be adjusted using the least-squares method. In this chapter, the observation equation method for adjusting differential leveling observations by least squares is developed, and several examples are given to illustrate the adjustment procedures.

12.2 OBSERVATION EQUATION

 To apply the method of least squares in leveling adjustments, a prototype observation equation is first written for any elevation difference. Figure 12.1 illustrates the functional relationship for an observed elevation difference between two stations I and J. The equation is expressed as:

$$E_j - E_i = \Delta Elev_{ij} + v_{\Delta Elev_{ij}} \tag{12.1}$$

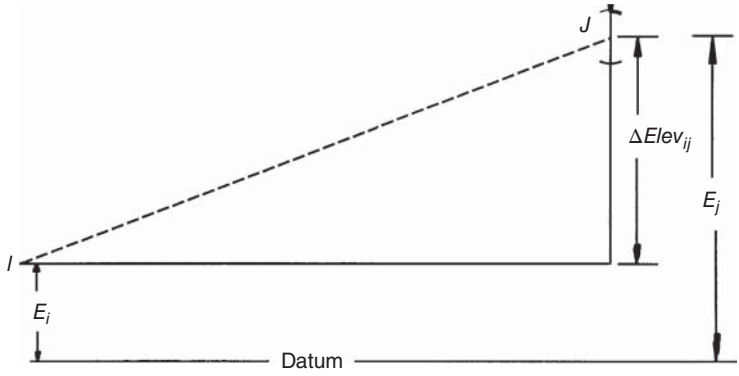
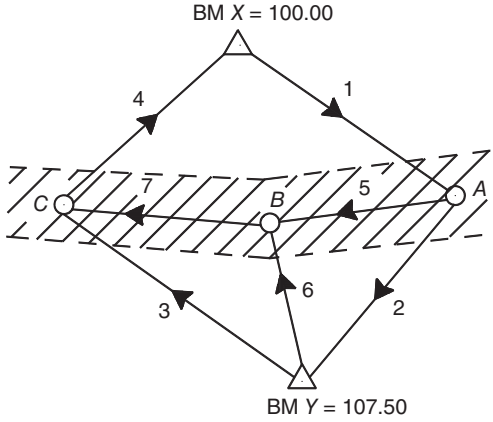


FIGURE 12.1 A differential leveling observation.

This prototype observation equation relates the unknown elevations of any two stations, I and J, with the differential leveling observation $\Delta Elev_{ij}$ and its residual $\Delta v_{Elev_{ij}}$. This equation is fundamental in performing least-squares adjustments of differential level nets.

12.3 UNWEIGHTED EXAMPLE

In Figure 12.2, a leveling network and its survey data are shown. Assume that all observations are equal in weight. In the figure, arrows indicate the direction of leveling, and thus, for line 1, leveling proceeds from benchmark X to A with an observed elevation difference of +5.10 ft. By substituting it into prototype



Line	Observed Elevation Difference
1	5.10
2	2.34
3	–1.25
4	–6.13
5	–0.68
6	–3.00
7	1.70

FIGURE 12.2 Interlocking leveling network.

Equation (12.1), an observation equation is written for each observation in Figure 12.2. The resulting equations are

$$\begin{array}{rrrrrrrr} A & & & -BMX = & 5.10 & + v_1 \\ -A & & & +BMX = & 2.34 & + v_2 \\ & & C & -BMX = & -1.25 & + v_3 \\ & & -C & +BMX = & -6.13 & + v_4 \\ -A & B & & = & -0.68 & + v_5 \\ & B & & -BMY = & -3.00 & + v_6 \\ & -B & C & = & 1.70 & + v_7 \end{array} \tag{12.2}$$

Rearranging so that the known benchmarks are on the right side of the equations and substituting in their elevations yields

$$\begin{array}{rrrrrr} A & & = & +105.10 & + v_1 \\ -A & & = & -105.16 & + v_2 \\ & & C = & +106.25 & + v_3 \\ & & -C = & -106.13 & + v_4 \\ -A & +B & = & -0.68 & + v_5 \\ & B & = & +104.50 & + v_6 \\ & -B & C = & 1.70 & + v_7 \end{array} \tag{12.3}$$

In this example, there are three unknowns, A, B, and C. In matrix form, Equations (12.2) are written as

$$AX + B = L + V \tag{12.4a}$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad X = \begin{bmatrix} A \\ B \\ C \end{bmatrix} \quad B = \begin{bmatrix} -100.00 \\ +107.50 \\ -107.50 \\ 100.00 \\ 0 \\ -107.50 \\ 0 \end{bmatrix} \quad L = \begin{bmatrix} 5.10 \\ 2.34 \\ -1.25 \\ -6.13 \\ -0.68 \\ -3.00 \\ 1.70 \end{bmatrix} \quad V = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{bmatrix}$$

In Equation (12.4a), the B matrix is a vector of the constants (benchmarks) collected from the left side of the equation, and L is a collection of leveling observations. The right side of the Equation (12.3) is equal to $L - B$. It is a collection of the constants in the observation equations and is often referred to as the constants matrix, L. Since the benchmarks can also be thought of as observations, this

combination of benchmarks and leveling observation is referred to as L in this book, and Equation (12.4a) is simplified as

$$AX = L + V \tag{12.4b}$$

Also note in the A matrix that when an unknown does not appear in an equation, its coefficient is zero. Since this is an unweighted example, according to Equations (11.31a) and (11.31b), the normal equations are

$$(A^{\mathrm{T}}A) = NX = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} \text{ and } A^{\mathrm{T}}L = \begin{bmatrix} 210.94 \\ 102.12 \\ 214.08 \end{bmatrix} \tag{12.5}$$

Using Equation (11.32), the solution of Equation (12.5) is

$$X = N^{-1}A^{\mathrm{T}}L = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 210.94 \\ 102.12 \\ 214.08 \end{bmatrix}$$

$$= \begin{bmatrix} 0.38095 & 0.14286 & 0.04762 \\ 0.14286 & 0.42857 & 0.14286 \\ 0.04762 & 0.14286 & 0.38095 \end{bmatrix} \begin{bmatrix} 210.94 \\ 102.12 \\ 214.08 \end{bmatrix} = \begin{bmatrix} 105.14 \\ 104.48 \\ 106.19 \end{bmatrix} \tag{12.6}$$

From Equation (12.6), the most probable elevations for A, B, and C are 105.14, 104.48, and 106.19, respectively. The rearranged form of Equation (12.4b) is used to compute the residuals as

$$V = AX - L \tag{12.7}$$

From Equation (12.7), the matrix solution for V is

$$V = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 105.141 \\ 104.483 \\ 106.188 \end{bmatrix} - \begin{bmatrix} 105.10 \\ -105.16 \\ 106.25 \\ -106.13 \\ -0.68 \\ 104.50 \\ 1.70 \end{bmatrix} = \begin{bmatrix} 0.041 \\ 0.019 \\ -0.062 \\ -0.058 \\ 0.022 \\ -0.017 \\ 0.005 \end{bmatrix}$$

12.4 WEIGHTED EXAMPLE

 In Section 10.6, it was shown that relative weights for adjusting level lines are inversely proportional to the lengths of the lines:

$$w = \frac{1}{\text{length}} \tag{12.8}$$

TABLE 12.1 Weights for Example in Section 12.2

Line	Length (mi)	Relative Weights
1	4	12/4 = 3
2	3	12/3 = 4
3	2	12/2 = 6
4	3	12/3 = 4
5	2	12/2 = 6
6	2	12/2 = 6
7	2	12/2 = 6

The application of weights to the least-squares adjustment of the level circuit is illustrated by including the variable line lengths for the unweighted example of Section 12.3. These line lengths for the leveling network of Figure 12.2 and their corresponding relative weights are given in Table 12.1. For convenience, each length is divided into the constant 12, so that integer *relative weights* are obtained. (Note that this is an unnecessary step in the adjustment.) The observation equations are now formed as in Section 12.3, except that in the weighted case, each equation is multiplied by its weight:

$$
\begin{aligned}
w_1\,(A\qquad\qquad\quad) &= w_1\,(+105.10) + w_1 v_1\\
w_2\,(-A\qquad\qquad\quad) &= w_2\,(-105.16) + w_2 v_2\\
w_3\,(\qquad\qquad C) &= w_3\,(+106.25) + w_3 v_3\\
w_4\,(\qquad\quad -C) &= w_4\,(-106.13) + w_4 v_4\\
w_5\,(-A\ +B\qquad\quad) &= w_5\quad(-0.68) + w_5 v_5\\
w_6\,(\qquad B\qquad\quad) &= w_6\,(+104.50) + w_6 v_6\\
w_7\,(\qquad -B\ +C) &= w_7\quad(+1.70) + w_7 v_7
\end{aligned}
\tag{12.9}
$$

After dropping the residual terms in Equation (12.9), they can be written in terms of matrices as:

$$
\begin{bmatrix}
3 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 4 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 6 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 4 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 6 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 6 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 6
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0\\
-1 & 0 & 0\\
0 & 0 & 1\\
0 & 0 & -1\\
-1 & 1 & 0\\
0 & 1 & 0\\
0 & -1 & 0
\end{bmatrix}
\begin{bmatrix} A\\ B\\ C \end{bmatrix}
=
\begin{bmatrix}
3 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 4 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 6 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 4 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 6 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 6 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 6
\end{bmatrix}
\begin{bmatrix}
105.10\\ -105.16\\ 106.25\\ -106.13\\ -0.68\\ 104.50\\ 1.70
\end{bmatrix}
\tag{12.10}
$$

Applying Equation (11.34), we find that the normal equations are

$$(A^{\mathrm{T}}WA)X = NX = A^{\mathrm{T}}WL \tag{12.11}$$

where

$$N = \begin{bmatrix} 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & -6 & 0 \\ -6 & 18 & -6 \\ 0 & -6 & 16 \end{bmatrix}$$

$$A^{\mathrm{T}}WL = \begin{bmatrix} 740.02 \\ 612.72 \\ 1072.22 \end{bmatrix}$$

By using Equation (11.35), the solution for the X matrix is

$$X = N^{-1}(A^{\mathrm{T}}WL) = \begin{bmatrix} 0.0933 & 0.0355 & 0.0133 \\ 0.0355 & 0.0770 & 0.0289 \\ 0.0133 & 0.0289 & 0.0733 \end{bmatrix} \begin{bmatrix} 740.02 \\ 612.72 \\ 1072.22 \end{bmatrix} = \begin{bmatrix} 105.150 \\ 104.489 \\ 106.197 \end{bmatrix} \tag{12.12}$$

The residual equation [Equation (12.7)] is now applied to compute the residuals as:

$$V = AX - L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 105.150 \\ 104.489 \\ 106.197 \end{bmatrix} - \begin{bmatrix} 105.10 \\ -105.16 \\ 106.25 \\ -106.13 \\ -0.68 \\ 104.50 \\ 1.70 \end{bmatrix} = \begin{bmatrix} 0.050 \\ 0.010 \\ -0.053 \\ -0.067 \\ 0.019 \\ -0.011 \\ 0.008 \end{bmatrix}$$

It should be noted that these adjusted values (X matrix) and residuals (V matrix) differ slightly from those obtained in the unweighted adjustment of Section 12.3. This illustrates the effect of weights in an adjustment. Although the differences in this example are small, for precise level circuits, it is both logical and wise to use a weighted adjustment since a correct stochastic model will place the errors back in the observations that most likely produced the errors.

12.5 REFERENCE STANDARD DEVIATION

Equation (10.20) expressed the standard deviation for a weighted set of observations as:

$$S_0 = \sqrt{\frac{\sum wv^2}{n-1}} \tag{12.13}$$

However, Equation (12.13) applies to a set of multiple observations for a single quantity where each observation has a different weight. Often, observations are obtained that involve several unknown parameters that are related functionally like those in Equations (12.3) or (12.9). For these types of observations, the standard deviation in the unweighted case is

$$S_0 = \sqrt{\frac{\sum v^2}{m-n}} = \sqrt{\frac{\sum v^2}{r}}, \text{ which in matrix form is } S_0 = \sqrt{\frac{V^{\mathrm{T}}V}{r}} \tag{12.14}$$

In Equation (12.14), $\sum v^2$ is expressed in matrix form as $V^{\mathrm{T}}V$, m the number of observations, and n the number of unknowns. There are $r = m - n$ redundant measurements or degrees of freedom.

The standard deviation for the weighted case is

$$S_0 = \sqrt{\frac{\sum wv^2}{m-n}} = \sqrt{\frac{\sum wv^2}{r}}, \text{ which in matrix form is } S_0 = \sqrt{\frac{V^{\mathrm{T}}WV}{r}} \tag{12.15}$$

where $\sum wv^2$ in matrix form is $V^{\mathrm{T}}WV$.

Since these standard deviations relate to the overall adjustment and not a single quantity, they are referred to as *reference standard deviations*. Computations of the reference standard deviations for both unweighted and weighted examples are illustrated below.

12.5.1 Unweighted Example

In the example of Section 12.3, there are $7-3$, or 4 degrees of freedom. Using the residuals given in Equation (12.7) and using Equation (12.14), the reference standard deviation in the unweighted example is

$$S_0 = \sqrt{\frac{(0.041)^2 + (0.019)^2 + (-0.062)^2 + (-0.058)^2 + (0.022)^2 + (-0.017)^2 + (0.005)^2}{7-3}}$$
$$= \pm 0.05 \tag{12.16}$$

This can be computed using the matrix expression in Equation (12.14) as:

$$V^{\mathrm{T}}V = [0.041 \;\; 0.019 \; -0.062 \; -0.058 \;\; 0.022 \; -0.017 \;\; 0.005] \begin{bmatrix} 0.041 \\ 0.019 \\ -0.062 \\ -0.058 \\ 0.022 \\ -0.017 \\ 0.005 \end{bmatrix} = 0.010$$

$$S_0 = \sqrt{\frac{V^{\mathrm{T}}V}{r}}$$

$$= \sqrt{\frac{0.010}{4}} = \pm 0.05 \tag{12.17}$$

12.5.2 Weighted Example

Notice that the weights are used when computing the reference standard deviation in Equation (12.15). That is, each residual is squared and multiplied by its weight, and thus the reference standard deviation computed using nonmatrix methods is

$$S_0 = \sqrt{\frac{V^{\mathrm{T}}WV}{m-n}} = \sqrt{\frac{\sum wv^2}{m-n}}$$

$$= \sqrt{\frac{3(0.050)^2 + 4(0.010)^2 + 6(-0.053)^2 + 4(-0.067)^2 + 6(0.019)^2 + 6(-0.011)^2 + 6(0.008)^2}{7-3}}$$

$$= \sqrt{\frac{0.04598}{4}} = \pm 0.107 \tag{12.18}$$

It is left as an exercise to verify this result by solving the matrix expression of Equation (12.18).

12.6 ANOTHER WEIGHTED ADJUSTMENT

Example 12.1 The level net shown in Figure 12.3 is observed with the following results (the elevation differences and standard deviations are given in meters, and the elevation of A is 437.596 m):

From	To	ΔElev (m)	σ (m)	From	To	ΔElev (m)	σ (m)
A	B	10.509	0.006	D	A	−7.348	0.003
B	C	5.360	0.004	B	D	−3.167	0.004
C	D	−8.523	0.005	A	C	15.881	0.012

What are the most probable values for the elevations of B, C, and D?

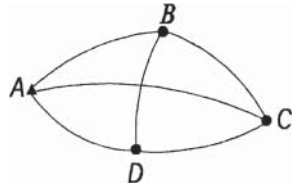


FIGURE 12.3 Differential leveling network for Example 12.1.

SOLUTION

Step 1: Write the observation equations without their weights:

$$\begin{array}{llllll}
(1) & +B & & = & A + 10.509 + v_1 = & 448.105 + v_1 \\
(2) & -B & +C & = & & 5.360 + v_2 \\
(3) & & -C & +D = & & -8.523 + v_3 \\
(4) & & & -D = & -A - 7.348 + v_4 = & -444.944 + v_4 \\
(5) & -B & & +D = & & -3.167 + v_5 \\
(6) & & +C & = & A + 15.881 + v_6 = & 453.477 + v_6
\end{array}$$

Step 2: Rewrite observation equations in matrix form $AX = L + V$ as:

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 448.105 \\ 5.360 \\ -8.523 \\ -444.944 \\ -3.167 \\ 453.477 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{bmatrix} \tag{12.19}$$

Step 3: In accordance with Equations (11.4) and (11.6), form the weight matrix as:

$$W = \begin{bmatrix} \frac{1}{0.006^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{0.004^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{0.005^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{0.003^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{0.004^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{0.012^2} \end{bmatrix} \tag{12.20}$$

From which

$$W = \begin{bmatrix} 27,778 & 0 & 0 & 0 & 0 & 0 \\ 0 & 62,500 & 0 & 0 & 0 & 0 \\ 0 & 0 & 40,000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 111,111 & 0 & 0 \\ 0 & 0 & 0 & 0 & 62,500 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6{,}944 \end{bmatrix} \tag{12.21}$$

Step 4: Compute the normal equations using Equation (11.34)

$$(A^{\mathrm{T}}WA)X = NX = A^{\mathrm{T}}WL \tag{12.22}$$

where

$$N = \begin{bmatrix} 152,778 & -62,500 & -62,500 \\ -62,500 & 109,444 & -40,000 \\ -62,500 & -40,000 & 213,611 \end{bmatrix} \quad X = \begin{bmatrix} B \\ C \\ D \end{bmatrix}$$

$$A^{\mathrm{T}}WL = \begin{bmatrix} 12,310,298.611 \\ 3,825,065.833 \\ 48,899,364.722 \end{bmatrix}$$

Step 5: Solve for the X matrix using Equation (11.35) yields

$$X = \begin{bmatrix} 448.1087 \\ 453.4685 \\ 444.9436 \end{bmatrix} \tag{12.23}$$

Step 6: Compute the residuals using the matrix expression $V = AX - L$.

$$V = \begin{bmatrix} 448.1087 \\ 5.3598 \\ -8.5249 \\ -444.9436 \\ -3.1651 \\ 453.4685 \end{bmatrix} - \begin{bmatrix} 448.105 \\ 5.360 \\ -8.523 \\ -444.944 \\ -3.167 \\ 453.477 \end{bmatrix} = \begin{bmatrix} 0.0037 \\ -0.0002 \\ -0.0019 \\ 0.0004 \\ 0.0019 \\ -0.0085 \end{bmatrix} \tag{12.24}$$

Step 7: Calculate the reference standard deviation for the adjustment using the matrix expression of Equation (12.15).

$$V^{\mathrm{T}}WV = \begin{bmatrix} 0.0037 & -0.0002 & -0.0019 & 0.0004 & 0.0019 & -0.0085 \end{bmatrix} \times W \begin{bmatrix} 0.0037 \\ -0.0002 \\ -0.0019 \\ 0.0004 \\ 0.0019 \\ -0.0085 \end{bmatrix} = [1.26976] \tag{12.25}$$

Since the number of system redundancies is the number of observations minus the number of unknowns, $r = 6 - 3 = 3$, and thus

$$S_0 = \sqrt{\frac{1.26976}{3}} = \pm 0.6575 \tag{12.26}$$

Step 8: Tabulate the results showing the adjusted elevation differences, their residuals, and final adjusted elevations.

From	To	Adjusted ΔElevation	Residual	Station	Adjusted Elevation
A	*B*	10.513	0.004	*A*	437.596
B	*C*	5.360	0.000	*B*	448.109
C	*D*	−8.525	−0.002	*C*	453.468
D	*A*	−7.348	0.000	*D*	444.944
B	*D*	−3.165	0.002		
A	*C*	15.872	−0.009		

12.7 SOFTWARE

The example files presented in this chapter are solved in spreadsheet format in the file *Chapter 12.xls*, which is available on the companion website for this book. This file demonstrates how least-squares solutions can be solved in a spreadsheet. Additionally, for Example 12.1 an example of the data file created in the spreadsheet is developed for copying to the MATRIX software, which is also available on the website. This file is shown in Figure 12.4. All the examples shown in this chapter are solved in the Mathcad® file *C12.xmcd*, which is available on the companion website. This file demonstrates how to solve differential leveling problems using a higher-level programming language. For Example 12.1, the file shows how to read a differential leveling and create formatted results in Mathcad®.

```
Example 12.1
A
6       3
1       0       0
-1      1       0
0       -1      1
0       0       -1
-1      0       1
0       1       0
W
6       6
27777.77778     0       0       0       0       0
0       62500   0       0       0       0
0       0       40000   0       0       0
0       0       0       111111.1111     0       0
0       0       0       0       62500   0
0       0       0       0       0       6944.444444
L
6       1
448.105
5.360
-8.523
-444.944
-3.167
453.477
```

FIGURE 12.4 MATRIX file for Example 12.1.

Furthermore, files for solving Example 12.1 in MATRIX and ADJUST are available on the companion website. The data files for both programs are simple text files with specific formats, which can be created with the editors supplied in each program. The data file for MATRIX is called *Matrix file for Example 12-1.Mdat* and is shown in Figure 12.4. It has the following format. The first line is a title line and can contain a description of the file contents. The second line contains the name of the first matrix, which is *A* in this file. This is followed by the dimensions of the matrix, then each row of the *A* matrix. Following the entry of the *A* matrix, the weight matrix *W*, and the constants matrix *L* are entered in a similar fashion. A space delimits each entry in the file. These entries can also be delimited by a tab or a comma. The tab is a common delimiter for data copied from a spreadsheet. This feature allows for the quick solution of least-squares problems.

Figure 12.5 shows the format for ADJUST file used to solve Example 12.1. The format of this file is a file description line, followed by a line containing the

```
Example 12.1
1 6 4
A 437.596
A B 10.509 0.006
B C 5.360 0.004
C D -8.523 0.005
D A -7.348 0.003
B D -3.167 0.004
A C 15.881 0.012
```

FIGURE 12.5 ADJUST data file for Example 12.1.

number of control benchmarks (1), elevation differences (6), and the total number of stations (4) in the file. Following these lines are lines containing the control benchmark identifiers and elevations. All station identifiers in this file can have 10 alphanumeric characters, but may not contain a space, comma, or tab, which are reserved as text delimiters between entries in the file. Following the entry of control benchmarks, the observed elevation differences are listed. Each elevation difference is entered as *I J* ΔElev, which is the *from* station, *to* station, and observed difference in elevation. When the standard deviation, number of setups between stations, or distance between stations is known, it can be entered at the end of each observation's line for use in a weighted adjustment. Once this file is saved, the differential leveling least-squares adjustment option can be run from the *programs* menu of ADJUST. As shown in Figure 12.6, you can select the appropriate adjustment options for the data. If distances or the number of setups is used to develop a stochastic model in a weighted adjustment, the first option in Figure 12.6 should be selected. If standard deviations or an unweighted adjustment is desired, the first option should not be selected. Many of the remaining options for this program will be discussed in later chapters. However, if the *print matrices* option is selected, the software will create a separate file having a *mat* extension, which will contain several matrices, including the *A* and *L*. These matrices can be used to check those developed by the reader in the solution of a problem.

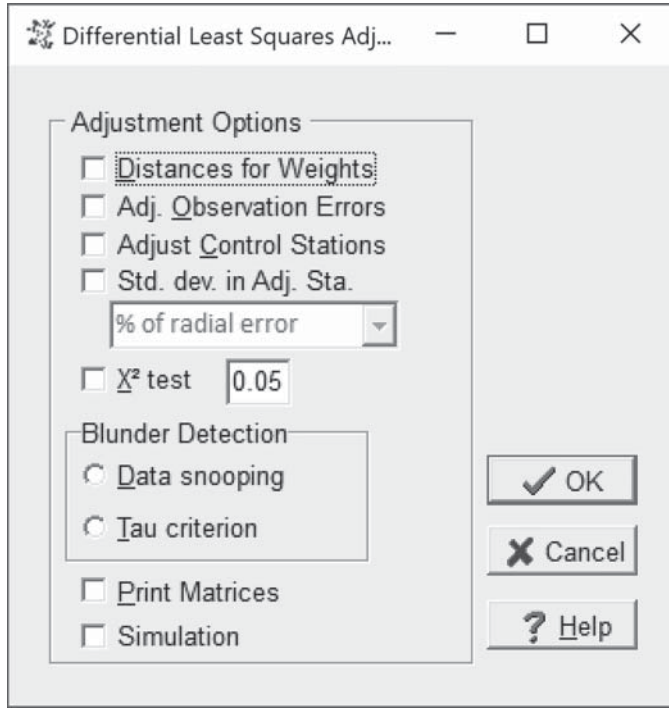


FIGURE 12.6 Differential leveling least-squares options in ADJUST.

PROBLEMS

Note: For problems requiring least-squares adjustment, if a computer program is not distinctly specified for use in the problem, it is expected that the least-squares algorithm will be solved using the program MATRIX, which is available on the companion website for this book. Partial answers to problems marked with an asterisk can be found in Appendix G.

***12.1** For the leveling network in the accompanying figure, calculate the most probable elevations for X and Y. Use an unweighted least-squares adjustment with the observed values given in the accompanying table. Assume units of feet.

Line	$\Delta Elev$ (ft)
1	+8.55
2	+6.95
3	+2.31
4	+2.80
5	+4.80

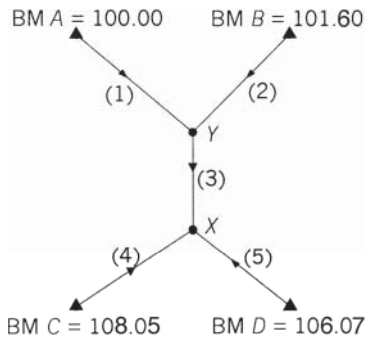


FIGURE P12.1.

***12.2** For Problem 12.1, compute the reference standard deviation and tabulate the adjusted observations and their residuals.

12.3 Repeat Problem 12.1 using ADJUST.

12.4 For the leveling network shown in the accompanying figure, calculate the most probable elevations for X, Y, and Z. The observed values and line lengths are given in the table. Apply appropriate weights in the computations.

Line	$\Delta Elev$ (ft)	Length (ft)
1	+43.23	3500
2	−34.32	3500
3	+35.86	2500
4	−27.00	2500
5	−8.94	3000
6	8.94	3000

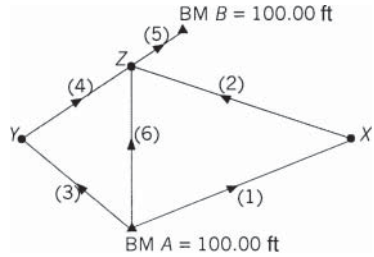


FIGURE P12.4

12.5 For Problem 12.4, compute the reference standard deviation, and tabulate the adjusted observations and their residuals.

12.6 Repeat Problem 12.4 and 12.5 using the following standard deviations for the lines.

Line	S (ft)
1	±0.060
2	±0.060
3	±0.051
4	±0.051
5	±0.024
6	±0.057

(a) Tabulate the most probable values for the adjusted benchmarks.

(b) Tabulate the adjusted observations and their residuals.

(c) What is the reference standard deviation for the adjustment?

12.7 Repeat Problem 12.6 for the following set of data.

Line	$\Delta Elev$ (ft)	S (ft)	Line	$\Delta Elev$ (ft)	S (ft)
1	13.07	±0.030	4	−3.85	±0.025
2	−8.34	±0.030	5	−4.74	±0.016
3	8.56	±0.025	6	4.75	±0.028

***12.8** A line of differential level is run from benchmark Oak (elevation 753.01) to station 13+00 on a proposed alignment. It continued along the alignment to 19+00. Rod readings were taken on stakes at each full station. The circuit then closed on the benchmark Bridge having an elevation of 764.95 ft. The observed elevation differences, in order, are −3.03, 4.10, 4.03, 7.92, 7.99, −6.00, −6.02, and 2.98 ft. A third tie between benchmark Rock (elevation of 772.39 ft) and station 16+00 is observed as −6.34 ft. What are the

(a) Most probable values for the adjusted elevations?

(b) Reference standard deviation for the adjustment?

(c) Adjusted observations and their residuals?

12.9 Use ADJUST to solve Problem 12.8.

12.10 Repeat Problem 12.8 using the differences in elevation between Oak (534.56 ft) and Bridge (546.50 ft) of −3.03, 4.10, 4.03, 7.92, 7.99, −6.00, −6.02, 2.98 ft, and the elevation difference between Rock (553.91 ft) and station 1600 is −6.34 ft.

12.11 Repeat Problem 12.10 using standard deviations for each observation of ±0.029, ±0.015, ±0.015, ±0.015, ±0.015, ±0.015, ±0.015, ±0.021, and ±0.032 ft, respectively.

12.12 If the elevation of *A* is 28.243 m, adjust the following leveling data using the weighted least-squares method.

From	To	Δ*Elev* (m)	*S* (cm)
A	*B*	−7.902	±4.2
B	*C*	2.102	±6.0
C	*D*	7.389	±6.0
D	*E*	0.587	±6.0
E	*F*	−2.023	±7.4
F	*G*	−4.400	±6.0
G	*A*	4.024	±7.4
A	*C*	−5.790	±10.4
C	*F*	6.070	±9.5
F	*D*	1.459	±7.4

(a) What are the most probable values for the elevations of the stations?

(b) What is the reference standard deviation?

(c) Tabulate the adjusted observations and their residuals.

12.13 Use ADJUST to solve Problem 12.12.

12.14 Repeat Problem 12.12 using the following observations and an elevation for A of 854.687 m.

From	To	$\Delta Elev$ (m)	S (m)
A	B	6.098	±0.0124
B	C	1.547	±0.0159
C	D	−4.047	±0.0147
D	E	−5.917	±0.0134
E	F	−4.696	±0.0159
F	G	4.258	±0.0170
G	A	2.710	±0.0120
A	C	7.651	±0.0190
C	F	−14.835	±0.0190
F	D	10.619	±0.0190

12.15 If the elevation of Station 1 is 342.384 m, use weighted least squares to adjust the following leveling.

From	To	$\Delta Elev$ (m)	S (m)	From	To	$\Delta Elev$ (m)	S (m)
1	2	−6.767	±0.004	2	3	15.232	±0.004
3	4	33.095	±0.007	4	5	−11.873	±0.004
5	6	−8.450	±0.005	6	1	−21.230	±0.004
1	7	6.002	±0.004	7	3	2.446	±0.007
2	7	12.785	±0.006	7	6	15.222	±0.003
6	3	−12.776	±0.007	3	8	−9.745	±0.007
8	5	30.956	±0.008	6	8	−22.519	±0.007
8	4	42.344	±0.009				

(a) What are the most probable values for the elevations for the stations?

(b) What is the adjustment's reference standard deviation?

(c) Tabulate the adjusted observations and their residuals.

12.16 Use ADJUST to solve Problem 12.15.

12.17 Precise leveling procedures were applied with a level that can be read to within ±0.2 mm/1 m and had a bubble sensitivity of ±0.2″. The line of sight was held to within ±1″ of horizontal, and the sight distances were 20 ± 3 m in length. Use these specifications and Equation (9.20) to compute standard deviations and hence weights. The elevation of A is 35.896 m. Adjust the network using a weighted least squares.

From	To	$\Delta Elev$ (m)	Number of Setups	From	To	$\Delta Elev$ (m)	Number of Setups
A	B	12.383	15	M	D	−38.238	42
B	C	−16.672	18	C	M	30.338	39
C	D	−7.903	25	M	B	−13.676	23
D	A	12.190	53	A	M	26.058	30

(a) What are the most probable values for the elevations of the stations?

(b) What is the reference standard deviation for the adjustment?

(c) Tabulate the adjusted observations and their residuals.

12.18 Repeat Problem 12.17 using the number of setups for weighting following the procedures discussed in Section 12.4.

12.19 In Problem 12.17, the estimated error in reading the rod is 1.5 mm/100 m. The bubble sensitivity of the instrument is 1 mm/km, and the average sight distance is 30 m using Equation (9.21). What are the

(a) Estimated standard errors for the observations?

(b) Most probable values for the elevations of the stations?

(c) Reference standard deviation for the adjustment?

(d) Tabulate the adjusted observations and their residuals?

12.20 Show that $\sum v^2 = V^{\mathrm{T}}V$.

12.21 Show that $\sum wv^2 = V^{\mathrm{T}}WV$.

PROGRAMMING PROBLEMS

12.22 Write a program that reads a file of differential leveling observations and writes the matrices A, W, and L in a format suitable for the MATRIX program. Using this package, solve Problem 12.14.

12.23 Write a computational package that reads the matrices A, W, and L, computes the least-squares solution for the unknown station elevations, and writes a file of adjusted elevation differences and their residuals. Using this package, solve Problem 12.14.

12.24 Write a computational package that reads a file of differential leveling observations, computes the least-squares solution for the adjusted station elevations, and writes a file of adjusted elevation differences and their residuals. Using this package, solve Problem 12.14.

CHAPTER 13

PRECISIONS OF INDIRECTLY DETERMINED QUANTITIES

13.1 INTRODUCTION

Following an adjustment, it is important to know the estimated errors in both the derived quantities and the adjusted observations. For example, after adjusting a level net as described in Chapter 12, the uncertainties in both computed benchmark elevations and adjusted elevation differences can be determined. In Chapter 5, error-propagation formulas were developed for indirectly measured quantities that were functionally related to observed values. In this chapter, error-propagation formulas are developed for the quantities computed in a least-squares solution.

13.2 DEVELOPMENT OF THE COVARIANCE MATRIX

 Consider an adjustment involving weighted observation equations like those in the level circuit example of Section 12.4. The matrix form for the system of weighted observation equations is

$$WAX = WL + WV \tag{13.1}$$

and the least-squares solution of the weighted observation equations is given by

$$X = (A^{\mathrm{T}}WA)^{-1}A^{\mathrm{T}}WL \tag{13.2}$$

In this equation, X contains the most probable values for the unknowns, whereas the true values are X_{true}. The true values differ from X by some small amount ΔX, such that

$$X + \Delta X = X_{\text{true}} \tag{13.3}$$

where ΔX represents the true errors in the adjusted values.

Consider now a small incremental change, ΔL, in the observed values, L, which changes X to its true value, $X + \Delta X$. Then Equation (13.2) becomes

$$X + \Delta X = (A^{\mathrm{T}}WA)^{-1}A^{\mathrm{T}}W(L + \Delta L) \tag{13.4}$$

Expanding Equation (13.4) yields

$$X + \Delta X = (A^{\mathrm{T}}WA)^{-1}A^{\mathrm{T}}WL + (A^{\mathrm{T}}WA)^{-1}A^{\mathrm{T}}W\Delta L \tag{13.5}$$

Note, in Equation (13.2), that $X = (A^{\mathrm{T}}WA)^{-1}A^{\mathrm{T}}WL$, and thus, subtracting this from Equation (13.5) yields

$$\Delta X = (A^{\mathrm{T}}WA)^{-1}A^{\mathrm{T}}W\Delta L \tag{13.6}$$

Recognizing ΔL as the errors in the observations, Equation (13.6) can be rewritten as

$$\Delta X = (A^{\mathrm{T}}WA)^{-1}A^{\mathrm{T}}WV \tag{13.7}$$

where the vector of residuals V is substituted for ΔL. Now let

$$B = (A^{\mathrm{T}}WA)^{-1}A^{\mathrm{T}}W \tag{13.8}$$

Then

$$\Delta X = BV \tag{13.9}$$

Multiplying both sides of Equation (13.9) on the right by their transposes results in

$$\Delta X\Delta X^{\mathrm{T}} = (BV)(BV)^{\mathrm{T}} \tag{13.10}$$

Applying the matrix property $(BV)^{\mathrm{T}} = V^{\mathrm{T}}B^{\mathrm{T}}$ to Equation (13.10) yields

$$\Delta X\Delta X^{\mathrm{T}} = BVV^{\mathrm{T}}B^{\mathrm{T}} \tag{13.11}$$

The expanded left side of Equation (13.11) is

$$\Delta X \Delta X^{\mathrm{T}} = \begin{bmatrix} \Delta x_1^2 & \Delta x_1 \Delta x_2 & \Delta x_1 \Delta x_3 & \cdots & \Delta x_1 \Delta x_n \\ \Delta x_2 \Delta x_1 & \Delta x_2^2 & \Delta x_2 \Delta x_3 & \cdots & \Delta x_2 \Delta x_n \\ \Delta x_3 \Delta x_1 & \Delta x_3 \Delta x_2 & \Delta x_3^2 & \cdots & \Delta x_3 \Delta x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \Delta x_n \Delta x_1 & \Delta x_n \Delta x_2 & \Delta x_n \Delta x_3 & \cdots & \Delta x_n^2 \end{bmatrix} \tag{13.12}$$

Also, the expanded right side of Equation (13.11) is

$$B \begin{bmatrix} v_1^2 & v_1 v_2 & v_1 v_3 & \cdots & v_1 v_m \\ v_2 v_1 & v_2^2 & v_2 v_3 & \cdots & v_2 v_m \\ v_3 v_1 & v_3 v_2 & v_3^2 & \cdots & v_3 v_m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ v_m v_1 & v_m v_2 & v_m v_3 & \cdots & v_m^2 \end{bmatrix} B^{\mathrm{T}} \tag{13.13}$$

Assume that it is possible to repeat the entire sequence of observations many times, say a times, and that each time a slightly different solution occurs, yielding a different set of Xs. Averaging these sets, the left side of Equation (13.11) becomes

$$\frac{1}{a} \sum (\Delta X)(\Delta X)^{\mathrm{T}} = \begin{bmatrix} \dfrac{\sum \Delta x_1^2}{a} & \dfrac{\sum \Delta x_1 \Delta x_2}{a} & \cdots & \dfrac{\sum \Delta x_1 \Delta x_n}{a} \\ \dfrac{\sum \Delta x_2 \Delta x_1}{a} & \dfrac{\sum \Delta x_2^2}{a} & \cdots & \dfrac{\sum \Delta x_2 \Delta x_n}{a} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\sum \Delta x_n \Delta x_1}{a} & \dfrac{\sum \Delta x_n \Delta x_2}{a} & \cdots & \dfrac{\sum \Delta x_n^2}{a} \end{bmatrix} \tag{13.14}$$

If a is large, the terms in Equation (13.14) are the variances and covariances as defined in Equation (6.7), and Equation (13.14) can be rewritten as

$$\begin{bmatrix} S_{x_1}^2 & S_{x_1 x_2} & \cdots & S_{x_1 x_n} \\ S_{x_2 x_1} & S_{x_2}^2 & \cdots & S_{x_2 x_n} \\ \vdots & \vdots & \ddots & \vdots \\ S_{x_n x_1} & S_{x_n x_2} & \cdots & S_{x_n}^2 \end{bmatrix} = S_{xx}^2 \tag{13.15}$$

Also, considering a sets of observations, Equation (13.13) becomes

$$B\begin{bmatrix} \dfrac{\sum v_1^2}{a} & \dfrac{\sum v_1 v_2}{a} & \cdots & \dfrac{\sum v_1 v_m}{a} \\ \dfrac{\sum v_2 v_1}{a} & \dfrac{\sum v_2^2}{a} & \cdots & \dfrac{\sum v_2 v_m}{a} \\ \vdots & \vdots & \ddots & \\ \dfrac{\sum v_m v_1}{a} & \dfrac{\sum v_m v_2}{a} & \cdots & \dfrac{\sum v_m^2}{a} \end{bmatrix} B^{\mathrm{T}} \tag{13.16}$$

Recognizing the diagonal terms as variances of the quantities observed, S_l^2, off-diagonal terms as the covariances, $S_{l_i l_j}$, and the fact that the matrix is symmetric, Equation (13.16) can be rewritten as

$$B\begin{bmatrix} S_{l_1}^2 & S_{l_1 l_2} & \cdots & S_{l_1 l_m} \\ S_{l_2 l_1} & S_{l_2}^2 & \cdots & S_{l_2 l_m} \\ \vdots & \vdots & \ddots & \vdots \\ S_{l_m l_1} & S_{l_m l_2} & \cdots & S_{l_m}^2 \end{bmatrix} B^{\mathrm{T}} \tag{13.17}$$

In Section 10.1, it was shown that the weight of an observation is inversely proportional to its variance. Also, from Equation (10.5), the variance of an observation of weight w can be expressed in terms of the reference variance as

$$S_i^2 = \frac{S_0^2}{w_i} \tag{13.18}$$

Recall from Equation (10.3) that $W = Q^{-1} = \sigma_0^2 W \Sigma^{-1}$. Therefore, $\Sigma = \sigma_0^2 W^{-1}$, and by substituting Equation (13.18) into matrix (13.17) and replacing σ_0 with S_0 yields

$$S_0^2 B W_{ll}^{-1} B^{\mathrm{T}} \tag{13.19}$$

Substituting Equation (13.8) into Equation (13.19) gives

$$S_0^2 B W^{-1} B^{\mathrm{T}} = S_0^2 (A^{\mathrm{T}} W A)^{-1} A^{\mathrm{T}} W W^{-1} W^{\mathrm{T}} [(A^{\mathrm{T}} W A)^{-1}]^{\mathrm{T}} \tag{13.20}$$

Since the normal and weight matrices are symmetric, it follows that

$$[(A^{\mathrm{T}} W A)^{-1}]^{\mathrm{T}} = (A^{\mathrm{T}} W A)^{-1} \tag{13.21}$$

Also, since the weight matrix W is symmetric, $W^{\mathrm{T}} = W$, and thus Equation (13.20) reduces to

$$S_0^2 (A^{\mathrm{T}} W A)^{-1} (A^{\mathrm{T}} W A)(A^{\mathrm{T}} W A)^{-1} = S_0^2 (A^{\mathrm{T}} W A)^{-1} \tag{13.22}$$

Equation (13.15) is the left side of Equation (13.11), for which Equation (13.22) is the right. That is,

$$S_x^2 = S_0^2(A^{\mathrm{T}}WA)^{-1} = S_0^2 N^{-1} = S_0^2 Q_{xx} \tag{13.23}$$

In least-squares adjustment, the matrix S_x^2 of Equation (13.23) is known as the *variance–covariance matrix*, or simply the *covariance matrix*, and Q_{xx} is called the *cofactor* matrix for the adjusted unknown parameters. Diagonal elements of the cofactor matrix, when multiplied by S_0^2, yield variances of the adjusted quantities, and the off-diagonal elements multiplied by S_0^2 yield covariances. From Equation (13.23), the estimated standard deviation S_i for any unknown parameter having been computed from a system of observation equations is expressed as

$$S_i = S_0\sqrt{q_{x_i x_i}} \tag{13.24}$$

where $q_{x_i x_i}$ is the diagonal element (from the ith row and ith column) of the Q_{xx} matrix, which, as noted in Equation (13.23), is equal to the inverse of the matrix of normal equations. Since the normal equation matrix is symmetric, its inverse is also symmetric, and thus the covariance matrix for the adjusted unknown parameters, $S_{xx} = S_0^2 Q_{xx}$, is also a symmetric matrix (i.e., element ij = element ji).

Note that an estimate for the reference variance, σ_0^2, may be computed using either Equation (12.14) or (12.15), depending on whether the observations are unweighted or weighted. However, it should be remembered that this is only an estimate for the *a priori* (before the adjustment) value of the reference variance. The validity of this estimate can be checked using a χ^2 test, as discussed in Chapter 5. If it is a valid estimate for σ_0^2, the *a priori* value for the reference variance, which is 1 typically,[1] can and should be used in the postadjustment statistics computations discussed in this and the following chapters. Thus, the reference variance computed from the adjustment, S_0^2, should be used only when the null hypothesis of the χ^2 test is rejected. However, many software packages always use the computed reference variance rather than the *a priori* value. When the χ^2 test fails to reject the null hypothesis and the *a priori* reference variance is not used in the computations, this only causes small differences, typically in the computed statistical values. Thus, it is considered valid to use the *a posteriori* (after the adjustment) value for the reference variance in the postadjustment statistical computations.

13.3 NUMERICAL EXAMPLES

The results of the level net adjustment in Section 12.3 will be used to illustrate the computation of estimated errors for the adjusted unknowns. From Equation (12.6),

[1]When weights are determined as $1/\sigma^2$, the implicit assumption is that the *a priori* reference variance value for σ_0^2 is 1.

the N^{-1} matrix, which is also the Q_{xx} matrix, is

$$Q_{xx} = \begin{bmatrix} 0.38095 & 0.14286 & 0.04762 \\ 0.14286 & 0.42857 & 0.14286 \\ 0.04762 & 0.14286 & 0.38095 \end{bmatrix}$$

Also, from Equation (12.17), $S_0 = \pm 0.05$. Now by Equation (13.24), the estimated standard deviations for the unknown benchmark elevations A, B, and C are

$$S_A = S_0\sqrt{q_{AA}} = \pm 0.05\sqrt{0.38095} = \pm 0.031 \text{ ft}$$

$$S_B = S_0\sqrt{q_{BB}} = \pm 0.05\sqrt{0.42857} = \pm 0.033 \text{ ft}$$

$$S_C = S_0\sqrt{q_{CC}} = \pm 0.05\sqrt{0.38095} = \pm 0.031 \text{ ft}$$

Note that this is an unweighted adjustment, and thus the χ^2 test is not applicable here. This is also true for differential leveling adjustments that use the number of setups or distances as a weighting criterion, as discussed in Section 10.6. In the unweighted adjustment or weighted by distances or number of setups, S_0 tends to go to zero as the fit of observations improves. However, as can be seen from Equation (5.2), which is repeated here as $\chi^2 = \frac{vS^2}{\sigma^2}$, the value for χ^2 cannot be computed if σ^2 is assumed to be zero. In any event, the value for σ^2 is unknown, and thus the χ^2 statistic is also unknown. When the weighting uses observational standard deviations estimated by Equation (9.20) or (9.21), the *a priori* reference variance, σ_0^2, is implicitly set equal to 1 when weighting is $1/\sigma_i^2$.

Recalling the weighted example in Section 12.4 and Equation (12.12), the Q_{xx} matrix is

$$Q_{xx} = \begin{bmatrix} 0.0933 & 0.0355 & 0.0133 \\ 0.0355 & 0.0770 & 0.0289 \\ 0.0133 & 0.0289 & 0.0733 \end{bmatrix}$$

Since the adjustment was weighted using distances, we must use the postadjustment (*a posteriori*) computed reference standard deviation of $S_0 = \pm 0.107$ when determining the estimated errors in the computed elevations for the benchmarks A, B, and C:

$$S_A = S_0\sqrt{q_{AA}} = \pm 0.107\sqrt{0.0933} = \pm 0.033 \text{ ft}$$

$$S_B = S_0\sqrt{q_{BB}} = \pm 0.107\sqrt{0.0770} = \pm 0.030 \text{ ft}$$

$$S_C = S_0\sqrt{q_{CC}} = \pm 0.107\sqrt{0.0733} = \pm 0.029 \text{ ft}$$

These standard deviations are at approximately the 68% probability level, and if other percentage errors are desired, these values should be multiplied by their respective t values, as discussed in Chapter 3.

It should be noted that in the weighted example, if variances had been used to compute the weights as $1/\sigma_i^2$, then a χ^2 test could have been performed to check if the computed reference variance S_0^2 was statistically equal to σ_0^2. If they were determined to be statistically equal, that is, H_0 is not rejected, then the *a priori* value for σ_0^2 of 1 can be substituted for S_0^2 in Equation (13.23).

13.4 STANDARD DEVIATIONS OF COMPUTED QUANTITIES



In Section 6.1, the generalized law of propagation of variances was developed. Recalled here for convenience, Equation (6.13) was written as

$$\Sigma_{\hat{l}\hat{l}} = A\Sigma_{xx}A^{\mathrm{T}} \tag{a}$$

where $\hat{l}$ represents the adjusted observations, $\Sigma_{\hat{l}\hat{l}}$ the covariance matrix of the adjusted observations, Σ_{xx} the covariance matrix of the unknown parameters (i.e., $\sigma_0^2 Q_{xx}$), and A the coefficient matrix of the observations. Rearranging Equation (10.2) and using sampling statistics there results $\Sigma_{xx} = S_0^2 Q_{xx}$. Also from Equation (13.23), $S_{xx} = S_0^2 Q_{xx} = S_0^2 (A^{\mathrm{T}} W A)^{-1}$, and thus $\Sigma_{xx} = S_{xx}$, where S_{xx} is an estimate for Σ_{xx}. Substituting this equality into Equation (*a*), the estimated standard deviations of the adjusted observations are

$$\Sigma_{\hat{l}\hat{l}} = S_{\hat{l}\hat{l}} = A\Sigma_{xx}A^{\mathrm{T}} = AS_0^2(A^{\mathrm{T}}WA)^{-1}A^{\mathrm{T}} = S_0^2 A Q_{xx} A^{\mathrm{T}} = S_0^2 Q_{\hat{l}\hat{l}} \tag{13.25}$$

where $AQ_{xx}A^{\mathrm{T}} = Q_{\hat{l}\hat{l}}$ is the *cofactor matrix of the adjusted observations*.

Example 13.1 Consider the unweighted leveling adjustment in Section 12.3. By Equation (13.25), the estimated standard deviations in the adjusted elevation differences are given by

$$S_{\hat{l}}^2 = 0.050^2 \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0.38095 & 0.14286 & 0.04762 \\ 0.14286 & 0.42857 & 0.14286 \\ 0.04762 & 0.14286 & 0.38095 \end{bmatrix} \times \begin{bmatrix} 1 & -1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1 \end{bmatrix} \tag{13.26}$$

Performing the required matrix multiplications in Equation (13.26) yields

$$S_{\hat{l}}^2 = 0.050^2 \times \begin{bmatrix} 0.38095 & -0.38095 & 0.04762 & -0.04762 & -0.23810 & 0.14286 & -0.09524 \\ -0.38095 & 0.38095 & -0.04762 & 0.04762 & 0.23810 & -0.14286 & 0.09524 \\ 0.04762 & -0.04762 & 0.38095 & -0.38095 & 0.09524 & 0.14286 & 0.23810 \\ -0.04762 & 0.04762 & -0.38095 & 0.38095 & -0.09524 & -0.14286 & -0.23810 \\ -0.23810 & 0.23810 & 0.09524 & -0.09524 & 0.52381 & 0.28571 & -0.19048 \\ 0.14286 & -0.14286 & 0.14286 & -0.14286 & 0.28571 & 0.42857 & -0.28571 \\ -0.09524 & 0.09524 & 0.23810 & -0.23810 & -0.19048 & -0.28571 & 0.52381 \end{bmatrix} \tag{13.27}$$

The estimated standard deviation of an observation is found by taking the square root of the corresponding diagonal element of the covariance matrix $S_{\hat{l}}^2$ (leveling from A to B). For instance, for the fifth observation, ΔAB, $S_{\hat{l}}(5,5)$ applies, and the estimated error in the adjusted elevation difference of that observation is

$$S_{\Delta AB} = \pm 0.050\sqrt{0.52381} = \pm 0.036 \text{ ft}$$

An interpretation of the meaning of the value just calculated is that there is a 68% probability that the true value is within the range ± 0.036 ft of the adjusted elevation difference ($l_5 + v_5 = -0.68 + 0.022 = -0.658$). That is, the true value lies between -0.658 ± 0.036 (-0.694, -0.622) with 68% probability.

Careful examination of the matrix manipulations involved in solving Equation (13.25) for Example 13.1 reveals that the arithmetic effort can be reduced significantly. In fact, to obtain the estimated standard deviation in the fifth element, only the fifth row of the coefficient matrix, A, which represents the elevation difference between A and B, needs to be used in the calculations. That row is [−1 1 0]. Thus, to determine the standard deviation in this observation, the following computations could be made:

$$\begin{aligned} S_{\Delta AB}^2 &= 0.050^2 \begin{bmatrix} -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.38095 & 0.14286 & 0.04762 \\ 0.14286 & 0.42857 & 0.14286 \\ 0.04762 & 0.14286 & 0.38095 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \\ &= 0.050^2 \begin{bmatrix} 0.23809 & -0.28571 & -0.09524 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \\ &= 0.050^2 [0.52381] \\ S_{\Delta AB} &= \pm 0.050\sqrt{0.52381} = \pm 0.036 \text{ ft} \end{aligned} \tag{13.28}$$

Note that this shortcut method produces the same value. Furthermore, because of the zero in the third position of this row from the coefficient matrix, the matrix

operations in Equation (13.28) could be further reduced to

$$S^2_{\Delta AB} = 0.050^2 \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} 0.38095 & 0.14286 \\ 0.14286 & 0.42857 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = 0.050^2[0.52381]$$

Another use for Equation (13.25) is in the computation of adjusted uncertainties for observations that were never made. For instance, in the example of Section 12.3, the elevation difference between benchmarks X and B was not observed. But from the results of the adjustment, this elevation difference is $104.48 - 100.00 = 4.48$ ft. The estimated error in this difference can be found by writing an observation equation for it (i.e., $B = X + \Delta XB$). This equation does not involve either A or C, and thus, in matrix form, this difference would be expressed as

$$\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \tag{13.29}$$

Using this row matrix in the same process as shown in Equation (13.28) yields

$$S^2_{\Delta XB} = 0.050^2 \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.38095 & 0.14286 & 0.04762 \\ 0.14286 & 0.42857 & 0.14286 \\ 0.04762 & 0.14286 & 0.38095 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0.050^2[0.42857]$$

Hence,

$$S_{\Delta XB} = \pm 0.050\sqrt{0.42857} = \pm 0.033 \text{ ft}$$

Again, recognizing the presence of the zeros in the row matrix, these computations can be simplified to

$$S^2_{\Delta XB} = 0.050^2[1][0.42857][1] = 0.050^2[0.42857]$$

The method illustrated above of eliminating unnecessary matrix computations is formally known as *matrix partitioning*.

Computing uncertainties of quantities that were not actually observed has application in many areas. For example, suppose in a triangulation adjustment, that the x and y coordinates of stations A and B are calculated, and the covariance matrix exists. Equation (13.25) could be applied to determine the estimated error in the length of line AB calculated from the adjusted coordinates of A and B. This is accomplished by relating the length AB to the unknown parameters as

$$\overline{AB} = \sqrt{(X_b - X_a)^2 + (Y_b - Y_a)^2} \tag{13.30}$$

This subject is discussed further in Chapter 14.

An important observation that should be made about the $Q_{\hat{l}\hat{l}}$ and Q_{xx} matrices is that only the coefficient matrix, A, is used in their formation. Since the A matrix contains coefficients that express the relationships of the unknowns to each other, it depends only on the *geometry* of the problem. The only other term in Equation (13.25) is the reference variance, and that depends on the quality of the observations. These are important concepts that will be revisited in Chapter 21 when the simulation and design of surveying networks is discussed.

PROBLEMS

Note: Partial answers to problems marked with an asterisk can be found in Appendix G.

13.1 The reference variance of an adjustment is 0.86. The cofactor matrix and unknown parameter matrix are

$$Q_{xx} = \begin{bmatrix} 0.5904 & 0.2146 & 0.0987 \\ 0.2146 & 0.6876 & -0.1678 \\ 0.0987 & -0.1678 & 0.6987 \end{bmatrix} \qquad X = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$$

What is the estimated error in the adjusted value for

***(a)** A?

(b) B?

(c) C?

13.2 In Problem 13.1, the adjustment had 9 degrees of freedom.

***(a)** Did the adjustment pass the χ^2 test at a 95% confidence level?

(b) Assuming it passed the χ^2 test in part a, what would be the estimated errors in the adjusted parameters?

For Problems 13.3–13.8, determine the estimated errors in the adjusted elevations:

***13.3** Problem 12.1.

13.4 Problem 12.4.

13.5 Problem 12.6.

13.6 Problem 12.8.

13.7 Problem 12.10.

13.8 Problem 12.11.

For each problem, calculate the estimated errors for the adjusted elevation differences.

***13.9** Problem 12.1.

13.10 Problem 12.4.

13.11 Problem 12.6.

13.12 Problem 12.8.

13.13 Problem 12.11.

13.14 Calculate the adjusted length $\overline{AD}$ and its estimated error given the accompanying sketch and observational data below (assume equal weights).

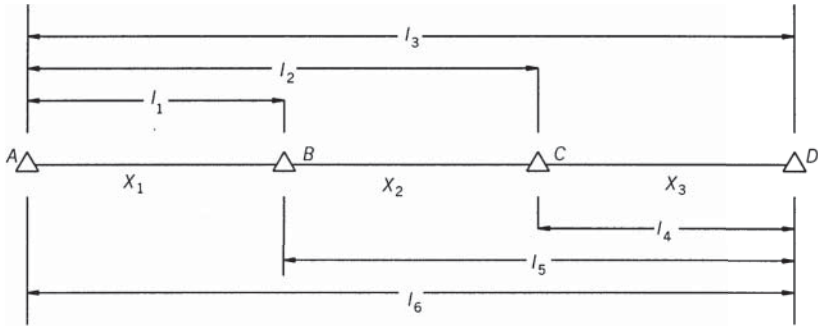


FIGURE P13.14

Length Observations

$l_1 = 99.99 \quad l_2 = 200.01 \quad l_3 = 300.01 \quad l_4 = 99.98 \quad l_5 = 200.01 \quad l_6 = 299.99$

13.15 Use the accompanying sketch and the data below to answer the following questions.

Elevation of BM $A = 384.597$ m					Elevation of BM $B = 396.875$ m				
Obs	From	To	ΔElev (m)	σ (m)	Obs	From	To	ΔElev (m)	σ (m)
1	BM *A*	*V*	−29.743	±0.0952	8	*Y*	*Z*	40.688	±0.1237
2	BM *B*	*V*	−42.046	±0.0968	9	*W*	*Z*	93.903	±0.1237
3	*V*	*X*	−21.329	±0.0935	10	*V*	*W*	−43.271	±0.1104
4	*V*	*Y*	9.484	±0.1118	11	BM *A*	*W*	−72.940	±0.0935
5	BM *B*	*Y*	−32.158	±0.0968	12	BM *B*	*X*	−63.418	±0.1132
6	BM *A*	*X*	−51.095	±0.1132	13	*W*	*X*	21.941	±0.0952
7	*Y*	*X*	−31.223	±0.0952	14	*X*	*Z*	71.808	±0.0984

What is the

(a) Most probable elevation for each of stations V, W, X, Y, and Z.

(b) Estimated error in each elevation.

(c) Adjusted observations, their residuals, and estimated errors?

(d) Elevation difference from benchmark V to station Z and its estimated error?

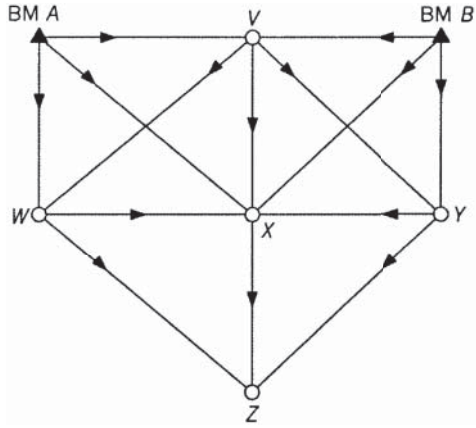


FIGURE P13.15

13.16 Perform a χ^2 test in Problem 13.15.

13.17 Repeat Problem 13.15(a)–(c) without observation 5.

13.18 Repeat Problem 13.15(a)–(c) without observations 3, 5, and 11.

13.19 Repeat Problem 13.17 without observations 7 and 13.

PROGRAMMING PROBLEMS

13.20 Adapt the program developed in Problem 12.17 to compute and tabulate the adjusted

(a) Elevations and their estimated errors.

(b) Elevation differences and their estimated errors.

13.21 Adapt the program developed in Problem 12.18 to compute and tabulate the adjusted

(a) Elevations and their estimated errors.

(b) Elevation differences and their estimated errors.

CHAPTER 14

ADJUSTMENT OF HORIZONTAL SURVEYS: TRILATERATION

14.1 INTRODUCTION

Horizontal surveys are performed for the purpose of determining precise relative horizontal positions of points. They have traditionally been accomplished by trilateration, triangulation, and traverse. These traditional types of surveys involve making distance, direction, and angle observations. With all types of surveys, errors occur while making these observations, and thus they must be analyzed and if acceptable must be adjusted. The following three chapters describe procedures for adjusting trilateration, triangulation, and traverse surveys, in that order.

The *global navigation satellite systems* (GNSS), such as GPS, have gradually been replacing these traditional procedures for conducting precise surveys. In fact, GNSS surveys not only yield horizontal positions but also give ellipsoidal heights as well. Thus, GNSS surveys provide three-dimensional surveys. Again, as with all observations, GNSS surveys contain errors and must be adjusted. In Chapter 17, we discuss the subject of GNSS surveying in more detail and illustrate methods for adjusting networks surveyed by this procedure.

Horizontal surveys, especially those covering a large extent, must account for the systematic effects of Earth curvature. One way this can be accomplished is to do the computations using coordinates from a mathematically rigorous map projection system such as the state plane coordinate system (SPCS), universal

transverse Mercator (UTM) system, or a local plane coordinate system that rigorously accounts for Earth curvature.[1] In the following chapters, methods are developed for adjusting horizontal surveys using parametric equations that are based on plane coordinates. In Chapter 23, a three-dimensional geodetic network adjustment is developed for traditional surveying observations including differential leveling, slant distances, and vertical angles.

It should be noted that if state plane coordinates are used, the numbers are usually rather large. Consequently, when they are used in mathematical computations, errors due to rounding and truncation can occur. This can be prevented by translating the origin of the coordinates prior to adjustment, a process that simply involves subtracting a constant value from all coordinates. Then, after the adjustment is finished, the true origin is restored by adding the constants to the adjusted values. This procedure is demonstrated with the following example.

Example 14.1 Assume that the SPCS 2022 state plane coordinates of three control stations to be used in a horizontal survey adjustment are as given below. Translate the origin.

Point	Easting (m)	Northing (m)
A	698,257.171	172,068.220
B	698,734.839	171,312.344
C	698,866.717	170,696.617

SOLUTION

Step 1: Many surveyors in the United States prefer to work in feet and some jobs require it. Thus, in this step, the eastings and northings, respectively, are converted to X and Y values in feet by multiplying by the appropriate exact conversion value. The factor for converting meters to feet is based on there being exactly 0.3048 m per foot. After making the conversions, the coordinates in feet are

Point	X (ft)	Y (ft)
A	2,290,869.98	564,528.28
B	2,292,437.14	562,048.37
C	2,292,869.81	560,028.27

[1]Map projection systems are presented in *Elementary Surveying: An Introduction to Geomatics*, 16th edition by Charles D. Ghilani. The software ADJUST not only has map projection software for the Lambert conformal, transverse Mercator, and oblique Mercator but also supports the Universal Traverse Mercator and has routines to compute and remove systematic errors caused by map projection distortions.

Step 2: To reduce the sizes of these numbers, an X constant is subtracted from each X coordinate and a Y constant is subtracted from each Y coordinate. For convenience, these constants are usually rounded to the nearest thousands of units and are normally selected to give the smallest possible coordinates without producing negative values. In this instance, 2,290,000 ft and 560,000 ft are used as the X constant and Y constant, respectively. Subtracting these values from the coordinates yields

Point	X' (ft)	Y' (ft)
A	869.98	4528.28
B	2437.14	2048.37
C	2869.81	28.27

These X' and Y' coordinates can then be used in the adjustment. After the adjustment is complete, the coordinates are translated back to their state plane values by reversing the steps described above, that is, by adding 2,290,000 ft to all adjusted X coordinates and adding 560,000 ft to all adjusted Y coordinates. Also, if desired, they can be converted back to meters.

In the horizontal adjustment problems solved later in this book, either translated state plane coordinates or local plane coordinates are used. It should be stated that these steps are unnecessary when using software and are only used here as a convenience to the reader. In this chapter, we concentrate on adjusting trilateration surveys, those involving only horizontal distance observations. This method of conducting horizontal surveys became common with the introduction of EDM instruments that enabled accurate distance observations to be made rapidly and economically. Trilateration is still possible using today's modern total station instruments, but as noted, the procedure is now giving way to traversing, networks, and GNSS surveys. However, the techniques discussed in this chapter also apply to any distance observations in traversing and networks.

14.2 DISTANCE OBSERVATION EQUATION



In adjusting trilateration surveys using the parametric least-squares method, observation equations are written that relate the observed quantities and their inherent random errors to the most probable values for the x and y coordinates (the parameters) of the stations involved. Referring to Figure 14.1, the following distance equation can be written for any observation l_{ij}

$$l_{ij} + v_l = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2} \tag{14.1}$$

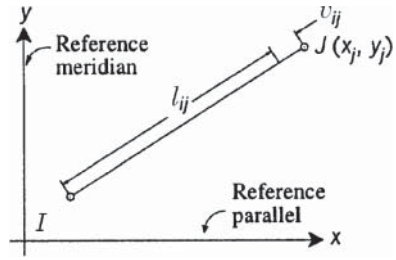


FIGURE 14.1 Observation of a distance.

In Equation (14.1), l_{ij} is the observed distance of a line between stations I and J; v_i is the residual in the observation l_{ij}; x_i and y_i are the most probable coordinate values for station I; and x_j and y_j are the most probable coordinate values for station J. Equation (14.1) is a nonlinear function involving the unknown parameters x_i, y_i, x_j, and y_j, that can be rewritten as

$$F(x_i, y_i, x_j, y_j) = l_{ij} + v_l \tag{14.2}$$

where

$$F(x_i, y_i, x_j, y_j) = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}.$$

As discussed in Section 11.10 and demonstrated in Appendix C, a system of nonlinear equations such as Equation (14.2) can be linearized and solved using a first-order Taylor series approximation. The linearized form of Equation (14.2) is

$$F(x_i, y_i, x_j, y_j) = F(x_{i_0}, y_{i_0}, x_{j_0}, y_{j_0}) + \left(\frac{\partial F}{\partial x_i}\right)_0 dx_i + \left(\frac{\partial F}{\partial y_i}\right)_0 dy_i + \left(\frac{\partial F}{\partial x_j}\right)_0 dx_j + \left(\frac{\partial F}{\partial y_j}\right)_0 dy_j \tag{14.3}$$

where $\left(\frac{\partial F}{\partial x_i}\right)_0$, $\left(\frac{\partial F}{\partial y_i}\right)_0$, $\left(\frac{\partial F}{\partial x_j}\right)_0$, and $\left(\frac{\partial F}{\partial y_j}\right)_0$ are the partial derivatives of F with respect to x_i, y_i x_j, and y_j, respectively, evaluated with the approximate coordinate values $x_{i_0}, y_{i_0}, x_{j_0}$, and y_{j_0}; x_i, y_i, x_j, and y_j are the unknown parameters; and dx_i, dy_i, dx_j, and dy_j are the corrections to the approximate coordinate values such that

$$\begin{aligned} x_i &= x_{i_0} + dx_i \qquad & y_i &= y_{i_0} + dy_i \\ x_j &= x_{j_0} + dx_j \qquad & y_j &= y_{j_0} + dy_j \end{aligned} \tag{14.4}$$

The evaluation of partial derivatives is straightforward and will be illustrated with $\partial F/\partial x_i$. Equation (14.2) can be rewritten as

$$F(x_i, y_i, x_j, y_j) = \left[(x_j - x_i)^2 + (y_j - y_i)^2\right]^{1/2} \tag{14.5}$$

Taking the derivative of Equation (14.5) with respect to x_i yields

$$\frac{\partial F}{\partial x_i} = \frac{1}{2}\left[(x_j - x_i)^2 + (y_j - y_i)^2\right]^{-1/2}[2(x_j - x_i)(-1)] \tag{14.6}$$

Simplifying Equation (14.6) yields

$$\frac{\partial F}{\partial x_i} = \frac{-(x_j - x_i)}{\sqrt{(x_j - x_i)^2 + (y_j - y_i)^2}} = \frac{x_i - x_j}{IJ} \tag{14.7}$$

Employing the same procedure, the remaining partial derivatives are

$$\frac{\partial F}{\partial y_i} = \frac{y_i - y_j}{IJ} \quad \frac{\partial F}{\partial x_j} = \frac{x_j - x_i}{IJ} \quad \frac{\partial F}{\partial y_j} = \frac{y_j - y_i}{IJ} \tag{14.8}$$

If Equations (14.7) and (14.8) are substituted into Equation (14.3) and the results substituted into Equation (14.2), the following prototype linearized distance observation equation obtained is

$$\left(\frac{x_i - x_j}{IJ}\right)_0 dx_i + \left(\frac{y_i - y_j}{IJ}\right)_0 dy_i + \left(\frac{x_j - x_i}{IJ}\right)_0 dx_j + \left(\frac{y_j - y_i}{IJ}\right)_0 dy_j = k_l + v_l \tag{14.9}$$

where the coefficients in parenthesis are evaluated at the approximate parameter values, $k_l = l_{ij} - IJ_0$, and $IJ_0 = F(x_{i_0}, y_{i_0}, x_{j_0}, y_{j_0}) = \sqrt{(x_{j_0} - x_{i_0})^2 + (y_{j_0} - y_{i_0})^2}$.

14.3 TRILATERATION ADJUSTMENT EXAMPLE



Even though the geometric figures used in trilateration are many and varied, they are equally adaptable to the observation equation method in a parametric adjustment. Consider the example shown in Figure 14.2, where the distances are observed from three stations with known coordinates to a common unknown

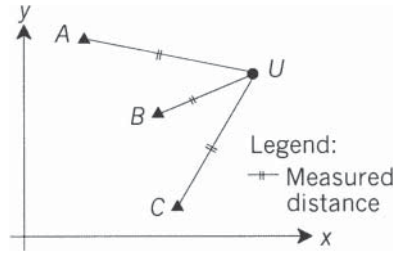


FIGURE 14.2 Trilateration example.

station U. Since the unknown station has two unknown coordinates and there are three observations, this yields one redundant observation. That is, the coordinates of station U could be determined using any two of the three observations. But, all three observations can be used simultaneously and adjusted by the method of least squares to determine the most probable value for the coordinates of the station based on these three observations.

The observation equations are developed by substituting into prototype Equation (14.9). For example, the equation for distance AU is formed by interchanging subscript I with A and subscript J with U in Equation (14.9). In a similar fashion, an equation can be created for each observed distance using the following subscript substitutions:

I	J
A	U
B	U
C	U

When one end of the observed line is a control station, its coordinates are fixed, and thus those terms can be dropped in prototype Equation (14.9).[2] This can be thought of as setting the dx and dy corrections for the control station equal to zero. In this example, station U always takes the position of J in the prototype equation, and thus only the coefficients corresponding to dx_j and dy_j are used. Using the appropriate substitutions, the following three linearized observation equations result:

$$\begin{aligned}
\frac{x_{u_0} - x_a}{AU_0} dx_u + \frac{y_{u_0} - y_a}{AU_0} dy_u &= (l_{AU} - AU_0) + v_{AU} \\
\frac{x_{u_0} - x_b}{BU_0} dx_u + \frac{y_{u_0} - y_b}{BU_0} dy_u &= (l_{BU} - AU_0) + v_{BU} \\
\frac{x_{u_0} - x_c}{CU_0} dx_u + \frac{y_{u_0} - y_c}{CU_0} dy_u &= (l_{CU} - CU_0) + v_{CU}
\end{aligned} \tag{14.10}$$

In Equation (14.10),

$$AU_0 = \sqrt{(x_{u_0} - x_a)^2 + (y_{u_0} - y_a)^2},$$

$$BU_0 = \sqrt{(x_{u_0} - x_b)^2 + (y_{u_0} - y_b)^2},$$

$$CU_0 = \sqrt{(x_{u_0} - x_c)^2 + (y_{u_0} - y_c)^2};$$

[2]The method of dropping control station coordinates from the adjustment is known as elimination of constraints. This method is covered in Chapter 20.

l_{AU}, l_{BU}, and l_{CU} are the observed distances; the v's are residuals; and x_{u_0} and y_{u_0} are initial coordinate values for station U. Equations (14.10) can be expressed in matrix form as

$$JX = K + V \tag{14.11}$$

where J is the Jacobian matrix of partial derivatives, X the matrix or unknown corrections dx_u and dy_u, K the matrix of constants (i.e., the observed lengths, minus their corresponding lengths computed from the initial approximate coordinates), and V the residual matrix. Equation (14.11) in expanded form is

$$\begin{bmatrix} \dfrac{x_{u_0} - x_a}{AU_0} & \dfrac{y_{u_0} - y_a}{AU_0} \\ \dfrac{x_{u_0} - x_b}{BU_0} & \dfrac{y_{u_0} - y_b}{BU_0} \\ \dfrac{x_{u_0} - x_c}{CU_0} & \dfrac{y_{u_0} - y_c}{CU_0} \end{bmatrix} \begin{bmatrix} dx_u \\ dy_u \end{bmatrix} = \begin{bmatrix} l_{AU} - AU_0 \\ l_{BU} - BU_0 \\ l_{CU} - CU_0 \end{bmatrix} + \begin{bmatrix} v_{AU} \\ v_{BU} \\ v_{CU} \end{bmatrix} \tag{14.12}$$

The Jacobian matrix can systematically be formed using the following steps:

Step 1: Head each column with an unknown value.

Step 2: Create a row for every observation.

Step 3: Substitute in the appropriate coefficient corresponding to the column into each row.

If this procedure is followed for this problem, the Jacobian matrix is

$$J = \begin{bmatrix} & dx_u & dy_u \\ AU \;\; | & \dfrac{\partial F}{dx_u} & \dfrac{\partial F}{dy_u} \\ BU \;\; | & \dfrac{\partial F}{dx_u} & \dfrac{\partial F}{dy_u} \\ CU \;\; | & \dfrac{\partial F}{dx_u} & \dfrac{\partial F}{dy_u} \end{bmatrix}$$

Once Equation (14.12) is created, the corrections of dx_u and dy_u, and thus the most probable coordinate values, x_u and y_u can be computed using Equation (11.37). Of course, due to the nonlinear nature of the equations, to obtain the final adjusted values, the solution must be iterated, as discussed in Section 11.10 and Appendix C.

Example 14.2 To clarify the computational procedure, a numerical example for Figure 14.2 is presented. Suppose that the observed distances l_{AU}, l_{BU}, and l_{CU} are 6049.00, 4736.83, and 5446.49 ft, respectively, and the control stations have

coordinates in units of feet of

$$x_a = 865.40 \quad x_b = 2432.55 \quad x_c = 2865.22$$
$$y_a = 4527.15 \quad y_b = 2047.25 \quad y_c = 27.15$$

(Note that these are the translated coordinates obtained in Example 14.1.) Compute the most probable coordinates for station U.

SOLUTION Perform the first iteration.

Step 1: Calculate approximate coordinates for station U.

(a) Calculate azimuth AB from the coordinate values of stations A and B.

$$\begin{aligned} Az_{AB} &= \tan^{-1}\left(\frac{x_b - x_a}{y_b - y_a}\right) + 180^\circ \\ &= \tan^{-1}\left(\frac{2432.55 - 865.40}{2047.25 - 4527.15}\right) + 180^\circ \\ &= 147^\circ 42' 34'' \end{aligned}$$

(b) Calculate the distance between Station A and B from their coordinate values.

$$\begin{aligned} AB &= \sqrt{(x_b - x_a)^2 + (y_b - y_a)^2} \\ &= \sqrt{(2432.55 - 865.20)^2 + (2047.25 - 4527.15)^2} \\ &= 2933.58 \text{ ft} \end{aligned}$$

(c) Calculate azimuth AU_0 using the law of cosines in triangle AUB.

$$c^2 = a^2 + b^2 - 2ab \cos C$$
$$\cos(UAB) = \frac{6049.00^2 + 2933.58^2 - 4736.83^2}{2(6049.00)(2933.58)}$$
$$UAB = 50^\circ 06' 50''$$
$$Az_{AU_0} = 147^\circ 42' 34'' - 50^\circ 06' 50'' = 97^\circ 35' 44''$$

(d) Calculate the coordinates for station U.

$$x_{u_0} = 865.40 + 6049.00 \sin 97^\circ 35' 44'' = 6861.325 \text{ ft}$$
$$y_{u_0} = 4527.15 + 6049.00 \cos 97^\circ 35' 44'' = 3727.596 \text{ ft}$$

Step 2: Calculate AU_0, BU_0, and CU_0. For this first iteration, AU_0 and BU_0 are exactly equal to their respective observed distances since x_{u_0} and y_{u_0} were calculated using these quantities. Thus

$$AU_0 = 6049.00 \quad BU_0 = 4736.83$$

$$CU_0 = \sqrt{(6861.325 - 2865.22)^2 + (3727.596 - 27.15)^2} = 5446.298 \text{ ft}$$

Step 3: Formulate the matrices.

(a) The elements of the Jacobian matrix in Equation (14.12) are[3]

$$j_{11} = \frac{6861.325 - 865.40}{6049.00} = 0.991 \qquad j_{12} = \frac{3727.596 - 4527.15}{6049.00} = -0.132$$

$$j_{21} = \frac{6861.325 - 2432.55}{4736.83} = 0.935 \qquad j_{22} = \frac{3727.596 - 2047.25}{4736.83} = 0.355$$

$$j_{31} = \frac{6861.325 - 2865.22}{5446.298} = 0.734 \qquad j_{32} = \frac{3727.596 - 27.15}{5446.298} = 0.679$$

(b) The elements of the K matrix in Equation (14.12) are

$$k_1 = 6049.00 - 6049.00 = 0.000$$

$$k_2 = 4736.83 - 4736.83 = 0.000$$

$$k_3 = 5446.49 - 5446.298 = 0.192$$

Step 4: The matrix solution using Equation (11.37) is[4]

$$X = (J^{\mathrm{T}}J)J^{\mathrm{T}}K$$

$$J^{\mathrm{T}}J = \begin{bmatrix} 0.991 & 0.935 & 0.734 \\ -0.132 & 0.355 & 0.679 \end{bmatrix} \begin{bmatrix} 0.991 & -0.132 \\ 0.935 & 0.355 \\ 0.735 & 0.679 \end{bmatrix} = \begin{bmatrix} 2.395 & 0.699 \\ 0.699 & 0.605 \end{bmatrix}$$

$$(J^{\mathrm{T}}J)^{-1} = \frac{1}{0.960} \begin{bmatrix} 0.605 & -0.699 \\ -0.699 & 2.395 \end{bmatrix}$$

$$J^{\mathrm{T}}K = \begin{bmatrix} 0.991 & 0.935 & 0.734 \\ -0.132 & 0.355 & 0.679 \end{bmatrix} \begin{bmatrix} 0.000 \\ 0.000 \\ 0.192 \end{bmatrix} = \begin{bmatrix} 0.141 \\ 0.130 \end{bmatrix}$$

$$X = \frac{1}{0.960} \begin{bmatrix} 0.605 & -0.699 \\ -0.699 & 2.395 \end{bmatrix} \begin{bmatrix} 0.141 \\ 0.130 \end{bmatrix} = \begin{bmatrix} -0.006 \\ 0.222 \end{bmatrix}$$

[3]Note that the denominators in the coefficients of Step 3 are distances computed from the approximate coordinates. Only distances computed in the first iteration will match the observed distances exactly. Do not use observed distances for the denominators of these coefficients. Failure to heed this warning will sometimes result in no solution or an incorrect solution to the problem.

[4]The values are rounded to three decimal places simply for publication purposes. When performing these adjustments, it is important to carry all digits during the adjustment and only round the final solution values.

The revised coordinates of U are

$$x_u = 6861.325 - 0.006 = 6861.319$$

$$y_u = 3727.596 + 0.222 = 3727.818$$

Now, perform the second iteration.

Step 1: Calculate AU_0, BU_0, and CU_0.

$$AU_0 = \sqrt{(6861.319 - 865.40)^2 + (3727.818 - 4527.15)^2} = 6048.965 \text{ ft}$$

$$BU_0 = \sqrt{(6861.319 - 2432.55)^2 + (3727.818 - 2047.25)^2} = 4736.909 \text{ ft}$$

$$CU_0 = \sqrt{(6861.319 - 2865.22)^2 + (3727.818 - 27.15)^2} = 5446.444 \text{ ft}$$

Notice that these computed distances no longer match their observed counterparts.

Step 2: Formulate the matrices. With these minor changes in the lengths, the J matrix (to three places) does not change and thus $(J^{\mathrm{T}}J)^{-1}$ does not change, either. However, the K matrix does change as shown by the following computations.

$$k_1 = 6049.00 - 6048.965 = \quad 0.035$$

$$k_2 = 4736.83 - 4736.909 = -0.079$$

$$k_3 = 5446.49 - 5446.444 = \quad 0.046$$

Step 3: Matrix solution

$$J^{\mathrm{T}}K = \begin{bmatrix} 0.991 & 0.935 & 0.734 \\ -0.132 & 0.355 & 0.679 \end{bmatrix} \begin{bmatrix} 0.035 \\ -0.079 \\ 0.046 \end{bmatrix} = \begin{bmatrix} -0.005 \\ -0.001 \end{bmatrix}$$

$$X = \frac{1}{0.960} \begin{bmatrix} 0.605 & -0.699 \\ -0.699 & 2.395 \end{bmatrix} \begin{bmatrix} -0.005 \\ -0.001 \end{bmatrix} = \begin{bmatrix} -0.002 \\ 0.001 \end{bmatrix}$$

The revised coordinates of U are

$$x_u = 6861.319 - 0.002 = 6861.317$$

$$y_u = 3727.818 + 0.001 = 3727.819$$

Satisfactory convergence is shown by the very small corrections in the second iteration. This problem has also been solved using the program ADJUST, which is available on the companion website for this book, and the text file shown in Figure 14.3. Values computed include the most probable coordinates for station U, their standard deviations, the adjusted lengths of the observed distances, and their residuals, and the reference variance and standard deviation. These results are tabulated as shown in Figure 14.4.

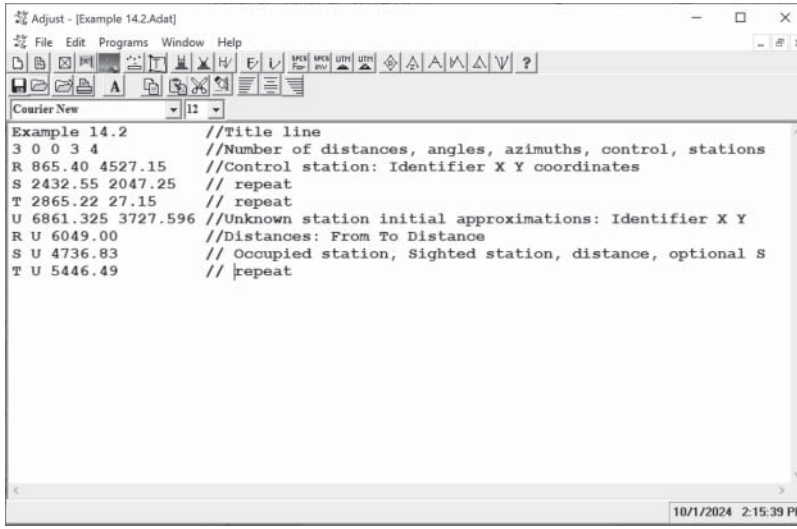


FIGURE 14.3 File format for Example 14.2 in ADJUST.

```
******************
Adjusted stations
******************
   Station              X             Y          Sx          Sy
================================================================
         U      6,861.317     3,727.819      0.0775      0.1542

*********************************
Adjusted Distance Observations
*********************************
   Station    Station
  Occupied    Sighted       Distance        V
================================================
         R          U      6,048.96      -0.037
         S          U      4,736.91       0.077
         T          U      5,446.44      -0.047

                  ****************************************
                          Adjustment Statistics
                  ****************************************
                                Iterations = 2
                              Redundancies = 1

                       Reference Variance = 0.00954
                              Reference So = ±0.10
                               Convergence!
```

FIGURE 14.4 ADJUST output for the trilateration in Example 14.2.

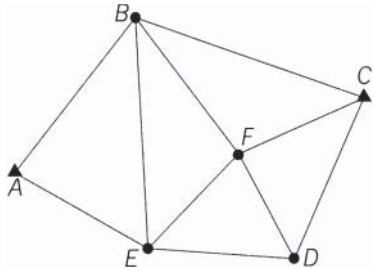


FIGURE 14.5 Trilateration network.

TABLE 14.1 Structure of the Normal Matrix for Complex Network in Figure 14.5

Distance	Unknown							
IJ	dx_b	dy_b	dx_d	dy_d	dx_e	dy_e	dx_f	dy_f
AB	dx_j	dy_j	0	0	0	0	0	0
AE	0	0	0	0	dx_j	dy_j	0	0
BC	dx_i	dy_i	0	0	0	0	0	0
BF	dx_i	dy_i	0	0	0	0	dx_j	dy_j
BE	dx_i	dy_i	0	0	dx_j	dy_j	0	0
CD	0	0	dx_j	dy_j	0	0	0	0
CF	0	0	0	0	0	0	dx_j	dy_j
DF	0	0	dx_i	dy_i	0	0	dx_j	dy_j
DE	0	0	dx_i	dy_i	dx_j	dy_j	0	0
EF	0	0	0	0	dx_i	dy_i	dx_j	dy_j

14.4 FORMULATION OF A GENERALIZED COEFFICIENT MATRIX FOR A MORE COMPLEX NETWORK



In the trilaterated network of Figure 14.5, all lines were observed. Assume that stations *A* and *C* are control stations. For this network, there are 10 observations and 8 unknowns. Stations *A* and *C* can be fixed by setting the terms dx_a, dy_a, dx_c, and dy_c to zero, which effectively drops these terms from the solution. The coefficient matrix formulated from prototype Equation (14.9) has nonzero elements, as indicated in Table 14.1. In this table, the appropriate coefficient from Equation (14.9) is indicated by its corresponding unknown terms of dx_i, dy_i, dx_j, or dy_j.

14.5 COMPUTER SOLUTION OF A TRILATERATED QUADRILATERAL

The quadrilateral shown in Figure 14.6 was adjusted using the MATRIX software. In this problem, points Bucky and Badger are control stations whose coordinates

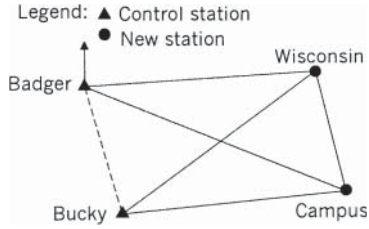


FIGURE 14.6 Quadrilateral network.

are held fixed. The five observed distances are

Line	Distance (ft)
Badger – Wisconsin	5870.302
Badger – Campus	7297.588
Wisconsin – Campus	3616.434
Wisconsin – Bucky	5742.878
Campus – Bucky	5123.760

The state plane control coordinates in units of feet for station Badger are $x = 2{,}410{,}000.000$ and $y = 390{,}000.000$, and for Bucky are $x = 2{,}411{,}820.000$ and $y = 386{,}881.222$.

Step 1: To solve this problem, approximate coordinates are first computed for stations Wisconsin and Campus. This is done following the procedures used in Section 14.3, with the resulting initial approximations being

$$\text{Wisconsin:} \quad x = 2{,}415{,}776.819 \quad y = 391{,}043.461$$

$$\text{Campus:} \quad x = 2{,}416{,}898.227 \quad y = 387{,}602.294$$

Step 2: Following prototype Equation (14.9) and the procedures outlined in Section 14.4, a table of coefficients is established. For the sake of brevity in Table 14.2, the following station assignments were made: Badger = 1, Bucky = 2, Wisconsin = 3, and Campus = 4.

After forming the J matrix, the K matrix is computed. This is done in a manner similar to step 3 of the first iteration in Example 14.2. The matrices were entered into a file following the format described in Section 12.7. Section 14.7 discusses in more detail the procedures used to solve these problems with the software on the companion website for this book. Following are the input data, matrices for the first and last iterations of this three-iteration solution, and the final results.

TABLE 14.2 Structure of the Coefficient or *J* Matrix for Example in Figure 14.6

	$dx_{\text{Wisconsin}}$	$dy_{\text{Wisconsin}}$	dx_{Campus}	dy_{Campus}
Badger – Wisconsin 1–3	$\frac{x_{3_0} - x_1}{(1-3)_0}$	$\frac{y_{3_0} - y_1}{(1-3)_0}$	0	0
Badger – Campus 1–4	0	0	$\frac{x_{4_0} - x_1}{(1-4)_0}$	$\frac{y_{4_0} - y_1}{(1-4)_0}$
Wisconsin – Campus 3–4	$\frac{x_{3_0} - x_{4_0}}{(3-4)_0}$	$\frac{y_{3_0} - y_{4_0}}{(3-4)_0}$	$\frac{x_{4_0} - x_{3_0}}{(3-4)_0}$	$\frac{y_{4_0} - y_{3_0}}{(3-4)_0}$
Wisconsin – Bucky 3–2	$\frac{x_{3_0} - x_{2_0}}{(3-2)_0}$	$\frac{y_{3_0} - y_{2_0}}{(3-2)_0}$	0	0
Campus – Bucky 4–2	0	0	$\frac{x_{4_0} - x_{2_0}}{(4-2)_0}$	$\frac{y_{4_0} - y_{2_0}}{(4-2)_0}$

```
*****************************************
Initial approximations for unknown stations
*****************************************
Station          X              Y
===================================
 Wisconsin  2,415,776.819   391,043.461
    Campus  2,416,898.227   387,602.294

Control Stations
~~~~~~~~~~~~~~~~
Station          X              Y
===================================
    Badger  2,410,000.000   390,000.000
     Bucky  2,411,820.000   386,881.222

**********************
Distance Observations
**********************
Occupied       Sighted       Distance
===================================
    Badger  Wisconsin      5,870.302
    Badger     Campus      7,297.588
 Wisconsin     Campus      3,616.434
 Wisconsin      Bucky      5,742.878
    Campus      Bucky      5,123.760
```

First Iteration Matrices

```
J Dim: 5x4                                          K Dim: 5x1 X Dim 4x1
========================================== ======== =========
    0.98408    0.17775    0.00000    0.00000   -0.00026   0.084751
    0.00000    0.00000    0.94457   -0.32832   -5.46135  -0.165221
   -0.30984    0.95079    0.30984   -0.95079   -2.84579  -5.531445
    0.68900    0.72477    0.00000    0.00000   -0.00021   0.959315
    0.00000    0.00000    0.99007    0.14058   -5.40507  =========
========================================== ========
```

```
JtJ Dim: 4x4
=========================================
    1.539122   0.379687  -0.096003   0.294595
    0.379687   1.460878   0.294595  -0.903997
   -0.096003   0.294595   1.968448  -0.465525
    0.294595  -0.903997  -0.465525   1.031552
=========================================

Inv( N ) Dim: 4x4
=========================================
    1.198436  -1.160169  -0.099979  -1.404084
   -1.160169   2.635174   0.194272   2.728324
   -0.099979   0.194272   0.583337   0.462054
   -1.404084   2.728324   0.462054   3.969873
=========================================
```

Final Iteration

```
J Dim: 5x4                                        K Dim:5x1 X Dim 4x1
============================================================
    0.98408    0.17772    0.00000    0.00000  -0.05468   0.000627
    0.00000    0.00000    0.94453   -0.32843   0.07901  -0.001286
   -0.30853    0.95121    0.30853   -0.95121  -0.03675  -0.000040
    0.68902    0.72474    0.00000    0.00000   0.06164   0.001814
    0.00000    0.00000    0.99002    0.14092  -0.06393  ========
================================================

JtJ Dim: 4x4
=========================================
    1.538352   0.380777  -0.095191   0.293479
    0.380777   1.461648   0.293479  -0.904809
   -0.095191   0.293479   1.967465  -0.464182
    0.293479  -0.904809  -0.464182   1.032535
=========================================

Qxx = Inv( N ) Dim: 4x4
=========================================
    1.198574  -1.160249  -0.099772  -1.402250
   -1.160249   2.634937   0.193956   2.725964
   -0.099772   0.193956   0.583150   0.460480
   -1.402250   2.725964   0.460480   3.962823
=========================================

Qll = J Qxx Jt Dim: 5x5
================================================
    0.838103   0.233921  -0.108806   0.182506  -0.189263
    0.233921   0.662015   0.157210  -0.263698   0.273460
    0.108806   0.157210   0.926875   0.122656  -0.127197
    0.182506  -0.263698   0.122656   0.794261   0.213356
    0.189263   0.273460  -0.127197   0.213356   0.778746
================================================
```

```
*****************
Adjusted stations
*****************
Station         X             Y            Sx       Sy
================================================
 Wisconsin 2,415,776.904  391,043.294    0.1488   0.2206
    Campus 2,416,892.696  387,603.255    0.1038   0.2705

*****************************
Adjusted Distance Observations
*****************************
Occupied Sighted        Distance       V        S
=============================================
    Badger Wisconsin     5,870.357    0.055   0.1244
    Badger    Campus     7,297.509   -0.079   0.1106
 Wisconsin    Campus     3,616.471    0.037   0.1308
 Wisconsin     Bucky     5,742.816   -0.062   0.1211
 Campus        Bucky     5,123.824    0.064   0.1199

-----Reference Standard Deviation = ±0.135905-----
Iterations » 3
```

Notes

1. As earlier noted, it is important that observed distances not be used in the denominator of the coefficient matrix, J. This is not only theoretically incorrect, but it can also cause slight differences in the final solution or even worse cause the system to diverge from any solution! Always compute distances based on the current approximate coordinates.
2. The final portion of the output lists the adjusted x and y coordinates of the stations, the reference standard deviation, the standard deviations of the adjusted coordinates, the adjusted line lengths, and their residuals.
3. The Q_{xx} matrix was listed on the last iteration only. It is needed for calculating the estimated errors of the adjusted coordinates using Equation (13.24) and is also necessary for calculating error ellipses. The subject of error ellipses is discussed in Chapter 19.

14.6 ITERATION TERMINATION

When programming a nonlinear least-squares adjustment, some criteria must be established to determine the appropriate point at which the iteration process should stop. Since it is possible to have a set of data that has no solution, it is also important to determine when this condition occurs. This section describes three methods used to indicate the appropriate time to end the iteration process.

14.6.1 Method of Maximum Iterations

The simplest procedure of iteration termination involves limiting the number of iterations to a predetermined maximum. The risk with this method is that if this maximum is too low, a solution may not be reached at the conclusion of the process, and if it is too high, time is wasted on unnecessary iterations. Although this method does not assure convergence, it can prevent the adjustment from continuing indefinitely, which could occur if the solution diverges. When good initial approximations are supplied for the unknown parameters, a limit of 10 iterations should be well beyond what is required for a solution since the method of least squares converges quadratically.

14.6.2 Maximum Correction

This method was used in earlier examples. It involves monitoring the absolute size of the corrections. When all corrections become negligibly small, the iteration process can be stopped. The term *negligible* is relative. For example, if distances are observed to the nearest foot, it would be foolish to assume that the size of the corrections will become less than some small fraction of a foot. Generally, *negligible* is interpreted as a correction that is less than one-half the least count of the smallest unit of measure. For instance, if all distances are observed to the nearest 0.01 ft, it would be appropriate to assume convergence when the absolute size of all corrections is less than 0.005 ft. While the solution may continue to converge with continued iterations, the work to get these corrections is not warranted based on the precision of the observations.

14.6.3 Monitoring the Adjustment's Reference Variance

The best method for determining convergence involves monitoring the reference variance and its changes between iterations. Since the method of least squares converges quadratically, the iteration process should be stopped if the reference variance increases. An increasing reference variance suggests a diverging solution, which happens when one of two things has occurred: (1) a large blunder exists in the data set and no solution is possible or (2) the maximum correction-size is less than the precision of the observations. In the second case, the best solution for the given data set has already been reached, and when another iteration is attempted, the solution will converge only to diverge on the next iteration. This *apparent bouncing* in the solution is caused by convergence limits being too stringent for the quality of the data.

By monitoring the reference variance, convergence and divergence can be detected. Convergence is assumed when the change in the reference variance falls below some predefined percentage. Typically, convergence can be assumed when the change in the reference variance is less than 1% between iterations. If the size of the reference variance increases, the solution is diverging and the iteration

process should also be stopped. It should be noted that monitoring changes in the reference variance will always show convergence or divergence in the solution, and thus it is better than any method discussed previously. However, all methods should be used in concert when performing an adjustment.

14.7 SOFTWARE

The hand or calculator computation of the equations presented in this chapter can be tedious and often filled with mistakes. This explains why the method of least squares was not implemented fully until the advent of the digital computer.

Figure 14.7 shows the spreadsheet used to solve Example 14.2. By naming cells, the matrices can be easily developed for use in the MATRIX software, which is available on the companion web site for this book. The spreadsheet allows updates to the approximate coordinates for station U. Since the computed values in the computed distances and the J and K matrices are linked to the changes in the final coordinates for U, these matrices are automatically updated as the corrections are entered. The matrices shown in the figure are then cut and pasted into a new file

Control Station Coordinates

X_A	865.40	X_B	2432.55	X_C	2865.22
Y_A	4527.15	Y_B	2047.25	Y_C	27.15

Unknown Station	**1**	**2**	**3**	**Final**
X_U	6861.325	–0.008209	–3.53E-06	**6861.3168**
Y_U	3727.596	0.2225678	–2.03E-06	**3727.8186**

Distances	Observed	**Computed**
AU	6049.00	**6048.963**
BU	4736.83	**4736.9068**
CU	5446.49	**5446.4426**

File to cut and paste into MATRIX
Example 14.2
J

3	**2**
0.9912306	**–0.132144**
0.9349491	**0.3547819**
0.7337077	**0.6794653**

K

3	**1**
0.037	
–0.077	
0.047	

FIGURE 14.7 Example of a spreadsheet that develops matrices for Example 14.2, which is used in MATRIX.

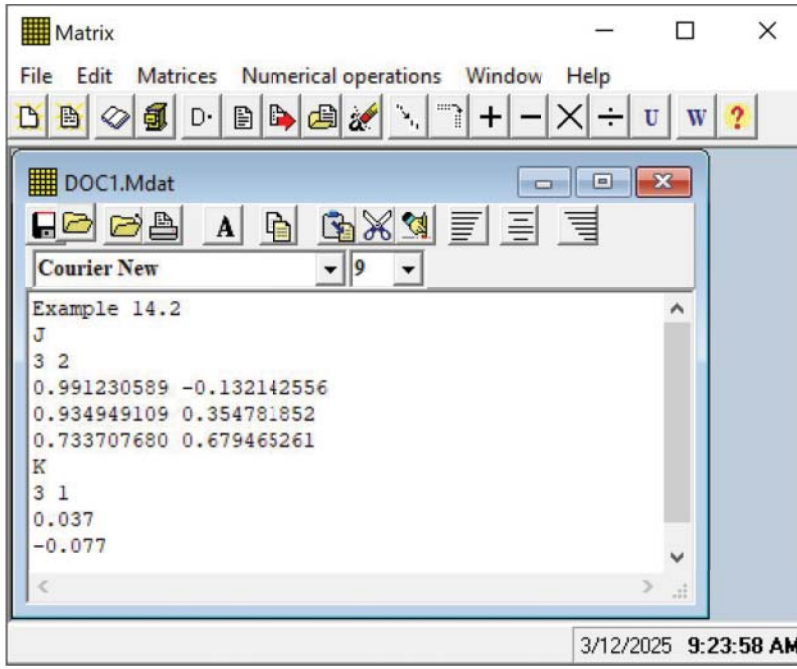


FIGURE 14.8 File format for Example 14.2 in MATRIX.

in MATRIX as shown in Figure 14.8. The MATRIX software computes the least squares solution to the problem, and the resultant corrections are copied and pasted back into the spreadsheet. This process is repeated until the corrections are negligibly small. The file format for MATRIX was discussed in Section 12.7. The file *chapter 14.xls* contains the spreadsheet for Example 14.2 and that presented in Section 14.5.

Figure 14.3 shows the file for Example 14.2 in the ADJUST software, which is available on the companion web site for this book (http://www.wiley.com/college/ghilani). In this figure, the explanatory information is shown to the right of the file and is indicated by the curly brackets {•}. This information should not be entered into the file. A zero is entered into the second line of the file to indicate that this problem did not have any angles or azimuths. The software uses free format reading rules. A comma, space, or tab character can be used to delimit the individual entries. This means that a space cannot be placed in a station identifier. A hyphen or underscore can be used in its place where necessary. Thus, station identifiers like "My Sta" will be interpreted as the two identifiers "My" and "Sta." The station identifier should be entered as "My_Sta" or "MySta to avoid reading errors. Notice that station identifiers in the distance observations are separated by a space. The first observed distance is entered as "A U 6049.00." Since Example 14.2 is unweighted,

the distances do not have standard deviation entries. If this was a weighted example, the standard deviation for each observation would follow the distance observation in each line.

Likewise, a Mathcad electronic book is available on the companion website for this book, which demonstrates a programmatic solution to both Examples 14.2 and 14.5. This software, while very brute force in this example, demonstrates common techniques that are used by commercial software and ADJUST to perform the least squares solutions.

PROBLEMS

Note: For problems requiring least-squares adjustment, if a computer program is not distinctly specified for use in the problem, it is expected that the least-squares algorithm will be solved using the program MATRIX, which is included on the book's companion website. Partial answers to problems marked with an asterisk can be found in Appendix G.

***14.1** The following values observed for the lines in Figure 14.2 are given:

$$AU = 2828.83 \text{ ft} \quad BU = 2031.55 \text{ ft} \quad CU = 2549.83 \text{ ft}$$

The control coordinates of A, B, and C are

Station	x (ft)	y (ft)
A	1418.17	4747.14
B	2434.53	3504.91
C	3234.86	2105.56

What are the most probable values for the adjusted coordinates of station U?

14.2 Do a weighted least-squares adjustment using the data in Problem 14.1 with weights based on the following observational errors.

$$AU = \pm 0.031 \text{ ft} \quad BU = \pm 0.025 \text{ ft} \quad CU = \pm 0.029 \text{ ft}$$

(a) Tabulate the most probable values for the adjusted coordinates of station U and their standard deviations.

*(b) What is the reference standard deviation of unit weight?

(c) Tabulate the adjusted distances, their residuals and standard deviations.

14.3 Do a least-squares adjustment for the following observed values in the accompanying figure.

$$AC = 2401.51 \text{ ft} \quad AD = 2296.96 \text{ ft} \quad BC = 1938.10 \text{ ft}$$
$$BD = 2492.94 \text{ ft} \quad CD = 1535.07 \text{ ft}$$

In the adjustment, hold the coordinates of Stations A and B (in units of feet) of

$$x_a = 10{,}000.00 \text{ ft and } y_a = 10{,}000.00 \text{ ft}$$

$$x_b = 10{,}258.82 \text{ ft and } y_b = 10{,}965.92 \text{ ft}$$

Use the initial approximations in feet for C and D of

$$x_c = 12{,}196.90, \quad y_c = 10{,}969.90, \quad \text{and} \quad x_d = 12{,}226.40, \quad y_d = 9{,}435.20$$

***(a)** Tabulate the most probable values and standard deviations for the adjusted coordinates of stations C and D.

(b) What is the reference standard deviation of unit weight?

(c) Tabulate the adjusted distances, their residuals and the standard deviations.

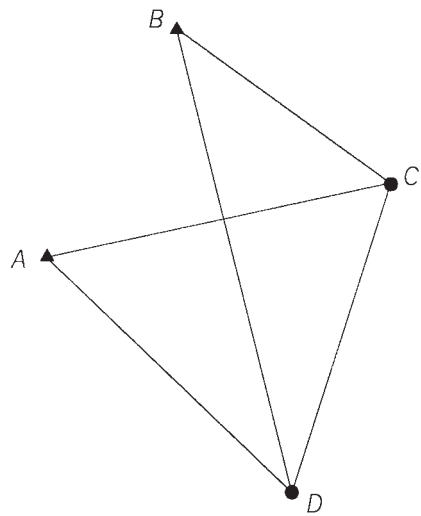


FIGURE P14.3

14.4 Repeat Problem 14.3 using a weighted least-squares adjustment, where the distance standard deviations are

$AC = \pm 0.014$ ft $\quad AD = \pm 0.014$ ft $\quad BC = \pm 0.014$ ft
$BD = \pm 0.015$ ft $\quad CD = \pm 0.015$ ft

14.5 Use the ADJUST software to do Problems 14.3 and 14.4. Explain any differences in the adjustments.

14.6 Using the trilaterated Figure 14.5 and the data below,

(a) Tabulate the most probable station coordinates and their standard deviations.

(b) Reference standard deviation of unit weight.

(c) Tabulate the adjusted distances, their residuals, and standard deviations.

Initial Approximations

Station	Easting (ft)	Northing (ft)
B	10,624.20	11,036.70
D	12,176.10	9,385.20
E	10,449.80	9,360.25
F	11,539.00	10,178.90

Control Stations

Station	Easting (ft)	Northing (ft)
A	10,000.00	10,000.00
C	12,487.08	10,528.65

Distance Observations

Station Occupied	Station Sighted	Distance (ft)	*S* (ft)
A	*B*	1210.10	0.052
B	*C*	1930.88	0.052
C	*D*	1185.02	0.051
D	*E*	1726.36	0.052
E	*A*	782.12	0.052
B	*E*	1685.48	0.052
B	*F*	1253.93	0.051
F	*E*	1362.49	0.051
F	*C*	1010.63	0.050
F	*D*	1017.84	0.050

14.7 Using the station coordinates and trilateration data given below:

***(a)** Tabulate the most probable coordinates for station *F* and their standard deviations.

(b) Reference standard deviation of unit weight.

(c) Tabulate the adjusted distances, their residuals, and standard deviations.

Control stations

Station	Easting (m)	Northing (m)
A	4423.671	3762.737
B	3799.456	4799.439
C	1936.591	4291.387
D	2247.584	3147.905
E	3973.864	3122.989

Initial approximations

Station	Easting (m)	Northing (m)
F	2884.71	3941.64

Distance Observations

From	To	Distance (m)	*S* (m)
A	*F*	1549.331	0.008
B	*F*	1254.039	0.007
C	*F*	1010.572	0.007
D	*F*	1017.807	0.007
E	*F*	1362.502	0.008

14.8 Repeat Problem 14.7 using the following data.

Control Stations

Station	Easting (ft)	Northing (ft)
A	3314.74	2626.96
B	3938.95	3663.66
C	5801.82	3155.61
D	5940.83	2012.13
E	3764.55	1987.21

Initial Approximations

Station	Easting (ft)	Northing (ft)
F	4853.70	2805.86

Distance observations

From	To	Distance (ft)	*S* (ft)
A	*F*	1549.30	0.009
B	*F*	1254.03	0.008
C	*F*	1010.56	0.007
D	*F*	1346.06	0.008
E	*F*	1362.52	0.008

14.9 Using the station coordinates and trilateration data given below, find:

(a) Most probable coordinates for the unknown stations and their standard deviations.

(b) Reference standard deviation of unit weight.

(c) The adjusted distances, their residuals, and standard deviations.

Control Stations

Station	X (m)	Y (m)
A	1548.316	1303.950
F	9104.265	1316.053

Initial Approximations

Station	X (m)	Y (m)
B	1650.376	4545.359
C	5498.195	4583.444
D	5451.779	1292.474
E	9099.866	4885.346

Distance Observations

From	To	Distance (m)	S (m)
A	*B*	3243.014	0.008
A	*C*	5133.893	0.012
B	*D*	3903.474	0.009
B	*C*	3848.012	0.009
B	*D*	5003.193	0.011
C	*D*	3291.303	0.008
C	*E*	3614.305	0.009
C	*F*	4866.172	0.011
D	*E*	5120.278	0.011
D	*F*	3652.550	0.009
E	*F*	3569.299	0.009

14.10 Using the station coordinates and trilateration data given below, find:

(a) Most probable coordinates for the unknown stations and their standard deviations.

(b) Reference standard deviation of unit weight.

(c) The adjusted distances, their residuals, and standard deviations.

Control Station

Station	X (ft)	Y (ft)
A	7,220.29	5,497.71
C	14,717.09	7,272.39

Initial Approximations

Station	X (ft)	Y (ft)
B	11,123.75	5,486.23
D	11,170.17	8,777.20
E	7,322.35	8,739.12

Distance Observations

From	To	Distance (ft)	S (ft)
A	B	3903.46	0.015
B	C	4012.78	0.015
C	D	3852.92	0.015
D	E	3848.01	0.015
A	D	5133.85	0.017
B	E	5003.21	0.017
B	D	3291.29	0.015
E	A	3242.99	0.015

14.11 Perform a χ^2 test at a 0.05 level of significance to check if the reference variance from Problem 14.4 is statistically equal to its a priori value of one.

14.12 Repeat Problem 14.11 checking the reference variance for Problem 14.9.

14.13 Repeat Problem 14.11 checking the reference variance for Problem 14.10.

14.14 Use the ADJUST software to do Problem 14.4.

14.15 Use the ADJUST software to do Problem 14.9.

14.16 The reference variance for the first iteration of a weighted least-squares adjustment is 1.586. On the second iteration, it is 1.573. Has the adjustment converged? Justify your response.

14.17 What is the standard deviation of unit weight for the adjustment in Problem 14.16?

14.18 Describe in detail the methods used to detect convergence in a nonlinear least-squares adjustment and the advantages and disadvantages of each.

PROGRAMMING PROBLEMS

14.19 Create a computational program that computes the distance, coefficients, and $k_{l_{ij}}$ in Equation (14.9) between stations I and J given their initial coordinate values. Use this spreadsheet to determine the matrix values necessary for solving Problem 14.9.

14.20 Create a computational program that reads a data file containing station coordinates and distances and generates the J, W, and K matrices that can be used by the MATRIX program. Demonstrate that this program works by using the data of Problem 14.9.

14.21 Create a computational program that reads a file containing the J, W, and K matrices and finds the most probable value for the station coordinates, the reference standard deviation, and the standard deviations of the station coordinates. Demonstrate that this program works by solving Problem 14.9.

14.22 Create a computational program that reads a file containing control station coordinates, initial approximations of unknown stations, and distance observations. The program should generate the appropriate matrices for a least-squares adjustment, do the adjustment, and print out the final adjusted coordinates, their standard deviations, the final adjusted distances, their residuals, and the standard deviations in the adjusted distances. Demonstrate that this program works by solving Problem 14.9.

CHAPTER 15

ADJUSTMENT OF HORIZONTAL SURVEYS: TRIANGULATION

15.1 INTRODUCTION

Prior to the development of electronic distance measuring equipment and the global navigation satellite systems, triangulation was the preferred method for extending horizontal control over long distances. The positions of widely spaced stations were computed from observed angles and a minimal number of observed distances called *baselines*. This method was used extensively by the National Geodetic Survey in extending much of the national network. Triangulation is still used by many surveyors in establishing horizontal control, although surveys that combine trilateration (distance observations) with triangulation (angle observations) are more common. In this chapter, methods are described for adjusting triangulation networks using the least-squares method.

A least-squares triangulation adjustment can use condition equations or observation equations written either in terms of azimuths or angles. In this chapter, the observation equation method is presented. The procedure involves a parametric adjustment where the parameters are coordinates in a plane rectangular system, such as state plane coordinates. In the examples, the specific types of triangulations known as intersections, resections, and quadrilaterals will be adjusted.

15.2 AZIMUTH OBSERVATION EQUATION

The azimuth equation in parametric form is

$$\text{azimuth} = \alpha + C \tag{15.1}$$

where $\alpha = \tan^{-1}\left(\frac{x_j - x_i}{y_j - y_i}\right)$, x_i and y_i are the coordinates of the occupied station I; x_j and y_j are the coordinates of the sighted station J; and C is a constant that depends on the quadrant in which point J lies, as shown in Figure 15.1. From Figure 15.1, Table 15.1 can be constructed that relates the algebraic sign of the computed angle α in Equation (15.1) [1] to the value of C and the value of the azimuth.

15.2.1 Linearization of the Azimuth Observation Equation

Referring to Equation (15.1), the observation equation for an observed azimuth of line IJ is

$$\tan^{-1}\left(\frac{x_j - x_i}{y_j - y_i}\right) + C = Az_{ij} + v_{Az} \tag{15.2}$$

where Az_{ij} is the observed azimuth from station I to station J; v_{Az} is the residual in the observed azimuth; x_i and y_i are the most probable values for the coordinates

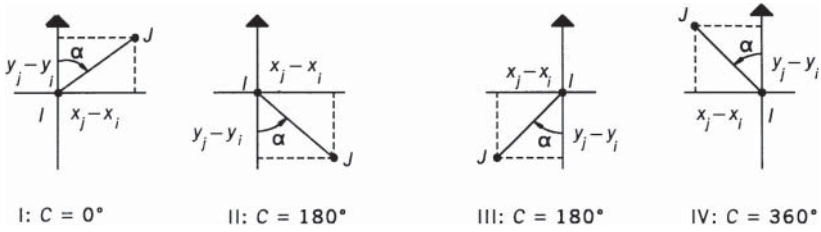


FIGURE 15.1 Relationship between the azimuth and the computed angle, α.

TABLE 15.1 Relationship between the Quadrant, C, and Azimuth

Quadrant	Sign $(x_j - x_i)$	Sign $(y_j - y_i)$	Sign α	C	Azimuth
I	+	+	+	0	A
II	+	−	−	180°	$\alpha + 180°$
III	−	−	+	180°	$\alpha + 180°$
IV	−	+	−	360°	$\alpha + 360°$

[1] An alternative form for computing the azimuth of a line is $Azimuth_{ij} = \text{atan2}(y_j - y_i, x_j - x_i) + D$, where $D = 360°$ if the results of the atan2() function is negative, otherwise, $D = 0$. The *a*tan2() function is often found in computer software such as spreadsheets and programming languages.

of station I; x_j and y_j are the most probable values for the coordinates of station J; and C is a constant with a value based on Table 15.1. Equation (15.2) is a nonlinear function involving variables x_i, y_i, x_j, and y_j, which can be rewritten as

$$F(x_i, y_i, x_j, y_j) = Az_{ij} + v_{Az} \tag{15.3}$$

where

$$F(x_i, y_i, x_j, y_j) = \tan^{-1}\left(\frac{x_j - x_i}{y_j - y_i}\right) + C.$$

As discussed in Section 11.10, nonlinear equations such as (15.3) can be linearized and solved using a first-order Taylor series approximation. The linearized form of Equation (15.3) is

$$\begin{aligned} F(x_i, y_i, x_j, y_j) = F(x_i, y_i, x_j, y_j)_0 &+ \left(\frac{\partial F}{\partial x_i}\right)_0 dx_i + \left(\frac{\partial F}{\partial y_i}\right)_0 dy_i \\ &+ \left(\frac{\partial F}{\partial x_j}\right)_0 dx_j + \left(\frac{\partial F}{\partial y_j}\right)_0 dy_j \end{aligned} \tag{15.4}$$

where $(\partial F/\partial x_i)_0$, $(\partial F/\partial y_i)_0$, $(\partial F/\partial x_j)_0$, and $(\partial F/\partial y_j)_0$ are the partial derivatives of F with respect to x_i, y_i, x_j, and y_j that are evaluated at the initial approximations and dx_i, dy_i, dx_j, and dy_j are the corrections applied to the initial approximations after each iteration, such that

$$x_i = x_{i_0} + dx_i, y_i = y_{i_0} + dy_i, x_j = x_{j_0} + dx_j, \text{and } y_j = y_{j_0} + dy_j \tag{15.5}$$

To determine the partial derivatives of Equation (15.4) requires the prototype equation for the derivative of $\tan^{-1}(u)$ with respect to x, which is

$$\frac{d}{dx}\tan^{-1}(u) = \frac{1}{1+u^2}\frac{du}{dx} \tag{15.6}$$

Using Equation (15.6), the procedure for determining the $\partial F/\partial x_i$ is demonstrated as follows:

$$\begin{aligned} \frac{\partial F}{\partial x_i} &= \frac{1}{1 + [(x_j - x_i)/(y_j - y_i)]^2}\frac{-1}{y_j - y_i} \\ &= \frac{-1(y_j - y_i)}{(x_j - x_i)^2 + (y_j - y_i)^2} \\ &= \frac{y_i - y_j}{IJ^2} \end{aligned} \tag{15.7}$$

By employing similar procedures, the remaining partial derivatives are

$$\frac{\partial F}{\partial y_i} = \frac{x_j - x_i}{IJ^2}, \quad \frac{\partial F}{\partial x_j} = \frac{y_j - y_i}{IJ^2}, \quad \frac{\partial F}{\partial y_j} = \frac{x_i - x_j}{IJ^2} \tag{15.8}$$

where $IJ^2 = (x_j - x_i)^2 + (y_j - y_i)^2$.

If Equations (15.7) and (15.8) are substituted into Equation (15.4) and the results are then substituted into Equation (15.3), the following prototype azimuth equation is obtained:

$$\left(\frac{y_i - y_j}{IJ^2}\right)_0 dx_i + \left(\frac{x_j - x_i}{IJ^2}\right)_0 dy_i + \left(\frac{y_j - y_i}{IJ^2}\right)_0 dx_j + \left(\frac{x_i - x_j}{IJ^2}\right)_0 dy_j = k_{Az_{ij}} + v_{Az_{ij}} \tag{15.9}$$

Both $k_{Az_{ij}} = Az_{ij} - \left[\tan^{-1}\left(\frac{x_j - x_i}{y_j - y_i}\right)_0 + C\right]$ and $IJ^2 = (x_j - x_i)_0^2 + (y_j - y_i)_0^2$ are evaluated using the approximate coordinate values of the unknown parameters.

15.3 ANGLE OBSERVATION EQUATION

Figure 15.2 illustrates the geometry for an angle observation. In Figure 15.2, B is the backsight station; I is the instrument station; and F is the foresight station. As shown in the figure, an angle observation equation can be written as the difference between two azimuth observations, and thus for clockwise angles:

$$\angle BIF = Az_{IF} - Az_{IB} = \tan^{-1}\left(\frac{x_f - x_i}{y_f - y_i}\right) - \tan^{-1}\left(\frac{x_b - x_i}{y_b - y_i}\right) + D = \theta_{bif} + v_\theta \tag{15.10}$$

where θ_{bif} is the observed clockwise angle; v_θ is the residual in the observed angle; x_b and y_b are the most probable values for the coordinates of the backsight station B; x_i and y_i are the most probable values for the coordinates of the instrument station I;

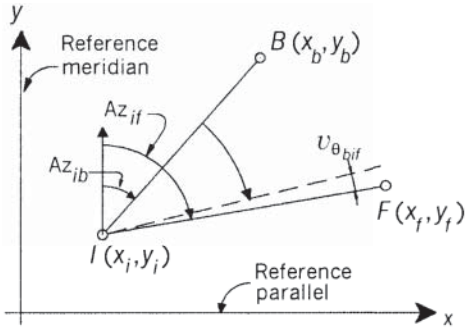


FIGURE 15.2 Relationship between an angle and two azimuths.

x_f and y_f are the most probable values for the coordinates of the foresighted station F; and D is a constant that depends on the quadrants in which the backsight and foresight occur. This term can be computed as the difference between the C terms from Equation (15.1) as applied to the backsight and foresight azimuths; that is,

$$D = C_{if} - C_{ib}$$

Equation (15.10) is a nonlinear function of x_b, y_b, x_i, y_i, x_f, and y_f that can be rewritten as

$$F(x_b, y_b, x_i, y_i, x_f, y_f) = \theta_{bif} + v_\theta \tag{15.11}$$

where

$$F(x_b, y_b, x_i, y_i, x_f, y_f) = \tan^{-1}\left(\frac{x_f - x_i}{y_f - y_i}\right) - \tan^{-1}\left(\frac{x_b - x_i}{y_b - y_i}\right) + D.$$

Equation (15.11), expressed as a linearized first-order Taylor series expansion, is

$$\begin{aligned} F(x_b, y_b, x_i, y_i, x_f, y_f) = F(x_b, y_b, x_i, y_i, x_f, y_f)_0 &+ \left(\frac{\partial F}{\partial x_b}\right)_0 dx_b \\ &+ \left(\frac{\partial F}{\partial y_b}\right)_0 dy_b + \left(\frac{\partial F}{\partial x_i}\right)_0 dx_i + \left(\frac{\partial F}{\partial y_i}\right)_0 dy_i \\ &+ \left(\frac{\partial F}{\partial x_f}\right)_0 dx_f + \left(\frac{\partial F}{\partial y_f}\right)_0 dy_f \end{aligned} \tag{15.12}$$

where $\partial F/\partial x_b$, $\partial F/\partial y_b$, $\partial F/\partial x_i$, $\partial F/\partial y_i$, $\partial F/\partial x_f$, and $\partial F/\partial y_f$ are the partial derivatives of F with respect to x_b, y_b, x_i, y_i, x_f, and y_f, respectively.

Evaluating partial derivatives of the function F and substituting into Equation (15.12), then substituting into Equation (15.11), results in the following equation:

$$\begin{aligned} &\left(\frac{y_i - y_b}{IB^2}\right)_0 dx_b + \left(\frac{x_b - x_i}{IB^2}\right)_0 dy_b + \left(\frac{y_b - y_i}{IB^2} - \frac{y_f - y_i}{IF^2}\right)_0 dx_i \\ &\quad + \left(\frac{x_i - x_b}{IB^2} - \frac{x_i - x_f}{IF^2}\right)_0 dy_i + \left(\frac{y_f - y_i}{IF^2}\right)_0 dx_f + \left(\frac{x_i - x_f}{IF^2}\right)_0 dy_f = k_\theta + v_\theta \end{aligned} \tag{15.13}$$

where

$$\begin{aligned} k_\theta &= \theta_{bif} - \theta_{bif_0} \\ \theta_{bif_0} &= \tan^{-1}\left(\frac{x_f - x_i}{y_f - y_i}\right)_0 - \tan^{-1}\left(\frac{x_b - x_i}{y_b - y_i}\right)_0 + D \end{aligned}$$

$$IB^2 = (x_b - x_i)^2 + (y_b - y_i)^2$$
$$IF^2 = (x_f - x_i)^2 + (y_f - y_i)^2$$

are evaluated at the approximate values for the unknown coordinates. Formulating the linearized angle observation equation can be thought of as the difference between two linearized azimuth equations. Using Equation (15.9) as a guide, the difference between the foresight and backsight azimuth is

$$\left(\frac{y_i - y_f}{IF^2}\right)_0 dx_i + \left(\frac{x_f - x_i}{IF^2}\right)_0 dy_i + \left(\frac{y_f - y_i}{IF^2}\right)_0 dx_f + \left(\frac{x_i - x_f}{IF^2}\right)_0 dy_f$$
$$- \left[\left(\frac{y_i - y_b}{IB^2}\right)_0 dx_i + \left(\frac{x_b - x_i}{IB^2}\right)_0 dy_i + \left(\frac{y_b - y_i}{IB^2}\right)_0 dx_b + \left(\frac{x_i - x_b}{IB^2}\right)_0 dy_b\right]$$
$$= k_{BIF} + v_\theta \qquad (a)$$

Rearranging and regrouping Equation (*a*) yields Equation (15.13). This is left as an exercise for the reader.

In formulating the angle observation equation, remember that I is always assigned to the instrument station, B the backsight, and F the foresight station. This station designation must be strictly followed in employing prototype Equation (15.13) as will be demonstrated in the numerical examples that follow and reproduced in the spreadsheet file *Chapter 15.xls*. As each is discussed, the reader may wish to refer to this file to see how the problem is implemented in a spreadsheet. Additionally, the file *C15.xmcd* demonstrates these examples in the higher-level programming language of Mathcad. These files can be found on the companion website for this book.

15.4 ADJUSTMENT OF INTERSECTIONS



When an unknown station is visible from two or more existing control stations, the angle intersection method is one of the simplest and sometimes most practical ways for determining the horizontal position of a station. For a unique computation, the method requires the observation of at least two horizontal angles from two control points. For example, angles θ_1 and θ_2 observed from control stations R and S in Figure 15.3 will enable a unique computation for the position of station U. If additional control is available, the computations for the unknown station's position can be strengthened by observing redundant angles such as angles θ_3 and θ_4 in Figure 15.3 and applying the method of least squares. In that case, for each of the four independent angles, a linearized observation equation is written in terms of the two unknown coordinates x_u and y_u.

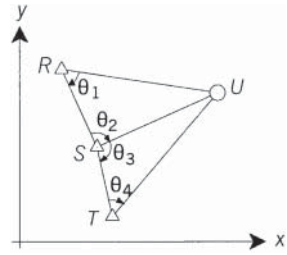


FIGURE 15.3 Intersection example.

Example 15.1 Using the method of least squares, compute the most probable coordinates of station U in Figure 15.3. The following equally weighted horizontal angles were observed from control stations R, S, and T:

$$\theta_1 = 50°06'50'' \qquad \theta_2 = 101°30'47'' \qquad \theta_3 = 98°41'17'' \qquad \theta_4 = 59°17'01''$$

The coordinates for the control stations R, S, and T are:

$$x_r = 865.40 \qquad x_s = 2432.55 \qquad x_t = 2865.22$$
$$y_r = 4527.15 \qquad y_s = 2047.25 \qquad y_t = 27.15$$

SOLUTION

Step 1: Determine initial approximations for the coordinates of station U.

(a) Using the coordinates of stations R and S, the distance RS is computed as

$$RS = \sqrt{(2432.55 - 865.40)^2 + (4527.15 - 2047.25)^2} = 2933.58 \text{ ft}$$

(b) From the coordinates of stations R and S, the azimuth of the line between R and S is determined using Equation (15.2). Then the initial azimuth of line RU is computed by subtracting θ_1 from the azimuth of line RS.

$$Az_{RS} = \tan^{-1}\left(\frac{x_s - x_r}{y_s - y_r}\right) + C = \tan^{-1}\left(\frac{865.40 - 2432.55}{4527.15 - 2047.25}\right) + 180° = 147°42'34''$$

$$Az_{RU_0} = 147°42'34'' - 50°06'50'' = 97°35'44''$$

(c) Using the law of sines with triangle RUS, an initial length for RU_0 can be calculated as

$$RU_0 = \frac{RS \sin\theta_2}{\sin(180° - \theta_1 - \theta_2)} = \frac{2933.58 \sin(100°30'47'')}{\sin(28°27'23'')} = 6049.00 \text{ ft}$$

(d) Using the azimuth and distance for RU_0 computed in steps 1(b) and 1(c), initial coordinates for station U are computed as

$$\begin{aligned} x_{u_0} &= x_r + RU_0 \sin Az_{RU_0} = 865.40 + 6049.00 \sin(97°35'44'') \\ &= 6861.35 \\ y_{u_0} &= y_r + RU_0 \cos Az_{RU_0} = 865.40 + 6049.00 \cos(97°35'44'') \\ &= 3727.59 \end{aligned}$$

(e) Using the appropriate coordinates, the initial distances for SU and TU are calculated as

$$SU_0 = \sqrt{(6861.35 - 2432.55)^2 + (3727.59 - 2047.25)^2} = 4736.83 \text{ ft}$$

$$TU_0 = \sqrt{(6861.35 - 2865.22)^2 + (3727.59 - 27.15)^2} = 5446.29 \text{ ft}$$

Step 2: Formulate the linearized equations. As in the trilateration adjustment, control station coordinates are held fixed during the adjustment by assigning zeros to their dx and dy values. Thus, these terms drop out of prototype Equation (15.13). In forming the observation equations, b, i, and f are assigned to the backsight, instrument, and foresight stations, respectively, for each angle. For example, with angle θ_1, B, I, and F are replaced by U, R, and S, respectively. By combining the station substitutions shown in Table 15.2 with prototype Equation (15.13), the following observation equations are written for the four observed angles:

$$\left(\frac{y_r - y_u}{RU^2}\right)_0 dx_u + \left(\frac{x_u - x_r}{RU^2}\right)_0 dy_u = \theta_1 - \left[\tan^{-1}\left(\frac{x_s - x_r}{y_s - y_r}\right) - \tan^{-1}\left(\frac{x_u - x_r}{y_u - y_r}\right)_0 + 0°\right] + v_1$$

$$\left(\frac{y_u - y_s}{SU^2}\right)_0 dx_u + \left(\frac{x_s - x_u}{SU^2}\right)_0 dy_u = \theta_2 - \left[\tan^{-1}\left(\frac{x_u - x_s}{y_u - y_s}\right)_0 - \tan^{-1}\left(\frac{x_r - x_s}{y_r - y_s}\right) + 0°\right] + v_2$$

$$\left(\frac{y_s - y_u}{SU^2}\right)_0 dx_u + \left(\frac{x_u - x_s}{SU^2}\right)_0 dy_u = \theta_3 - \left[\tan^{-1}\left(\frac{x_t - x_s}{y_t - y_s}\right) - \tan^{-1}\left(\frac{x_u - x_s}{y_u - y_s}\right)_0 + 180°\right] + v_3$$

$$\left(\frac{y_u - y_t}{TU^2}\right)_0 dx_u + \left(\frac{x_t - x_u}{TU^2}\right)_0 dy_u$$
$$= \theta_4 - \left[\tan^{-1}\left(\frac{x_u - x_t}{y_u - y_t}\right)_0 - \tan^{-1}\left(\frac{x_s - x_t}{y_s - y_t}\right) + 0^\circ\right] + v_4 \qquad (15.14)$$

TABLE 15.2 Substitutions

Angle	B	I	F
θ_1	U	R	S
θ_2	R	S	U
θ_3	U	S	T
θ_4	S	T	U

Substituting the appropriate values into Equations (15.14) and multiplying the left side of the equations by ρ to achieve unit consistency,[2] the following J and K matrices are formed:

$$J = \rho\begin{bmatrix} \dfrac{4527.15 - 3727.59}{6049.00^2} & \dfrac{6861.35 - 865.40}{6049.00^2} \\ \dfrac{3727.59 - 2047.25}{4736.83^2} & \dfrac{2432.55 - 6861.35}{4736.83^2} \\ \dfrac{2047.25 - 3727.59}{4736.83^2} & \dfrac{6861.35 - 2432.55}{4736.83^2} \\ \dfrac{3727.59 - 27.15}{5446.29^2} & \dfrac{2865.22 - 6861.35}{5446.29^2} \end{bmatrix} = \begin{bmatrix} 4.507 & 33.800 \\ 15.447 & -40.713 \\ -15.447 & 40.713 \\ 25.732 & -27.788 \end{bmatrix}$$

$$K = \begin{bmatrix} 50^\circ 06' 50'' - \left[\tan^{-1}\left(\dfrac{2432.55 - 865.40}{2047.25 - 4527.15}\right) - \tan^{-1}\left(\dfrac{6861.35 - 865.40}{3727.59 - 4527.15}\right) + 0^\circ\right] \\ 101^\circ 30' 47'' - \left[\tan^{-1}\left(\dfrac{6861.35 - 2432.55}{3727.59 - 2047.25}\right) - \tan^{-1}\left(\dfrac{865.40 - 2432.55}{4527.15 - 2047.25}\right) + 0^\circ\right] \\ 98^\circ 41' 17'' - \left[\tan^{-1}\left(\dfrac{2865.22 - 2432.55}{27.15 - 2047.25}\right) - \tan^{-1}\left(\dfrac{6861.35 - 2432.55}{3727.59 - 2047.25}\right) + 180^\circ\right] \\ 59^\circ 17' 01'' - \left[\tan^{-1}\left(\dfrac{6861.35 - 2865.22}{3727.59 - 27.15}\right) - \tan^{-1}\left(\dfrac{2432.55 - 2865.22}{2047.25 - 27.15}\right) + 0^\circ\right] \end{bmatrix}$$
$$= \begin{bmatrix} 0.00'' \\ 0.00'' \\ -0.69'' \\ -20.23'' \end{bmatrix}$$

[2]For these observations to be dimensionally consistent, the elements of the K and V matrices must be in radian measure, or in other words, the coefficients of the K and J elements must be in the same units. Since it is most common to work in the sexagesimal system, and since the magnitudes of the angle residuals are generally in the range of seconds, the units of the equations are converted to seconds by (1) multiplying the coefficients in the equation by ρ, which is the number of seconds per radian, or 206,264.8″/rad, and (2) computing the K elements in units of seconds.

Notice that the initial coordinates for x_{u_0} and y_{u_0} were calculated using θ_1 and θ_2, and thus their K-matrix values are zero for the first iteration. These values will change in subsequent iterations.

Step 3: Matrix solution. The least-squares solution is found by applying Equation (11.37).

$$J^{\mathrm{T}}J = \begin{bmatrix} 1159.7 & -1820.5 \\ -1820.5 & 5229.7 \end{bmatrix}$$

$$Q_{xx} = (J^{\mathrm{T}}J)^{-1} = \begin{bmatrix} 0.001901 & 0.000662 \\ 0.000662 & 0.000422 \end{bmatrix}$$

$$J^{\mathrm{T}}K = \begin{bmatrix} -509.9 \\ 534.1 \end{bmatrix}$$

$$X = (J^{\mathrm{T}}J)^{-1}(J^{\mathrm{T}}K) = \begin{bmatrix} 0.001901 & 0.000662 \\ 0.000662 & 0.000422 \end{bmatrix} \begin{bmatrix} -509.9 \\ 534.1 \end{bmatrix} = \begin{bmatrix} dx_u \\ dy_u \end{bmatrix}$$

$$dx_u = -0.62 \text{ ft and } dy_u = -0.11 \text{ ft}$$

Step 4: Add the corrections to the initial coordinates for station U.

$$x_u = x_{u_0} + dx_u = 6861.35 - 0.62 = 6860.73$$
$$y_u = y_{u_0} + dy_u = 3727.59 - 0.11 = 3727.48 \qquad (15.15)$$

Step 5: Repeat Steps 2 through 4 until negligible corrections occur. The next iteration produced negligible corrections, and thus Equations (15.15) produced the final adjusted coordinates for station U.

Step 6: Compute post-adjustment statistics. The residuals for the angles are

$$V = JX - K = \begin{bmatrix} 4.507 & 33.80 \\ 15.447 & -40.713 \\ -15.447 & 40.713 \\ 25.732 & -27.788 \end{bmatrix} \begin{bmatrix} -0.62 \\ -0.11 \end{bmatrix} - \begin{bmatrix} 0.00'' \\ 0.00'' \\ -0.69'' \\ -20.23'' \end{bmatrix} = \begin{bmatrix} -6.5'' \\ -5.1'' \\ 5.8'' \\ 7.3'' \end{bmatrix}$$

The reference standard deviation for the adjustment is computed using Equation (12.14) as

$$V^{\mathrm{T}}V = [-6.5 \; -5.1 \;\; 5.8 \;\; 7.3] \begin{bmatrix} -6.5 \\ -5.1 \\ 5.8 \\ 7.3 \end{bmatrix} = [155.2]$$

$$S_0 = \sqrt{\frac{V^{\mathrm{T}}V}{m-n}} = \sqrt{\frac{155.2}{4-2}} = \pm 8.8''$$

The estimated errors for the adjusted coordinates of station U, given by Equation (13.24), are

$$S_{x_u} = S_0\sqrt{Q_{x_u x_u}} = \pm 8.8\sqrt{0.001901} = \pm 0.38 \text{ ft}$$

$$S_{y_u} = S_0\sqrt{Q_{y_u y_u}} = \pm 8.8\sqrt{0.000422} = \pm 0.18 \text{ ft}$$

The estimated error in the position of station U is given by

$$S_u = \sqrt{S_x^2 + S_y^2} = \sqrt{0.38^2 + 0.18^2} = \pm 0.42 \text{ ft}$$

15.5 ADJUSTMENT OF RESECTIONS



Resection is a method used for determining the unknown horizontal position of an occupied station by observing a minimum of two horizontal angles to a minimum of three stations whose horizontal coordinates are known. If more than three stations are available, redundant observations are obtained, and the position of the unknown occupied station can be computed using the least-squares method. Like the intersection method, resection is suitable for locating an occasional station and is especially well adapted over inaccessible terrain. This method is commonly used for orienting total station instruments in locations favorable for staking projects by radiation using coordinates.

Consider the resection position computation for the occupied station U of Figure 15.4 having observed the three horizontal angles shown between stations P, Q, R, and S whose positions are known. To determine the position of station U, two angles could be observed. The third angle provides a check and allows a least-squares solution for computing the coordinates of station U.

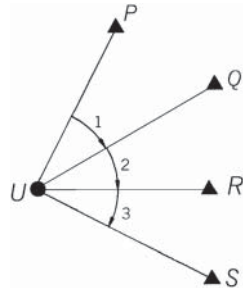


FIGURE 15.4 Resection example.

Using prototype Equation (15.13), a linearized observation equation can be written for each angle. In this problem, the vertex station is occupied and is the only unknown station. Thus, all coefficients in the Jacobian matrix follow the form used for the coefficients of dx_i and dy_i in prototype Equation (15.13). The method of least squares yields corrections, dx_u and dy_u, which gives the most probable coordinate values for station U.

15.5.1 Computing Initial Approximations in the Resection Problem

In Figure 15.4, only two angles are necessary to determine the coordinates of station U. Using stations P, Q, R, and U, a procedure to find the station U's approximate coordinate values is

Step 1: Let

$$\angle QPU + \angle URQ = G = 360^\circ - (\angle 1 + \angle 2 + \angle RQP) \tag{15.16}$$

Step 2: Using the sine law with triangle PQU,

$$\frac{QU}{\sin \angle QPU} = \frac{PQ}{\sin \angle 1} \tag{a}$$

and with triangle URQ,

$$\frac{QU}{\sin \angle URQ} = \frac{QR}{\sin \angle 2} \tag{b}$$

Step 3: Solving Equations (*a*) and (*b*) for QU and setting the resulting equations equal yields

$$\frac{PQ \sin \angle QPU}{\sin \angle 1} = \frac{QR \sin \angle URQ}{\sin \angle 2} \tag{c}$$

Step 4: From Equation (*c*), let H be defined as

$$H = \frac{\sin \angle QPU}{\sin \angle URQ} = \frac{QR \sin \angle 1}{PQ \sin \angle 2} \tag{15.17}$$

Step 5: From Equation (15.16),

$$\angle QPU = G - \angle URQ \tag{d}$$

Step 6: Solving Equation (15.17) for the $\sin \angle QPU$, and substituting Equation (d) into the result yields

$$\sin(G - \angle URQ) = H \sin \angle URQ \tag{e}$$

Step 7: From trigonometry

$$\sin(\alpha - \beta) = \sin\ \alpha\ \cos\ \beta - \cos\ \alpha\ \sin\ \beta$$

Applying this relationship to Equation (*e*),

$$\sin(G - \angle URQ) = \sin\ G\ \cos\angle URQ - \cos\ G\ \sin\angle URQ \qquad (f)$$

$$\sin(G - \angle URQ) = H\ \sin\angle URQ \qquad (g)$$

Step 8: Dividing Equation (*g*) by cos $\angle URQ$ and rearranging yields

$$\sin G = \tan\angle URQ(H + \cos G) \qquad (h)$$

Step 9: Solving Equation (*h*) for $\angle URQ$ gives

$$\angle URQ = \tan^{-1}\left(\frac{\sin(G)}{H + \cos(G)}\right) \qquad (15.18)$$

Step 10: From Figure 15.4,

$$\angle RQU = 180^\circ - (\angle 2 + \angle URQ) \qquad (15.19)$$

Step 11: Again, applying the sine law yields

$$RU = \frac{QR \sin\angle RQU}{\sin\angle 2} \qquad (15.20)$$

Step 12: Finally, the initial coordinates for station U are

$$x_u = x_r + RU\sin(Az_{RQ} - \angle URQ)$$
$$y_u = y_r + RU\cos(Az_{RQ} - \angle URQ) \qquad (15.21)$$

Example 15.2 The following data are obtained for Figure 15.4. Control stations *P*, *Q*, *R*, and *S* have the following (*x, y*) coordinates: *P* (1303.599, 1458.615), *Q* (1636.436, 1310.468), *R* (1503.395, 888.362), and *S* (1506.262, 785.061). The observed values for angles 1, 2, and 3 with standard deviations are

Backsight	Occupied	Foresight	Angle	S ($''$)
P	*U*	*Q*	30°29′33″	±5
Q	*U*	*R*	38°30′31″	±6
R	*U*	*S*	10°29′57″	±6

What are the most probable coordinates of station *U*?

SOLUTION Using the procedures described in Section 15.5.1, the initial approximations for the coordinates of station U are:

(a) From Equation (15.10),

$$\angle RQP = Az_{PQ} - Az_{QR} = 293°59'38.4'' - 197°29'38.4'' = 96°30'00.0''$$

(b) Substituting the appropriate angular values into Equation (15.16) yields

$$G = 360° - (30°29'33'' + 38°30'31'' + 96°30'00.0'') = 194°29'56''$$

(c) Substituting the appropriate station coordinates into Equation (15.1) yields

$$PQ = 364.318 \text{ and } QR = 442.576$$

(d) Substituting the appropriate values into Equation (15.17) yields H as

$$H = \frac{442.576 \sin(30°29'33'')}{364.318 \sin(38°30'31'')} = 0.990027302$$

(e) Substituting previously determined G and H into Equation (15.18), $\angle URQ$ is computed as

$$\angle URQ = \tan^{-1}\left(\frac{\sin(194°29'56'')}{0.990027302 + \cos(194°29'56'')}\right) + 180°$$
$$= -85°00'22.2'' + 180° = 94°59'36.3''$$

(f) Substituting the value of $\angle URQ$ into Equation (15.19), $\angle RQU$ is determined to be

$$\angle RQU = 180° - (38°30'31'' + 94°59'36.3'') = 46°29'52.7''$$

(g) From Equation (15.20), RU is

$$RU = \frac{442.576 \sin(46°29'52.7'')}{\sin(38°30'31'')} = 515.589$$

(h) Using Equation (15.1), the azimuth of RQ is

$$Az_{RQ} = \tan^{-1}\left(\frac{1636.436 - 1503.395}{1310.468 - 888.362}\right) + 0° = 17°29'38.4''$$

(i) From Figure (15.4), Az_{RU} is computed as

$$Az_{RQ} = 197°29'38.4'' - 180° = 17°29'38.4''$$
$$Az_{RU} = Az_{RQ} - \angle URQ = 360° + 17°29'38.4'' - 94°59'36.3''$$
$$= 282°30'02.2''$$

(j) Using Equation (15.21), the coordinates for station U are

$$x_u = 1503.395 + 515.589 \sin(Az_{RU}) = 1000.03$$
$$y_u = 888.362 + 515.589 \cos(Az_{RU}) = 999.96$$

For this problem, using prototype Equation (15.13), the J and K matrices are

$$J = \rho \begin{bmatrix} \left(\frac{y_p - y_u}{UP^2} - \frac{y_q - y_u}{UQ^2}\right)_0 & \left(\frac{x_u - x_p}{UP^2} - \frac{x_u - x_q}{UQ^2}\right)_0 \\ \left(\frac{y_q - y_u}{UQ^2} - \frac{y_r - y_u}{UR^2}\right)_0 & \left(\frac{x_u - x_q}{UQ^2} - \frac{x_u - x_r}{UR^2}\right)_0 \\ \left(\frac{y_r - y_u}{UR^2} - \frac{y_s - y_u}{US^2}\right)_0 & \left(\frac{x_u - x_r}{UR^2} - \frac{x_u - x_s}{US^2}\right)_0 \end{bmatrix}$$

$$K = \begin{bmatrix} (\angle 1 - \angle 1_0)'' \\ (\angle 2 - \angle 2_0)'' \\ (\angle 3 - \angle 3_0)'' \end{bmatrix}$$

Also, the weight matrix W is a diagonal matrix composed of the inverses of the variances for the observed angles, or

$$W = \begin{bmatrix} \frac{1}{5^2} & 0 & 0 \\ 0 & \frac{1}{6^2} & 0 \\ 0 & 0 & \frac{1}{6^2} \end{bmatrix}$$

Using the data given for the problem, together with the computed initial approximations, numerical values for the matrices were calculated and the adjustment performed using program ADJUST. A file named *Adjust file for Example 15-2.Adat* is on the companion web site. The following results were obtained after two iterations. The reader is encouraged to adjust these example problems using both the MATRIX and ADJUST programs supplied.

```
ITERATION 1
J MATRIX                          K MATRIX          X MATRIX
=========================      =========      =========
  184.993596    54.807717        −0.203359        −0.031107
  214.320813   128.785353        −0.159052         0.065296
   59.963802   -45.336838        −6.792817

ITERATION 2
J MATRIX                          K MATRIX          X MATRIX
=========================      =========      =========
  185.018081    54.771738         1.974063         0.000008
  214.329904   128.728773        −1.899346         0.000004
   59.943758   -45.340316        −1.967421

INVERSE MATRIX
======================
 0.00116318 -0.00200050
-0.00200050  0.00500943

*****************
Adjusted stations
*****************
Station       X              Y           Sx          Sy
=============================================================
  U        999.999      1,000.025     0.0206      0.0427

***************************
Adjusted Angle Observations
***************************
    Station   Station     Station
Backsighted  Occupied Foresighted         Angle          V       S (″)
=============================================================
      P          U          Q          30° 29′ 31″     −2.0″      2.3
      Q          U          R          38° 30′ 33″      1.9″      3.1
      R          U          S          10° 29′ 59″      2.0″      3.0

**********************
Adjustment Statistics
**********************
Redundancies = 1
Reference Variance = 0.3636
Reference So = ±0.60
```

15.6 ADJUSTMENT OF TRIANGULATED QUADRILATERALS



The quadrilateral is the basic figure for triangulation. Procedures like those used for adjusting intersections and resections are also used to adjust this figure. In fact, the parametric adjustment using the observation equation method can be applied to any triangulated geometric figure, regardless of its shape.

The procedure for adjusting a quadrilateral consists in first using a minimum number of the observed angles to solve triangles and computing initial values for the coordinates. Corrections to these initial coordinates are then calculated by applying the method of least squares. The procedure is iterated until the solution converges. This yields the most probable coordinate values. A statistical analysis of the results is then made. The following example illustrates the procedure.

Example 15.3 The following observations are supplied for Figure 15.5. Adjust this figure by the method of least squares. Assume equal weights for the angles. The observed angles (assume equal weights) are as follows:

$1 = 42°35'29.0''$ $3 = 79°54'42.1''$ $5 = 21°29'23.9''$ $7 = 31°20'45.8''$
$2 = 87°35'10.6''$ $4 = 18°28'22.4''$ $6 = 39°01'35.4''$ $8 = 39°34'27.9''$

The fixed coordinates are

$$x_A = 9{,}270.33 \quad y_A = 8{,}448.90 \quad x_D = 16{,}610.58 \quad y_D = 8{,}568.75$$

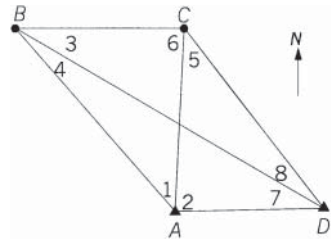


FIGURE 15.5 Quadrilateral example.

SOLUTION The coordinates of station *B* and *C* are to be computed in this adjustment. The Jacobian matrix has the form shown in Table 15.3. The subscripts *b*, *i*, and *f* of the *dx*'s and *dy*'s in Table 15.3 indicate whether stations *B* and *C* are the backsight, instrument, or foresight station in Equation (15.13). Of course, in developing the coefficient matrix, the appropriate station coordinate substitutions must be made to obtain each coefficient.

The reader is encouraged to review this example in the spreadsheet file *Chapter 15.xls*. In this spreadsheet, the cells are named, which allows the coefficients to be entered in an understandable manner. Additionally, the matrices are setup in a manner that allows them to be copied directly into MATRIX. In MATRIX, the *unweighted least-squares adjustment* option in the *numerical operations* menu is used to quickly obtain solution. The resulting *X* matrix is copied into the spreadsheet where it is placed in the appropriate cells to update the coordinates for stations *B* and *C*. Figure 15.4 shows the portion of the spreadsheet where the final coordinates are updated. Notice how the initial coordinate values are summed to the cell labeled as *Final*. Once the coordinates are updated, the

TABLE 15.3 Structure of the Coefficient or J Matrix in Example 15.3

	Unknowns			
Angle	dx_b	dy_b	dx_c	dy_c
1	$dx(b)$	$dy(b)$	$dx(f)$	$dy(f)$
2	0	0	$dx(b)$	$dy(b)$
3	$dx(i)$	$dy(i)$	$dx(b)$	$dy(b)$
4	$dx(i)$	$dy(i)$	0	0
5	0	0	$dx(i)$	$dy(i)$
6	$dx(f)$	$dy(f)$	$dx(i)$	$dy(i)$
7	$dx(f)$	$dy(f)$	0	0
8	$dy(b)$	$dy(b)$	$dx(f)$	$dy(f)$

J and K matrices are updated immediately. This process is repeated until the final solution is confirmed in the second iteration. Following the solution convergence, the V, S_{xx}, S_{ll}, and reference variance matrices are copied into the spreadsheet. These matrices are used to compute the post-adjustment statistics shown in Figures 15.6 and 15.7.

Example 15.3 set up for MATRIX

Control	A	D				
x	9270.330	15610.580				
y	8448.900	8568.750				
Unknowns	0	1	2	3	Final	S
x_B	2403.600	-0.01115	0.000		2403.589	±0.469
y_B	16275.400	0.04946	0.000		16275.449	±0.490
x_C	9649.800	0.06188	0.000		9649.862	±0.345
y_C	24803.500	0.03693	0.000		24803.537	±0.863

FIGURE 15.6 Portion of the spreadsheet for Example 15.3 listing the corrections to the unknown coordinates.

Angles	Observed			Adjusted			V	S
BAC	42°	35'	29.0"	42°	35'	31.1"	2.1"	±3.6"
CAD	87°	35'	10.6"	87°	35'	15.6"	5.0"	±4.3"
CBD	79°	54'	42.1"	79°	54'	37.9"	-4.2"	±4.3"
DBA	18°	28'	22.4"	18°	28'	21.0"	-1.4"	±3.4"
DCA	21°	29'	23.9"	21°	29'	25.7"	1.8"	±3.8"
ACB	39°	01'	35.4"	39°	01'	30.0"	-5.4"	±4.4"
ADB	31°	20'	45.8"	31°	20'	52.3"	6.5"	±4.2"
BDC	39°	34'	27.9"	39°	34'	26.4"	-1.5"	±3.5"
		S_0	±5.6					

FIGURE 15.7 Portion of the spreadsheet for Example 15.3 listing the post-adjustment results.

The following self-explanatory computer listing is from ADJUST. It was created using the file *Adjust file for Example 15-3.Adat*, which is on the companion web site (http://www.wiley.com/college/ghilani). As shown, a single iteration was satisfactory to achieve convergence, since the second iteration produced negligible corrections. Residuals, adjusted coordinates, their estimated errors, and adjusted angles are tabulated at the end of the listing.

```
****************************************
Initial approximations for unknown stations
****************************************
```

Station	X	Y
B	2,403.600	16,275.400
C	9,649.800	24,803.500

Control Stations

Station	X	Y
A	9,270.330	8,448.900
D	16,610.580	8,568.750

Angle Observations

Station Backsighted	Station Occupied	Station Foresighted	Angle
B	A	C	42°35′29.0″
C	A	D	87°35′10.6″
C	B	D	79°54′42.1″
D	B	A	18°28′22.4″
D	C	A	21°29′23.9″
A	C	B	39°01′35.4″
A	D	B	31°20′45.8″
B	D	C	39°34′27.9″

Iteration 1

J Matrix				K MATRIX	X MATRIX	
14.891521	-13.065362	12.605250	-0.292475	-1.811949	1	-0.011149
0.000000	0.000000	-12.605250	0.292475	-5.801621	2	0.049461
20.844399	-0.283839	-14.045867	11.934565	3.508571	3	0.061882
8.092990	1.414636	0.000000	0.000000	1.396963	4	0.036935
0.000000	0.000000	1.409396	-4.403165	-1.833544		
14.045867	11.934565	1.440617	-11.642090	5.806415		
6.798531	11.650726	0.000000	0.000000	-5.983393		
6.798531	-11.650726	11.195854	4.110690	1.818557		

Iteration 2

J Matrix				K MATRIX	X MATRIX	
14.891488	-13.065272	12.605219	-0.292521	-2.100998	1	0.000000
0.000000	0.000000	-12.605219	0.292521	-5.032381	2	-0.000000
20.844296	-0.283922	-14.045752	11.934605	4.183396	3	0.000000
8.092944	1.414588	0.000000	0.000000	1.417225	4	-0.000001
0.000000	0.000000	1.409357	-4.403162	-1.758129		
14.045752	11.934605	1.440533	-11.642083	5.400377		
6.798544	11.650683	0.000000	0.000000	-6.483846		
6.798544	-11.650683	11.195862	4.110641	1.474357		

INVERSE MATRIX

0.00700	-0.00497	0.00160	-0.01082
-0.00497	0.00762	0.00148	0.01138
0.00160	0.00148	0.00378	0.00073
-0.01082	0.01138	0.00073	0.02365

****
Adjusted stations
****

Station	X	Y	Sx	Sy
B	2,403.589	16,275.449	0.4690	0.4895
C	9,649.862	24,803.537	0.3447	0.8622

****
Adjusted Angle Observations
****

Station Backsighted	Station Occupied	Station Foresighted	Angle	V	S
B	A	C	42°35′31.1″	2.10″	3.65
C	A	D	87°35′15.6″	5.03″	4.33
C	B	D	79°54′37.9″	-4.18″	4.29
D	B	A	18°28′21.0″	-1.42″	3.36
D	C	A	21°29′25.7″	1.76″	3.79
A	C	B	39°°01′30.0″	-5.40″	4.37
A	D	B	31°20′52.3″	6.48″	4.24
B	D	C	39°34′26.4″	-1.47″	3.54

****
Adjustment Statistics
****

Iterations = 2
Redundancies = 4
Reference Variance = 31.42936404
Reference So = ±5.6062
Convergence!

PROBLEMS

Note: Partial answers to problems marked with an asterisk can be found in Appendix G.

***15.1** Given the following observations and control station coordinates to accompany figure, what are the most probable coordinates for station *E* using an unweighted least-squares adjustment?

Control Stations

Station	*X* (ft)	*Y* (ft)
A	10,000.00	10,000.00
B	11,498.58	10,065.32
C	12,432.17	11,346.19
D	11,490.57	12,468.51

Angle Observations

Backsight (*b*)	Occupied (*i*)	Foresight (*f*)	Angle	*S* (″)
E	*A*	*B*	90°59′57″	±2.6
A	*B*	*E*	40°26′02″	±2.2
E	*B*	*C*	88°08′55″	±2.2
B	*C*	*E*	52°45′02″	±2.1
E	*C*	*D*	51°09′55″	±2.2
C	*D*	*E*	93°13′14″	±2.3

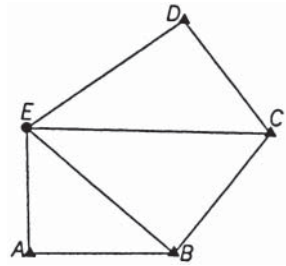


FIGURE P15.1

15.2 Repeat Problem 15.1 using a weighted least-squares adjustment with weights of $1/S^2$ for each angle. What are the:

(a) Most probable coordinates for station *E* and their standard deviations?

(b) Reference standard deviation of unit weight?

(c) Adjusted angles, their residuals, and standard deviations?

15.3 Given the following observations and control station coordinates to accompany Figure 15.3, what are the:

(a) Most probable coordinates for station *E* and their standard deviations?

(b) Reference standard deviation of unit weight?

(c) Adjusted angles, their residuals, and standard deviations?

Control Stations

Station	Easting (m)	Northing (m)
R	10,000.000	10,000.000
S	10,111.332	10,021.758
T	10,200.248	10,009.430

Angle Observations

Backsight (*b*)	Occupied (*i*)	Foresight (*f*)	Angle	*S* (″)
U	*R*	*S*	61°32′15″	±6.0
R	*S*	*U*	100°40′40″	±6.1
U	*S*	*T*	98°16′31″	±7.5
S	*T*	*U*	67°03′18″	±7.4

15.4 Given the following observed angles and control coordinates for a problem like the resection problem of Figure 15.4.

$$1 = 17°47'06'' \qquad 2 = 23°28'27'' \qquad 3 = 25°42'55''$$

Assuming equally weighted angles, what are the most probable coordinates for station *U*?

Control Stations

Station	*X* (ft)	*Y* (ft)
P	9,671.76	9,923.81
Q	9,780.99	9,893.19
R	9,905.93	9,787.40
S	10,073.47	9,835.56

15.5 If the estimated standard deviations for the angles in Problem 15.4 are $S_1 = \pm 8.6''$, $S_2 = \pm 8.3''$, and $S_3 = \pm 7.7''$, what are the:

(a) Most probable coordinates for station *U* and their standard deviations?

*(b) Reference standard deviation of unit weight?

(c) Adjusted angles, their residuals, and standard deviations?

15.6 Repeat Problem 15.5 using the following data.

Control Stations

Station	X (ft)	Y (ft)
P	9,912.04	10,276.03
Q	10,067.42	10,196.98
R	10,086.64	10,034.41
S	10,034.41	9,937.52

Angle Observations

Backsight (b)	Occupied (i)	Foresight (f)	Angle	S ($''$)
P	U	Q	25°43′00″	±1.9
Q	U	R	23°28′23″	±2.0
R	U	S	17°47′05″	±2.1

15.7 Using approximate (x, y) coordinates for station E of (10,086.500, 10,034.400), and given the following control coordinates and observed angles for an intersection problem:

Control Stations

Station	X (m)	Y (m)
A	9,559.500	9,735.000
B	10,534.500	9,738.500
C	10,490.000	10,393.500
D	9,579.500	7,491.846

Angle Observations

Backsight	Occupied	Foresight	Angle	S ($''$)
E	A	B	29°23′46″	±2.3
A	B	E	33°38′59″	±2.3
E	B	C	52°53′12″	±2.3
B	C	E	49°03′13″	±2.3
E	C	D	42°09′46″	±2.3
C	D	E	37°45′42″	±2.3
E	D	A	56°25′45″	±2.3
D	A	E	58°39′30″	±2.3

What are the:

(a) Most probable coordinates for station *E* and their standard deviations?

(b) Reference standard deviation of unit weight?

(c) Adjusted angles, their residuals, and standard deviations?

15.8 The following control station coordinates observed angles, and standard deviations apply to the quadrilateral in Figure 15.5.

Control Stations

Station	*X* (ft)	*Y* (ft)
A	9,684.03	9,442.35
D	10,606.11	9,430.28

Initial Approximations

Station	*X* (ft)	*Y* (ft)
B	9,155.50	10,652.00
C	9,993.00	10,708.00

Angle Observations

Backsight	Occupied	Foresight	Angle	*S* (″)
B	*A*	*C*	37°19′13″	±3.1
C	*A*	*D*	77°01′46″	±3.5
C	*B*	*D*	43°55′46″	±3.5
D	*B*	*A*	26°17′40″	±2.9
D	*C*	*A*	39°21′11″	±3.0
A	*C*	*B*	72°27′25″	±3.7
A	*D*	*B*	39°21′14″	±3.3
B	*D*	*C*	24°15′40″	±2.8

Do a weighted adjustment using the standard deviations to calculate weights. What are the

***(a)** Most probable coordinates for stations *B* and *C* and their standard deviations?

(b) Reference standard deviation of unit weight?

(c) Adjusted angles, their residuals, and standard deviations?

15.9 For the accompanying figure and the following observations, perform a weighted least-squares adjustment.

(a) Station coordinates and standard deviations.

(b) Angles, their residuals, and standard deviations.

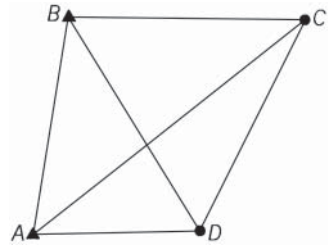


FIGURE P15.9

Control Stations

Station	X (m)	Y (m)
A	7167.532	7678.349
B	7166.000	8917.500

Initial Approximations

Station	X (m)	Y (m)
C	8288.886	8947.558
D	8089.606	7666.283

Angle Observations

Backsight	Occupied	Foresight	Angle	S (″)
B	*A*	*C*	41°31′53″	±2.2
C	*A*	*D*	49°17′19″	±2.2
C	*B*	*D*	55°06′00″	±2.2
D	*B*	*A*	36°21′41″	±2.2
D	*C*	*A*	32°37′16″	±2.2
A	*C*	*B*	47°00′20″	±2.2
B	*D*	*C*	45°16′22″	±2.2
A	*D*	*B*	52°49′00″	±2.2

15.10 Do Problem 15.9 using the additional horizon closure angles listed below.

Backsight	Occupied	Foresight	Angle	S (″)
D	*A*	*B*	269°10′48″	±2.2
A	*B*	*C*	268°32′46″	±2.2
B	*C*	*D*	280°22′28″	±2.2
C	*D*	*A*	261°54′31″	±2.2

(a) Station coordinates and standard deviations.

(b) Angles, their residuals, and standard deviations.

(c) Compare and discuss any differences or similarities between these results and those obtained in Problem 15.9.

15.11 The following observations were obtained for a triangulation chain shown in the accompanying figure.

Control Stations

Station	X (m)	Y (m)
A	7,167.532	7,678.349
B	8,448.606	7,692.283
G	6,961.500	11,972.500
H	8,376.497	11,972.500

Initial Approximations

Station	X (m)	Y(m)
C	7,166.000	8,917.500
D	8,288.880	8,947.550
E	7,055.500	10,401.500
F	8,158.000	10,495.000

Angle Observations

B	I	F	Angle	S (″)	B	I	F	Angle	S (″)
C	A	D	41°31′54.5″	±1.5	D	E	C	36°02′57.2″	±1.5
D	A	B	47°54′56.0″	±1.5	F	E	D	54°32′21.9″	±1.5
A	B	C	44°18′46.6″	±1.5	D	F	C	36°59′54.3″	±1.5
C	B	D	39°03′33.0″	±1.5	C	F	E	52°59′20.4″	±1.5
D	C	B	45°13′19.1″	±1.5	H	E	F	45°05′34.7″	±1.5
B	C	A	46°14′25.1″	±1.5	G	E	H	43°29′10.5″	±1.5
B	D	A	48°42′43.2″	±1.5	E	F	G	55°50′48.1″	±1.5
A	D	C	47°00′21.6″	±1.5	G	F	H	47°24′46.5″	±1.5
E	C	F	36°25′19.0″	±1.5	H	G	F	50°59′56.9″	±1.5
F	C	D	56°18′10.5″	±1.5	F	G	E	35°34′36.6″	±1.5
C	D	E	51°13′29.9″	±1.5	F	H	E	31°38′50.8″	±1.5
E	D	F	35°28′26.6″	±1.5	E	H	G	49°56′23.7″	±1.5

Use ADJUST to perform a weighted least-squares adjustment. Tabulate the final adjusted

(a) Station coordinates and their standard deviations.

(b) Angles, their residuals, and standard deviations.

15.12 Add the following horizon closure angles to the data in Problem 15.11 and use the program ADJUST to perform the adjustment.

B	I	F	Angle	S (″)
B	A	C	270°33′08.6″	±1.5
A	C	E	175°48′44.3″	±1.5
C	E	G	180°50′02.6″	±1.5
E	G	H	273°25′26.5″	±1.5
G	H	F	278°24′45.2″	±1.5
H	F	D	166°45′13.7″	±1.5
F	D	B	177°34′59.3″	±1.5
D	B	A	276°37′39.8″	±1.5

(a) Station coordinates and their standard deviations.

(b) Angles, their residuals, and standard deviations.

15.13 The following observations were obtained for a triangulation chain.

Control Stations

Station	*X* (ft)	*Y* (ft)
A	3,190.04	8,433.74
D	3,671.87	14,314.29
H	7,475.32	8,437.20
K	7,590.44	14,142.31

Initial Approximations

Station	*X* (ft)	*Y* (ft)
B	3,711.26	10,408.62
C	3,981.71	12,055.36
E	5,491.00	9,032.76
F	5,509.17	11,145.10
G	5,519.02	13,102.09
I	7,213.18	10,377.56
J	7,311.66	12,317.91

Angle Observations

B	*I*	*F*	Angle	*S* (″)	*B*	*I*	*F*	Angle	*S* (″)
B	*A*	*E*	60°37′23″	±1.7	*J*	*F*	*I*	57°17′59″	±1.8
E	*B*	*A*	67°04′41″	±1.7	*I*	*F*	*E*	66°14′38″	±1.8
F	*B*	*E*	59°58′55″	±1.7	*E*	*F*	*B*	67°13′52″	±1.8
C	*B*	*F*	58°23′48″	±1.9	*A*	*E*	*B*	52°17′56″	±1.6
F	*C*	*B*	68°32′07″	±2.0	*B*	*E*	*F*	52°47′11″	±1.7
G	*C*	*F*	65°02′32″	±1.9	*F*	*E*	*I*	51°31′17″	±1.7
D	*C*	*G*	63°33′32″	±1.8	*I*	*E*	*H*	51°41′29″	±1.7
G	*D*	*C*	48°54′53″	±1.7	*H*	*E*	*A*	148°42′06″	±1.7
D	*G*	*K*	120°03′38″	±1.7	*E*	*H*	*I*	65°35′58″	±1.8
K	*G*	*J*	50°17′30″	±1.7	*H*	*I*	*E*	59°42′34″	±1.8
J	*G*	*F*	66°39′44″	±1.8	*E*	*I*	*F*	62°14′01″	±1.8
F	*G*	*C*	55°27′39″	±1.8	*F*	*I*	*J*	68°39′25″	±1.9
C	*G*	*D*	67°31′30″	±1.8	*I*	*J*	*F*	54°02′38″	±1.8
B	*F*	*C*	53°04′04″	±1.9	*F*	*J*	*G*	56°40′36″	±1.8
C	*F*	*G*	59°29′48″	±1.9	*G*	*J*	*K*	75°03′42″	±1.9
G	*F*	*J*	56°39′38″	±1.8	*J*	*K*	*G*	54°38′50″	±1.8

Use ADJUST to perform a weighted least-squares adjustment. Tabulate the final adjusted

(a) Station coordinates and their standard deviations.

(b) Angles, their residuals, and standard deviations.

15.14 Add the following horizon closure angles to the data in Problem 15.13 and use the program ADJUST to perform the adjustment.

B	*I*	*F*	Angle	*S* (″)	*B*	*I*	*F*	Angle	*S* (″)
E	*A*	*B*	299°22′38″	±1.7	*G*	*K*	*J*	305°21′11″	±1.8
A	*B*	*C*	174°32′30″	±2.0	*K*	*J*	*I*	174°13′19″	±1.9
B	*C*	*D*	162°51′49″	±1.9	*J*	*I*	*H*	169°24′02″	±1.9
C	*D*	*G*	311°04′56″	±1.7	*I*	*H*	*E*	294°24′00″	±1.8

(a) Station coordinates and their standard deviations.

(b) Angles, their residuals, and standard deviations.

(c) Compare the results of the adjustment with that of Problem 15.13.

Use the ADJUST software to do the following problems.

15.15 Problems 15.2.

15.16 Problems 15.4.

15.17 Problem 15.5.

15.18 Problems 15.6.

15.19 Problems 15.8.

15.20 Show that Equation (15.9) can be rearranged and regrouped to develop Equation (15.13).

PROGRAMMING PROBLEMS

15.21 Write a computational program that computes the coefficients for prototype Equations (15.9) and (15.13), and their k values given the coordinates of the appropriate stations. Use this program to determine the matrix values necessary to do Problem 15.6.

15.22 Prepare a computational program that reads a file of station coordinates, observed angles, and their standard deviations and then

(a) Writes the data to a file in a formatted fashion.

(b) Computes the J, K, and W matrices.

(c) Writes the matrices to a file that is compatible with the MATRIX program.

(d) Test this program with Problem 15.6.

15.23 Write a computational program that reads a file containing the J, K, and W matrices and then

(a) Writes these matrices in a formatted fashion.

(b) Performs one iteration of either a weighted or unweighted least-squares adjustment of Problem 15.6.

(c) Writes the matrices used to compute the solution and the corrections to the station coordinates in a formatted fashion.

15.24 Write a computational program that reads a file of station coordinates, observed angles, and their standard deviations and then

(a) Writes the data to a file in a formatted fashion.

(b) Computes the J, K, and W matrices.

(c) Performs either a relative or equal-weight least-squares adjustment of Problem 15.6.

(d) Writes the matrices used to compute the solution, and tabulates the final adjusted station coordinates and their estimated errors, and the adjusted angles together with their residuals and their estimated errors.

15.25 Prepare a computational program that solves the resection problem. Use this program to compute the initial approximations for Problem 15.3.

CHAPTER 16

ADJUSTMENT OF HORIZONTAL SURVEYS: TRAVERSES AND HORIZONTAL NETWORKS

16.1 INTRODUCTION TO TRAVERSE ADJUSTMENTS

Of the many methods that exist for traverse adjustment, the characteristic that distinguishes the method of least squares from other methods is that distance, angle, and direction observations are adjusted simultaneously. Furthermore, the adjusted observations not only satisfy all geometrical conditions for the traverse, but they also provide the most probable values for the given set of data. Additionally, the observations can be weighted rigorously based on their estimated errors and adjusted accordingly. Given these facts, together with the computational power now provided by computers, it is hard to justify using any method other than least squares for all traverse adjustment work.

In this chapter, we describe methods for making traverse adjustments using the least-squares method. As was the case in triangulation adjustments, traverses can be adjusted using the least-squares method with either observation equations or conditional equations. Again, because of the relative ease with which the equations can be written and solved, the parametric observation equation approach will be discussed.

16.2 OBSERVATION EQUATIONS



When adjusting a traverse using parametric equations, an observation equation is written for each distance, direction, or angle. The necessary linearized observation equations developed previously are recalled in the following equations:

Distance observation equation:

$$\left[\frac{x_i - x_j}{IJ}\right]_0 dx_i + \left[\frac{y_i - y_j}{IJ}\right]_0 dy_i + \left[\frac{x_j - x_i}{IJ}\right]_0 dx_j + \left[\frac{y_j - y_i}{IJ}\right]_0 dy_j = k_l + v_l \quad (16.1)$$

Angle observation equation:

$$\left(\frac{y_i - y_b}{IB^2}\right)_0 dx_b + \left(\frac{x_b - x_i}{IB^2}\right)_0 dy_b + \left(\frac{y_b - y_i}{IB^2} - \frac{y_f - y_i}{IF^2}\right)_0 dx_i + \left(\frac{x_i - x_b}{IB^2} - \frac{x_i - x_f}{IF^2}\right)_0 dy_i + \left(\frac{y_f - y_i}{IF^2}\right)_0 dx_f + \left(\frac{x_i - x_f}{IF^2}\right)_0 dy_f = k_\theta + v_\theta \quad (16.2)$$

Azimuth observation equation:

$$\left(\frac{y_i - y_j}{IJ^2}\right)_0 dx_i + \left(\frac{x_j - x_i}{IJ^2}\right)_0 dy_i + \left(\frac{y_j - y_i}{IJ^2}\right)_0 dx_j + \left(\frac{x_i - x_j}{IJ^2}\right)_0 dy_j = k_{Az_{ij}} + v_{Az_{ij}} \quad (16.3)$$

The reader should refer to Chapters 14 and 15 to review the specific notation for these equations. As demonstrated with the examples that follow, the azimuth equation may or may not be used in traverse adjustments.

16.3 REDUNDANT EQUATIONS

As noted earlier, one observation equation can be written for each observed angle, distance, or direction in a closed traverse. Thus, if there are n sides in the traverse, there are n distances and $n + 1$ angles, assuming one angle exists for the orientation of the traverse. For example, each closed traverse in Figure 16.1 has four sides, four distances, and five angles. Each traverse also has three points whose positions are unknown, and each point introduces two unknown coordinates into the solution. Thus, there are a maximum of $2(n - 1)$ unknowns for any simple closed traverse. From the foregoing, no matter the number of sides, there will always be a minimum of $r = (n + n + 1) - 2(n - 1) = 3$ redundant equations for any simple closed traverse.



FIGURE 16.1 (*a*) Polygon and (*b*) link traverses.

That is, every simple closed traverse that is fixed both positionally and rotationally in space has a minimum of three redundant equations.

16.4 NUMERICAL EXAMPLE

Example 16.1 To illustrate a least squares traverse adjustment, the simple link traverse shown in Figure 16.2 will be used. The spreadsheet file *Chapter 16.xls* containing this problem can be found on the companion website for this book. The observational data are:

Distances (ft)	Angles
$RU = 200.00 \pm 0.05$	$\theta_1 = 240°00' \pm 30''$
$US = 100.00 \pm 0.08$	$\theta_2 = 150°00' \pm 30''$
	$\theta_3 = 240°01' \pm 30''$

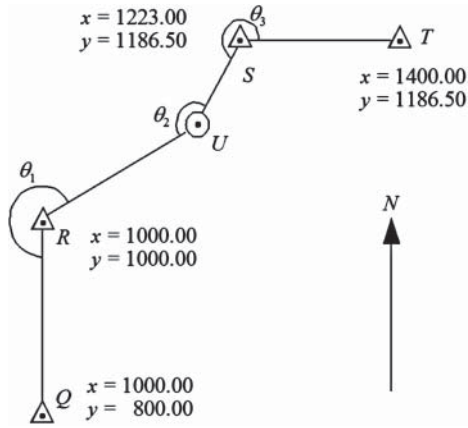


FIGURE 16.2 Simple link travers(e)

SOLUTION

Step 1: Calculate initial approximations for the unknown station coordinates.

$$x_{u_0} = 1000.00 + 200.00 \sin 60^\circ = 1173.20 \text{ ft}$$
$$y_{u_0} = 1000.00 + 200.00 \cos 60^\circ = 1100.00 \text{ ft}$$

Step 2: Formulate the X and K matrices. The traverse in this problem contains only one unknown station with two unknown coordinates. The elements of the X matrix consist of the corrections to the coordinates of U, which are the dx_u and dy_u terms. They are the unknown corrections to be applied to the initial approximations for the coordinates of station U. The values in the K matrix are derived by subtracting computed quantities, based on initial coordinates, from their respective observed quantities. Note that since the first and third observations were used to compute initial approximations for station U, their K-matrix values will be zero in the first iteration.

$$X = \begin{bmatrix} dx_u \\ dy_u \end{bmatrix} \qquad K = \begin{bmatrix} k_{RU} \\ k_{SU} \\ k_{\theta_1} \\ k_{\theta_2} \\ k_{\theta_3} \end{bmatrix} = \begin{bmatrix} 200.00 \text{ ft} - 200.00 \text{ ft} \\ 100.00 \text{ ft} - 99.81 \text{ ft} \\ 240^\circ 00' 00'' - 240^\circ 00' 00'' \\ 150^\circ 00' 00'' - 149^\circ 55' 51'' \\ 240^\circ 01' 00'' - 240^\circ 04' 12'' \end{bmatrix} = \begin{bmatrix} 0.00 \text{ ft} \\ 0.19 \text{ ft} \\ 0'' \\ 249'' \\ -192'' \end{bmatrix}$$

Step 3: Calculate the Jacobian matrix. Since the observation equations are nonlinear, the Jacobian matrix must be formed to obtain the solution. The J matrix is formed using prototype Equations (16.1) for distances and (16.2) for angles. As explained in Section 15.4, since the units of the K matrix that relate to the angles are in seconds, the angle coefficients of the J matrix must be multiplied by ρ to also obtain units of seconds.

In developing the J matrix using prototype Equations (16.1) and (16.2), subscript substitutions were, as shown in Table 16.1. Substitutions of numerical values

TABLE 16.1 Subscript Substitution

Observation	Subscript Substitution
Length RU	$R = i, U = j$
Length US	$U = i, S = j$
Angle θ_1	$Q = b, R = i, U = f$
Angle θ_2	$R = b, U = i, S = f$
Angle θ_3	$U = b, S = i, T = f$

and computation of the J matrix follow:

$$J = \begin{bmatrix} \frac{1173.20-1000.00}{200.00} & \frac{1100.00-1000.00}{200.00} \\ \frac{1173.20-1223.00}{99.81} & \frac{1100.00-1186.50}{99.81} \\ \left(\frac{1100.00-1000.00}{200.00^2}\right)\rho & \left(\frac{1000.00-1173.20}{200.00^2}\right)\rho \\ \left(\frac{1000.00-1100.00}{200.00^2} - \frac{1186.50-1100.00}{99.81^2}\right)\rho & \left(\frac{1173.20-1000.00}{200.00^2} - \frac{1173.20-1223.00}{99.81^2}\right)\rho \\ \left(\frac{1186.50-1100.00}{99.81^2}\right)\rho & \left(\frac{1173.50-1223.00}{99.81^2}\right)\rho \end{bmatrix}$$

$$J = \begin{bmatrix} 0.866 & 0.500 \\ -0.499 & -0.867 \\ 515.7 & -893.2 \\ -2306.6 & 1924.2 \\ 1709.9 & -1031.1 \end{bmatrix}$$

Step 4: Formulate the W matrix. The fact that distance and angle observations have differing observational units and are combined in an adjustment is resolved by using relative weights that are based on observational variances in accordance with Equation (10.6). This weighting makes the observation equations consistent dimensionally. If weights are not used in traverse adjustments (i.e., equal weights are assumed), the least-squares problem will generally either give unreliable results or result in a system of equations that has no solution. Since weights influence the correction size that each observation will receive, it is extremely important to use variances that correspond closely to the observational errors. The error propagation procedures discussed in Chapter 6 aid in the determination of the reasonable estimated errors for each observation. Repeating Equation (10.6), the distance and angle weights for this problem are

$$\text{distances: } w_l = \frac{1}{S_l^2} \text{ and angles: } w_\theta = \frac{1}{S_\theta^2} \tag{16.4}$$

Again, the units of the weight matrix must match those of the J and K matrices. That is, the angular weights must be in the same units of measure (seconds) as the counterparts in the other two matrices. Based on the estimated errors in the observations, the W matrix, which is diagonal, is

$$W = \begin{bmatrix} \frac{1}{0.05^2} & & & & & (\text{zeros}) \\ & \frac{1}{0.08^2} & & & & \\ & & \frac{1}{30^2} & & & \\ & & & \frac{1}{30^2} & & \\ (\text{zeros}) & & & & \frac{1}{30^2} & \end{bmatrix}$$

$$W = \begin{bmatrix} 400.00 & & & & & (\text{zeros}) \\ & 156.2 & & & & \\ & & 0.0011 & & & \\ & & & 0.0011 & & \\ (\text{zeros}) & & & & & 0.0011 \end{bmatrix}$$

Step 5: Solve the matrix system. This problem is iterative and was solved according to Equation (11.39) using the MATRIX program. The MATRIX data file was created using the spreadsheet file *Chapter 16.xls*. The first iteration yielded the following corrections to the initial coordinates:

$$dx_u = -0.11 \text{ ft}$$

$$dy_u = -0.01 \text{ ft}$$

Note that a second iteration produced zeros for dx_u and dy_u. The reader is encouraged to use the MATRIX or ADJUST program to duplicate these results. The ADJUST data file *Adjust file for Example 16-1.dat* is shown in Figure 16.3. It has the format described in Section 14.7.

Step 6: Compute the a posteriori adjustment statistics. Also, from the program MATRIX, the residuals and reference standard deviation are

$$v_{RU} = -0.11 \text{ ft} \qquad v_{US} = -0.12 \text{ ft} \qquad v_{\theta_1} = -49''$$

$$v_{\theta_2} = -17'' \qquad v_{\theta_3} = 6'' \qquad S_0 = \pm 1.82$$

A χ^2 test was used, as discussed in Section 5.4, to see if the a posteriori reference variance differed significantly from its a priori value of 1^2.[1] The test revealed

[1]Since weights are calculated using the $w_i = \sigma_0^2/\sigma_i^2$ when weights are $1/\sigma_i^2$ this implies an a priori value of 1^2 for the reference variance (see Chapter 10).

```
Example 16.1
2 3 0 4 5
Q 1000.00 800.00
R 1000.00 1000.00
S 1223.00 1186.50
T 1400.00 1186.50
U 1173.20 1100.00
R U 200.00 0.05
U S 100.00 0.08
Q R U 240 00 00 30
R U S 150 00 00 30
U S T 240 01 00 30
```

FIGURE 16.3 ADJUST the data file for Example 16.1.

that there was no statistically significant difference between the a posteriori value of 1.82^2 and its a priori value of 1^2 at a 99% confidence level, and thus, the a priori value was used for the reference variance when computing the standard deviations of the coordinates. By applying Equation (13.24), the estimated errors in the adjusted coordinates are

$$S_{x_U} = 1.00\sqrt{0.00053} = \pm 0.023 \text{ ft}$$

$$S_{y_U} = 1.00\sqrt{0.000838} = \pm 0.029 \text{ ft}$$

If the χ^2 test had been performed at a 0.05 level of significance, the test would have rejected the reference variance being equal to its a priori value. Thus, at a 95% confidence level, the a posteriori reference variance should have been used to compute the post-adjustment statistics. Many software packages simply provide the values computed using the a posteriori value for the reference variance. If the computed a posteriori value of ± 1.82 had been used to compute the standard deviations in the coordinates, the resulting values of *Sx* and *Sy* would be ± 0.042 ft and ± 0.053 ft, respectively. Since most surveyors are only concerned about having reliable coordinate values, this discrepancy is of little consequence in everyday practice.

The following are the results from the program ADJUST.

```
*****************************************
Initial approximations for unknown stations
*****************************************
Station      Northing       Easting
==================================
      U      1,100.00      1,173.20

Control Stations
~~~~~~~~~~~~~~~~~~
Station       Easting      Northing
==================================
      Q      1,000.00        800.00
      R      1,000.00      1,000.00
      S      1,223.00      1,186.50
      T      1,400.00      1,186.50

Distance Observations
~~~~~~~~~~~~~~~~~~~~
   Station   Station
  Occupied   Sighted     Distance         S
=============================================
         R         U       200.00     0.050
         U         S       100.00     0.080

Angle Observations
~~~~~~~~~~~~~~~~
      Station      Station       Station
    Backsight     Occupied     Foresight         Angle      S
=================================================================
            Q            R             U   240° 00' 00"    30"
            R            U             S   150° 00' 00"    30"
            U            S             T   240° 01' 00"    30"

First Iteration Matrices
          J  Dim: 5x2         K  Dim: 5x1   X Dim: 2x1
========================    =============   ===========
     0.86602     0.50001       0.00440       -0.11
    -0.49894    -0.86664       0.18873       -0.01
   515.68471  -893.16591       2.62001      ===========
 -2306.62893  1924.25287     249.36438
  1790.94422 -1031.08696    -191.98440
========================    =============
```

```
W  Dim: 5x5
================================================
 400.00000   0.00000   0.00000   0.00000   0.00000
   0.00000 156.25000   0.00000   0.00000   0.00000
   0.00000   0.00000   0.00111   0.00000   0.00000
   0.00000   0.00000   0.00000   0.00111   0.00000
   0.00000   0.00000   0.00000   0.00000   0.00111
================================================

N  Dim: 2x2                    Qxx  Dim: 2x2
==========================     ==========================
10109.947301-7254.506002           0.000530    0.000601
-7254.506002 6399.173533           0.000601    0.000838
==========================     ==========================

Final Iteration
J  Dim: 5x2                    K  Dim: 5x1      X Dim: 2x1
==============================  =============    ============
     0.86591     0.50020         0.10723          0.0000
    -0.49972    -0.86619         0.12203          0.0000
   516.14929  -893.51028        48.62499         ============
 -2304.96717  1925.52297        17.26820
  1788.81788 -1032.01269        -5.89319
==============================  =============

N  Dim: 2x2                     Qxx  Dim: 2x2
==========================     ==============================
10093.552221-7254.153057           0.000532    0.000602
-7254.153057 6407.367420           0.000602    0.000838
==========================     ==============================

J Qxx Jt  Dim: 5x5
============================================================
    0.001130   -0.001195   -0.447052    0.055151    0.391901
   -0.001195    0.001282    0.510776   -0.161934   -0.348843
   -0.447052    0.510776  255.118765 -235.593233  -19.525532
    0.055151   -0.161934 -235.593233  586.956593 -351.363360
    0.391901   -0.348843  -19.525532 -351.363360  370.888892
============================================================

*****************
Adjusted stations
*****************
Station      Northing       Easting       S-N       S-E
============================================================
      U      1,099.99      1,173.09     0.053     0.042

******************************
```

```
Adjusted Distance Observations
******************************
   Station   Station
  Occupied   Sighted      Distance         V         S
============================================================
         R         U        199.89     -0.11     0.061
         U         S         99.88     -0.12     0.065

****************************
Adjusted Angle Observations
****************************
     Station     Station     Station
   Backsight    Occupied   Foresight       Angle    V       S
================================================================
           Q           R           U   239°59′11″  -49″   29.0″
           R           U           S   149°59′43″  -17″   44.1″
           S           S           T   240°01′06″    6″   35.0″

--Reference Standard Deviation = ±1.82- - - - -
Iterations » 2
```

16.5 MINIMUM AMOUNT OF CONTROL



All adjustments require some form of control, and failure to supply enough will result in an indeterminate solution. A traverse requires a minimum of one control station to fix it in position and one line of known direction to fix it in angular orientation. When a traverse has the minimum amount of control, it is said to be *minimally constrained*. It is not possible to adjust a traverse without this minimum. If minimal constraint is not available, necessary control values can be assumed, and the computational process carried out in an arbitrary space. This enables the observed data to be tested for blunders and errors. In Chapter 21, we discuss minimally constrained adjustments.

A *free network adjustment* involves using a *pseudo-inverse* to solve systems that have less than the minimum amount of control. This material is beyond the scope of this book. Readers interested in this subject should consult Bjerhammar (1973) or White (1987) in the bibliography at the end of the book.

16.6 ADJUSTMENT OF NETWORKS

With the introduction of the EDM instrument, and particularly the total station, the speed and reliability of making angle and distance observations have increased greatly. This has led to observational systems that do not conform to the basic

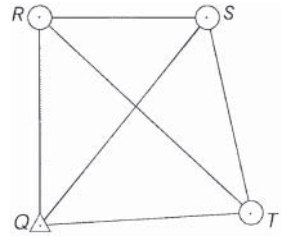


FIGURE 16.4 Horizontal network.

systems of trilateration, triangulation, or traverse. For example, it is common to collect more than the minimum observations at a station during a horizontal control survey. This creates what is called a *complex network*, referred to more commonly as a *network*. The least-squares solution of a network is similar to that of a traverse. That is, observation equations are written for each observation using the prototype equations given in Section 16.2. Coordinate corrections are found using Equation (11.39), and a posteriori error analysis is carried out.

Example 16.2 A network survey was conducted for the project shown in Figure 16.4. Station Q has control coordinates of (1000.00, 1000.00) and the azimuth of line QR is $0°06'24.5''$ with an estimated error of $\pm 0.001''$. The observations and their estimated errors are listed in Table 16.2. Adjust this survey by the method of least squares.

SOLUTION Using standard traverse coordinate computation methods, the initial approximations for station coordinates (x, y) were determined to be

R: (1003.07, 2640.00) S: (2323.07, 2638.46) T: (2661.74, 1096.08)

Under each station heading in the observation columns of Table 16.2, a letter representing the appropriate prototype equation dx and dy coefficients appears. For example, for the first distance QR, station Q is substituted for i in prototype Equation (16.1) and station R replaces j. For the first angle, observed at Q from R to S, station R takes on the subscript b, Q becomes i, and S is substituted for f in prototype Equation (16.2).

Table 16.3 shows the structure of the coefficient matrix for this adjustment and indicates by subscripts where the nonzero values occur. In this table, the column headings are the elements of the unknown X matrix dx_r, dy_r, dx_s, dy_s, dx_t, and dy_t. Note that since station Q is a control station, its corrections are set to zero, and thus, dx_q and dy_q are not included in the adjustment. Note also in this table that the elements that have been left blank are zeros.

To fix the orientation of the network, the direction of course QR is included as an observation, but with a very small, estimated error, $\pm 0.001''$. The last row

TABLE 16.2 Data for Example 16.2

Distance Observations

Occupied, *i*	Sighted, *j*	Distance (ft)	*S* (ft)
Q	*R*	1640.016	0.026
R	*S*	1320.001	0.024
S	*T*	1579.123	0.025
T	*Q*	1664.524	0.026
Q	*S*	2105.962	0.029
R	*T*	2266.035	0.030

Angle Observations

Backsight, *b*	Instrument, *i*	Foresight, *f*	Angle	*S* (″)
R	*Q*	*S*	38°48′50.7″	4.0
S	*Q*	*T*	47°46′12.4″	4.0
T	*Q*	*R*	273°24′56.5″	4.4
Q	*R*	*S*	269°57′33.4″	4.7
R	*S*	*T*	257°32′56.8″	4.7
S	*T*	*Q*	279°04′31.2″	4.5
S	*R*	*T*	42°52′51.0″	4.3
S	*R*	*Q*	90°02′26.7″	4.5
Q	*S*	*R*	51°08′45.0″	4.3
T	*S*	*Q*	51°18′16.2″	4.0
Q	*T*	*R*	46°15′02.0″	4.0
R	*T*	*S*	34°40′05.7″	4.0

Azimuth Observations

From, *i*	To, *j*	Azimuth	*S* (″)
Q	*R*	0°06′24.5″	0.001

of Table 16.3 shows the inclusion of this constrained observation using prototype Equation (16.3). Since for azimuth *QR*, only the foresight station, *R*, is an unknown, only the coefficients for the foresight station *j* are included in the coefficient matrix.

The following are the necessary matrices for the first iteration when doing the weighted least-squares solution of the problem. Note that the numbers have been truncated to five decimal places for publication purposes only. Following these initial matrices, the results of the adjustment are listed, as determined by program ADJUST using the file *Adjust file for Example 16-2.dat*, which is on the companion website (http://www.wiley.com/college/ghilani). Additionally, this problem is solved in the Mathcad® file *c16.xmcd* and the matrices for the problem are developed in the spreadsheet *Chapter 16.xls*. Readers are encouraged to explore these files.

$$J = \begin{bmatrix}
0.00187 & 1.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 \\
-1.00000 & 0.00117 & 1.00000 & -0.00117 & 0.00000 & 0.00000 \\
0.00000 & 0.00000 & -0.21447 & 0.97673 & 0.21447 & -0.97673 \\
0.00000 & 0.00000 & 0.00000 & 0.00000 & 0.99833 & 0.05772 \\
0.00000 & 0.00000 & 0.62825 & 0.77801 & 0.00000 & 0.00000 \\
-0.73197 & 0.68133 & 0.00000 & 0.00000 & 0.73197 & -0.68133 \\
-125.77078 & 0.23544 & 76.20105 & -61.53298 & 0.00000 & 0.00000 \\
0.00000 & 0.00000 & -76.20105 & 61.53298 & 7.15291 & -123.71223 \\
125.77078 & -0.23544 & 0.00000 & 0.00000 & -7.15291 & 123.71223 \\
-125.58848 & 156.49644 & -0.18230 & -156.26100 & 0.00000 & 0.00000 \\
-0.18230 & -156.26100 & 127.76269 & 184.27463 & -127.58038 & -28.01362 \\
0.00000 & 0.00000 & -127.58038 & -28.01362 & 134.73329 & -95.69861 \\
61.83602 & -89.63324 & 0.18230 & 156.26100 & -62.01833 & -66.62776 \\
125.58848 & -156.49644 & 0.18230 & 156.26100 & 0.00000 & 0.00000 \\
0.18230 & 156.26100 & -76.38335 & -94.72803 & 0.00000 & 0.00000 \\
0.00000 & 0.00000 & -51.37934 & -89.54660 & 127.58038 & 28.01362 \\
62.01833 & 66.62776 & 0.00000 & 0.00000 & -69.17123 & 57.08446 \\
-62.01833 & -66.62776 & 127.58038 & 28.01362 & -65.56206 & 38.61414 \\
125.77078 & -0.23544 & 0.00000 & 0.00000 & 0.00000 & 0.00000
\end{bmatrix}$$

TABLE 16.3 Format for Coefficient Matrix J of Example 16.2.

	Unknowns					
Observations	dx_r	dy_r	dx_s	dy_s	dx_t	dy_t
QR	*j*	*j*				
RS	*i*	*i*	*j*	*j*		
ST			*i*	*i*	*j*	*j*
TQ					*i*	*i*
QS			*j*	*j*		
RT	*i*	*i*			*j*	*j*
∠*RQS*	*b*	*b*	*f*	*f*		
∠*SQT*			*b*	*b*	*f*	*f*
∠*TQR*	*f*	*f*			*b*	*b*
∠*QRS*	*i*	*i*	*f*	*f*		
∠*RST*	*b*	*b*	*i*	*i*	*f*	*f*
∠*STQ*			*b*	*b*	*i*	*i*
∠*SRT*	*i*	*i*	*b*	*b*	*f*	*f*
∠*SRQ*	*i*	*i*	*b*	*b*		
∠*QSR*	*f*	*f*	*i*	*i*		
∠*TSQ*			*i*	*i*	*b*	*b*
∠*QTR*	*f*	*f*			*i*	*i*
∠*RTS*	*b*	*b*	*f*	*f*	*i*	*i*
Az_{QR}	*j*	*j*				

The weight matrix is

$$W = \begin{bmatrix} \frac{1}{0.026^2} & & &

The following is a summary of results from ADJUST.

```
Number of Control Stations      ≫ 1
Number of Unknown Stations      ≫ 3
Number of Distance observations ≫ 6
Number of Angle observations    ≫ 12
Number of Azimuth observations  ≫ 1

****************************************
Initial approximations for unknown stations
****************************************
Station         X              Y
==================================
      R      1,003.06      2,640.01
      S      2,323.07      2,638.47
      T      2,661.75      1,096.07

Control Stations
~~~~~~~~~~~~~~~~~~~~~~
Station           X             Y
================================
    Q      1,000.00      1,000.00

*********************
Distance Observations
*********************
   Station   Station
  Occupied   Sighted       Distance          S
=============================================
         Q         R      1,640.016      0.026
         R         S      1,320.001      0.024
         S         T      1,579.123      0.025
         T         Q      1,664.524      0.026
         Q         S      2,105.962      0.029
         R         T      2,266.035      0.030

******************
Angle Observations
******************
     Station    Station       Station
 Backsighted   Occupied   Foresighted             Angle  S (″)
=============================================================
           R          Q             S     38° 48′ 50.7″    4.0
           S          Q             T     47° 46′ 12.4″    4.0
           T          Q             R    273° 24′ 56.5″    4.4
           Q          R             S    269° 57′ 33.4″    4.7
           R          S             T    257° 32′ 56.8″    4.7
           S          T             Q    279° 04′ 31.2″    4.5
```

S	R	T	42° 52′ 51.0″	4.3
S	R	Q	90° 02′ 26.7″	4.5
Q	S	R	51° 08′ 45.0″	4.3
T	S	Q	51° 18′ 16.2″	4.0
Q	T	R	46° 15′ 02.0″	4.0
R	T	S	34° 40′ 05.7″	4.0

****
Azimuth Observations
****

Station Occupied	Station Sighted	Azimuth	S
Q	R	0° 06′ 24.5″	0.0″

Iteration: 1

K MATRIX	X MATRIX
0.0031	-0.002906
0.0099	-0.035262
0.0229	-0.021858
0.0007	0.004793
0.0053	0.003996
0.0196	-0.014381
0.0090	
0.5988	
0.2077	
2.3832	
1.4834	
1.4080	
1.0668	
2.4832	
0.0742	
3.4092	
22.1423	
2.4502	
0.3572	

Iteration: 2

K MATRIX	X MATRIX
0.0384	0.000000
0.0176	-0.000000
0.0155	-0.000000
0.0039	0.000000
0.0087	-0.000000
0.0104	0.000000

```
    2.0763
    0.6962
    2.3725
    0.6444
    0.5048
    3.3233
    1.3435
    0.7444
    3.7319
    5.2271
   18.5154
    0.7387
    0.0000

INVERSE MATRIX
~~~~~~~~~~~~~~~~~~~
0.00000000  0.00000047  0.00000003  0.00000034  0.00000005  0.00000019
0.00000047  0.00025290  0.00001780  0.00018378  0.00002767  0.00010155
0.00000003  0.00001780  0.00023696 -0.00004687  0.00006675 -0.00008552
0.00000034  0.00018378 -0.00004687  0.00032490  0.00010511  0.00022492
0.00000005  0.00002767  0.00006675  0.00010511  0.00027128  0.00011190
0.00000019  0.00010155 -0.00008552  0.00022492  0.00011190  0.00038959

*****************
Adjusted stations
*****************
   Station             X             Y        Sx        Sy
============================================================
         R      1,003.06      2,639.97     0.000     0.016
         S      2,323.07      2,638.45     0.015     0.018
         T      2,661.75      1,096.06     0.016     0.020

******************************
Adjusted Distance Observations
******************************
   Station   Station
  Occupied   Sighted       Distance          V          S
=============================================================
         Q         R      1,639.978    -0.0384     0.0159
         R         S      1,320.019     0.0176     0.0154
         S         T      1,579.138     0.0155     0.0158
         T         Q      1,664.528     0.0039     0.0169
         Q         S      2,105.953    -0.0087     0.0156
         R         T      2,266.045     0.0104     0.0163
```

```
**************************
Adjusted Angle Observations
**************************
```

Backsighted	Station Occupied	Foresighted	Angle	V	S (″)
R	Q	S	38° 48′ 52.8″	2.08″	1.75
S	Q	T	47° 46′ 13.1″	0.70″	1.95
T	Q	R	273° 24′ 54.1″	-2.37″	2.40
Q	R	S	269° 57′ 34.0″	0.64″	2.26
R	S	T	257° 32′ 57.3″	0.50″	2.50
S	T	Q	279° 04′ 34.5″	3.32″	2.33
S	R	T	42° 52′ 52.3″	1.34″	1.82
S	R	Q	90° 02′ 26.0″	-0.74″	2.26
Q	S	R	51° 08′ 41.3″	-3.73″	1.98
T	S	Q	51° 18′ 21.4″	5.23″	2.04
Q	T	R	46° 15′ 20.5″	18.52″	1.82
R	T	S	34° 40′ 05.0″	-0.74″	1.72

```
****************************
Adjusted Azimuth Observations
****************************
```

Station Occupied	Station Sighted	Azimuth	V	S
Q	R	0° 06′ 24.5″	0.00″	0.00″

```
                ****************************************
                        Adjustment Statistics
                ****************************************
                           Iterations = 2
                         Redundancies = 13
                      Reference Variance = 2.20
                         Reference So = ±1.5
              Passed X² test at 99.0% significance level!
                        X² lower value = 3.57
                      X² upper value = 29.82
        The a priori value of 1 used in computations involving
                        the reference variance
                             Convergence!
```

16.7 χ^2 TEST: GOODNESS-OF-FIT

At the completion of a least-squares adjustment, the significance of the computed reference variance S_0^2 can be checked statistically. This check is often referred to as a *goodness-of-fit* test since the computation of S_0^2 is based on $\sum v^2$. That is, as

TABLE 16.4 Two-Tailed χ^2 Test on S_0^2

H_0: $S^2 = 1$

H_a: $S^2 \neq 1$

Test Statistic:

$$\chi^2 = \frac{vS^2}{\sigma^2} = \frac{13(2.2)}{1} = 28.6$$

Rejection region:

$$28.6 = \chi^2 > \chi^2_{0.005,13} = 29.82$$

$$28.6 = \chi^2 < \chi^2_{0.995,13} = 3.565$$

the residuals become larger, so will the reference variance computed, and thus the model computed deviates more from the observed values. However, the size of the residuals is not the only contributing factor to the size of the reference variance in a weighted adjustment. The stochastic model also plays a role in the size of this value. Thus, when the χ^2 test indicates that the null hypothesis should be rejected, it may be due to a blunder in the data or an incorrect decision by the operator in selecting the stochastic model for the adjustment. In Chapter 21, these matters are discussed in greater detail. For now, the reference variance of the adjustment of Example 16.2 will be checked.

In Example 16.2, there are 13 degrees of freedom, and the computed reference variance, S_0^2, is 2.2. In Chapter 10, it was shown that the a priori value for the reference variance was 1. A check can be made to compare the computed value for the reference variance against its a priori value using a two-tailed χ^2 test. For this adjustment, a significance level of 0.01 was selected. The procedures for doing the test were outlined in Section 4.10, and the results for this example are shown in Table 16.4. Since $\alpha/2$ is 0.005 and the adjustment had 13 redundant observations, the critical $\chi^2_{0.005,13}$ value from the table is 29.82. Similarly, its lower-tailed value is 3.57. It can be seen that the computed χ^2 value is less than the upper-tailed tabular value and is greater than the lower-tailed tabular value. Thus, the test fails to reject the null hypothesis, H_0. As explained in Step 6 of Section 16.4, the a priori value of 1 for S_0^2 can and should be used when computing the standard deviations for the station coordinates and observations since the computed value for the reference variance is only an estimate of the a priori value.

PROBLEMS

Note: For problems below requiring least-squares adjustment, if a computer program is not distinctly specified for use in the problem, it is expected that the least-squares method will be solved using a spreadsheet and the program MATRIX, which is included on the book's companion website. Partial answers to problems marked with an asterisk are given in Appendix G.

16.1 Using the following control, adjust the data given for Figure P16.1. The control station coordinates in units of feet are

***A*:** $x = 10{,}623.00$ $y = 13{,}035.14$ ***C*:** $x = 17{,}236.42$ $y = 13{,}530.35$

Mk1: $x = 12{,}284.34$ $y = 10{,}575.08$ **Mk2:** $x = 21{,}118.21$ $y = 13{,}594.25$

Distance Observations

From	To	Length (ft)	S (ft)
A	*B*	4813.41	±0.017
B	*C*	4681.87	±0.017

Angle Observations

BS	Occ	FS	Angle	S (″)
Mk1	*A*	*B*	254°51′39″	±2.3
A	*B*	*C*	271°24′23″	±2.2
B	*C*	Mk2	136°49′26″	±2.2

(a) What is the standard deviation of unit weight, S_0?

*__(b)__ List the adjusted coordinates of station *B* and their standard deviations.

(c) Tabulate the adjusted observations, their residuals, and their standard deviations.

(d) List the inverted normal matrix used in the last iteration.

(e) Perform a χ^2 test on the reference variance at a 0.05 level of significance.

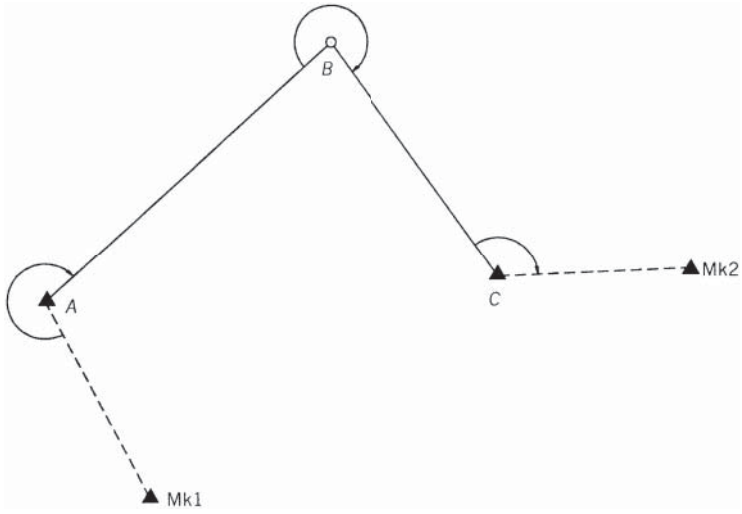


FIGURE P16.1

16.2 Adjust the closed traverse in Figure P16.2 by the method of least squares. The data are given below.

Observed Angles

Angle	Value	S (″)
XAB	73°12′35″	±2.1
BAC	46°46′39″	±2.1
CBA	91°50′01″	±2.1
ACB	41°23′18″	±2.1

Observed Distances

Course	Distance (m)	S (m)
AB	4801.318	±0.015
BC	5291.608	±0.016
CA	7258.097	±0.022

Control Stations

Station	X (m)	Y (m)
X	9,297.125	15,287.540
A	11,757.188	11,389.776

Unknown Stations

Station	X (m)	Y (m)
B	14,904.153	15,015.974
C	19,009.585	11,677.316

***(a)** What is the reference standard deviation, S_0?

(b) List the adjusted coordinates of the unknown stations and their standard deviations.

(c) Tabulate the adjusted observations, their residuals, and their standard deviations.

(d) List the inverted normal matrix used in the last iteration.

(e) Perform a χ^2 test on the reference variance at a 0.05 level of significance.

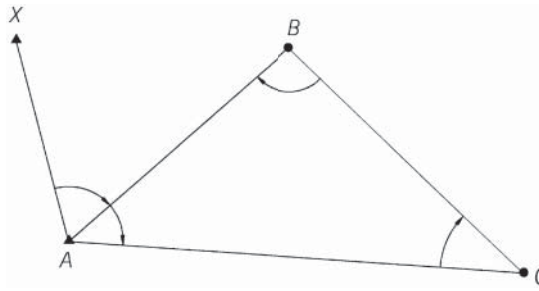


FIGURE P16.2

16.3 Adjust the network shown in Figure P16.3 by the method of least squares. The data are listed as follows:

Control Station

Station	X (m)	Y (m)
A	11,725.240	11,853.035

Unknown Stations

Station	X (m)	Y (m)
B	11,581.470	16,006.390
C	14,920.128	16,038.339
D	15,367.412	11,773.163

Distance Observations

Course	Distance (m)	S (m)
AB	4155.864	0.013
BC	3338.817	0.011
CD	4288.568	0.014
DA	3643.026	0.012
AC	5265.355	0.016

Angle Observations

Angle	Value	S ($''$)
CAD	53°53′57″	6.6
BAC	39°20′21″	4.0
CBA	88°33′57″	7.6
ACB	52°05′43″	7.3
DCA	43°20′38″	3.9
ADC	82°45′24″	6.7

Azimuth Observation: $AB = 358°01'04'' \pm 1.4''$

(a) What is the reference standard deviation, S_0?

(b) List the adjusted coordinates of the unknown stations and their standard deviations.

(c) Tabulate the adjusted observations, their residuals, and their standard deviations.

(d) List the inverted normal matrix used in the last iteration.

(e) Perform a χ^2 test on the reference variance at a 0.05 level of significance.

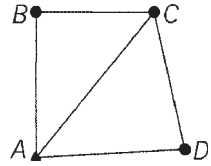


FIGURE P16.3

16.4 Perform a weighted least-squares adjustment using the following distance observations and the angle observations given in Problem 15.3.

From	To	Distance (m)	S
R	U	364.985	±0.003
S	U	326.530	±0.003
T	U	350.884	±0.003

(a) What is the reference standard deviation, S_0?

*__(b)__ List the adjusted coordinates of the unknown station and their standard deviations.

(c) Tabulate the adjusted observations, their residuals, and their standard deviations.

(d) List the inverted normal matrix used in the last iteration.

(e) Perform a χ^2 test on the reference variance at a 0.05 level of significance.

16.5 Perform a weighted least-squares adjustment using the following distance observations and the angle observations given in Problem 15.6.

Occupied	Sighted	Distance (ft)	S (ft)
P	U	380.08	±0.017
Q	U	398.90	±0.017
R	U	326.50	±0.017
S	U	364.99	±0.017

(a) What is the reference standard deviation, S_0?

(b) List the adjusted coordinates of the unknown station and their standard deviations.

(c) Tabulate the adjusted observations, their residuals, and their standard deviations.

(d) List the inverted normal matrix used in the last iteration.

(e) Perform a χ^2 test on the reference variance at a 0.05 level of significance.

16.6 Perform a weighted least-squares adjustment using the following distance observations and the angle observations given in Problem 15.8.

From	To	Distance (ft)	S (ft)
A	B	1320.05	0.017
B	C	839.34	0.016
C	D	1417.17	0.017
A	C	1302.81	0.017
B	D	1896.54	0.017

(a) What is the reference standard deviation, S_0?

(b) List the adjusted coordinates of the unknown station and their standard deviations.

(c) Tabulate the adjusted observations, their residuals, and their standard deviations.

(d) List the inverted normal matrix used in the last iteration.

(e) Perform a χ^2 test on the reference variance at a 0.05 level of significance.

16.7 Using the program ADJUST, do a weighted least-squares adjustment using the following distance observations and the angle observations given in Problem 15.9.

From	To	Distance (m)	S (m)
A	*B*	1239.161	0.006
B	*C*	1123.291	0.006
C	*D*	1296.681	0.006
D	*A*	922.143	0.005
A	*C*	1693.609	0.007
B	*D*	1555.178	0.006

(a) What is the reference standard deviation, S_0?

(b) List the adjusted coordinates of the unknown stations and their standard deviations.

(c) Tabulate the adjusted observations, their residuals, and their standard deviations.

(d) Perform a χ^2 test on the reference variance at a 0.05 level of significance.

16.8 Repeat Problem 16.7 using the additional data in Problem 15.10.

16.9 Using the program ADJUST, do a weighted least-squares adjustment using the following distance observations and the angle observations given in Problem 15.11.

From	To	Distance (m)	S (m)	From	To	Distance (m)	S (m)
A	*D*	1693.610	0.007	*D*	*E*	1906.616	0.007
A	*C*	1239.149	0.006	*D*	*F*	1552.976	0.006
B	*C*	1773.763	0.007	*E*	*F*	1106.461	0.006
B	*D*	1265.382	0.006	*E*	*G*	1573.817	0.007
C	*D*	1123.296	0.006	*E*	*H*	1053.530	0.008
C	*E*	1488.106	0.006	*F*	*G*	2052.571	0.007
C	*F*	1863.470	0.007	*F*	*H*	1493.571	0.006

(a) What is the reference standard deviation, S_0?

(b) List the adjusted coordinates of the unknown stations and their standard deviations.

(c) Tabulate the adjusted observations, their residuals, and their standard deviations.

(d) Perform a χ^2 test on the reference variance at a 0.05 level of significance.

16.10 Using program ADJUST, do a weighted least-squares adjustment using the following distance observations and the angle observations given in Problems 15.13 and 15.14.

From	To	Distance (ft)	S (ft)	From	To	Distance (ft)	S (ft)
A	*E*	2377.65	0.014	*C*	*F*	1778.14	0.013
A	*B*	2042.50	0.014	*F*	*J*	2150.44	0.014
B	*E*	2249.55	0.014	*I*	*J*	1942.84	0.014
E	*H*	2071.74	0.014	*J*	*G*	1956.66	0.014
E	*I*	2185.06	0.014	*J*	*K*	1845.58	0.013
E	*F*	2112.41	0.014	*G*	*K*	2317.92	0.014
H	*I*	1957.96	0.014	*F*	*G*	1957.04	0.014
F	*I*	1868.89	0.013	*C*	*G*	1859.82	0.013
B	*F*	1942.92	0.014	*D*	*G*	2209.38	0.014
B	*C*	1668.81	0.013	*C*	*D*	2280.09	0.014

(a) What is the reference standard deviation, S_0?

(b) List the adjusted coordinates of the unknown station and their standard deviations.

(c) Tabulate the adjusted observations, their residuals, and their standard deviations.

(d) Perform a χ^2 test on the reference variance at a 0.05 level of significance.

16.11 Using the distances and their uncertainties in Problems 14.3 and 14.4, and the following angles, perform a weighted least-squares adjustment.

BS	Occupied	FS	Angle	S (″)
B	*A*	*C*	51°10′41″	±2.7
C	*A*	*D*	38°03′24″	±1.9
C	*B*	*D*	38°00′03″	±2.0
C	*B*	*A*	105°07′09″	±2.8
D	*C*	*A*	67°16′51″	±2.2
A	*C*	*B*	23°42′07″	±2.0
A	*D*	*B*	23°38′49″	±1.9
B	*D*	*C*	51°00′57″	±2.1

(a) What is the reference standard deviation, S_0?

(b) List the adjusted coordinates of the unknown stations and their standard deviations.

(c) Tabulate the adjusted observations, their residuals, and their standard deviations.

(d) Perform a χ^2 test on the reference variance at a 0.05 level of significance.

16.12 Using the distances from Problem 14.10 and the angles listed below, do a weighted least-squares adjustment.

BS	Occupied	FS	Angle	S (″)
E	*A*	*B*	88°21′51″	±1.7
A	*B*	*E*	40°23′10″	±1.6
E	*B*	*D*	50°15′15″	±1.6
E	*B*	*C*	113°00′51″	±1.6
B	*C*	*D*	49°25′13″	±1.6
C	*D*	*B*	67°25′13″	±1.6
B	*D*	*E*	88°37′28″	±1.6
D	*E*	*B*	41°07′16″	±1.6
B	*E*	*A*	51°15′00″	±1.6

(a) What is the reference standard deviation, S_0?

(b) List the adjusted coordinates of the unknown stations and their standard deviations.

(c) Tabulate the adjusted observations, their residuals, and their standard deviations.

(d) Perform a χ^2 test on the reference variance at a 0.05 level of significance.

16.13 Do a weighted least-squares adjustment using the following.

Control Stations

Station	E (m)	*N* (m)
A	834.133	666.605

Unknown Stations

Station	E (m)	*N* (m)
B	1608.340	909.299
C	1624.576	1566.347
D	536.804	1622.029
E	1156.384	1276.568

Distance Observations

Course	Distance (m)	*S* (m)
AB	811.353	±0.005
BC	657.246	±0.005
CD	1089.197	±0.006
DA	1000.609	±0.005
EA	689.862	±0.005
EB	582.365	±0.005
EC	550.605	±0.005
ED	709.380	±0.005

Azimuth Observation

Course	Value	S
AB	72°35′43″	2.3″

Angle Observations

Angle	Value	*S* (″)
DAE	45°08′06″	±2.4
EAB	44°44′51″	±2.4
ABE	56°30′13″	±2.5
EBC	52°19′01″	±2.6
BCE	56°49′46″	±2.6
ECD	34°41′07″	±2.5
CDE	26°12′46″	±2.4
EDA	43°34′13″	±2.4
BEA	78°45′04″	±2.6
CEB	70°51′07″	±2.6
DEC	119°06′07″	±2.6
AED	91°17′42″	±2.5

(a) What is the reference standard deviation, S_0?

(b) List the adjusted coordinates of the unknown stations and their standard deviations.

(c) Tabulate the adjusted observations, their residuals, and their standard deviations.

(d) Perform a χ^2 test on the reference variance at a 0.05 level of significance.

16.14 Using ADJUST, do a weighted least-squares adjustment using the following data.

Control Stations

Station	X (ft)	Y (ft)
A	51,020.44	50,977.05
D	60,369.82	54,974.24

Unknown Stations

Station	X (ft)	Y (ft)
B	55,949.19	50,943.17
C	60,268.19	51,383.54
E	54,814.39	54,483.06
F	49,885.65	54,533.87

Distance Observations

Course	Length (ft)	S (ft)
AB	4928.85	±0.019
BC	4341.36	±0.017
CD	3592.15	±0.016
DE	5577.10	±0.020
EC	6273.01	±0.022
EB	3717.32	±0.016
EA	5165.90	±0.019
EF	4929.11	±0.019
FA	3733.46	±0.016

Angle Observations

Angle	Value	S (″)
FAE	64°57′13″	±3.1
EAB	43°08′06″	±3.0
ABE	71°49′53″	±3.1
EBC	101°57′08″	±3.1
BCE	35°25′56″	±3.0
ECD	62°00′38″	±3.1
CDE	83°20′34″	±3.1
DEC	34°39′45″	±3.0
CEB	42°36′54″	±3.1
BEA	65°02′09″	±3.1
AEF	43°19′54″	±3.0
EFA	71°42′50″	±3.1

(a) What is the reference standard deviation, S_0?

(b) List the adjusted coordinates of the unknown stations and their standard deviations.

(c) Tabulate the adjusted observations, their residuals, and their standard deviations.

(d) Perform a χ^2 test on the reference variance at a 0.05 level of significance.

16.15 Repeat Problem 16.14 using the following additional data.

Angle	Value	S (″)
BAF	251°54′42″	1.1
CBA	186°12′54”	1.1
DCB	262°33′26”	1.1
EDC	276°40′27”	1.1
FED	174°21′26”	1.0
AFE	288°17′07”	1.1

PROGRAMMING PROBLEMS

16.16 Write a computational program that reads a file of station coordinates and observations and then

(a) Writes the data to a file in a formatted fashion.

(b) Computes the J, K, and W matrices.

(c) Writes the matrices to a file that is compatible with the MATRIX program.

(d) Demonstrate this program with Problem 16.4.

16.17 Write a program that reads a file containing the J, K, and W matrices and then

(a) Writes these matrices in a formatted fashion.

(b) Performs one iteration in Problem 16.4.

(c) Writes the matrices used to compute the solution and tabulates the corrections to the station coordinates in a formatted fashion.

16.18 Write a program that reads a file of station coordinates and observations and then

(a) Writes the data to a file in a formatted fashion.

(b) Computes the J, K, and W matrices.

(c) Performs a weighted least-squares adjustment of Problem 16.4.

(d) Writes the matrices used in computations in a formatted fashion to a file.

(e) Computes the final adjusted station coordinates, their estimated errors, the adjusted observations, their residuals, and their estimated errors, and writes them to a file in a formatted fashion.

16.19 Develop a computational program that creates the coefficient, weight, and constant matrices for a network. Write the matrices to a file in a format usable by the MATRIX program supplied with this book. Demonstrate its use with Problem 16.4.

CHAPTER 17

ADJUSTMENT OF GNSS NETWORKS

17.1 INTRODUCTION

For nearly eight decades, NASA and the U.S. military have been engaged in a space research program to develop a precise positioning and navigation system. The first-generation system, called *TRANSIT*, used six satellites and was based on the *Doppler* principle. TRANSIT was made available for commercial use in 1967, and shortly thereafter its use in surveying began. The establishment of a worldwide network of control stations was among its earliest and most valuable applications. Point positioning using TRANSIT required very lengthy observing sessions, and its accuracy was at the 1-m level. Thus, in surveying, it was suitable only for control work on networks consisting of widely spaced points. It was not satisfactory for everyday surveying applications such as traversing or engineering layout.

Encouraged by the success of TRANSIT, a new research program was developed that ultimately led to the creation of the NAVigation Satellite Tracking and Ranging (NAVSTAR) Global Positioning System (GPS). This second-generation positioning and navigation system utilizes a nominal constellation of 24 orbiting satellites. The accuracy of GPS was a substantial improvement over that of the TRANSIT system, and the disadvantage of lengthy observing sessions was also

eliminated. Although developed for military applications, civilians, including surveyors, also found uses for the GPS.

Since its introduction, GPS has been used extensively. It has been found to be reliable, efficient, and capable of yielding extremely high accuracies. GPS observations can be taken day or night and in any weather conditions. A significant advantage of GPS is that visibility between surveyed points is not necessary. Thus, the time-consuming process of clearing lines of sight is avoided. Although most of the early applications of GPS were in control work, improvements have now made the system convenient and practical for use in virtually every type of survey, including property surveys, topographic mapping, and construction staking.

Today, several countries have satellite positioning systems like GPS. These combined systems are now known as *global navigation satellite systems* (GNSS). In keeping with these changes, this book has adopted GNSS in reference to all satellite-positioning systems.

This chapter provides a brief introduction to GNSS surveying. We explain the basic observations involved in the system, discuss errors in those observations, describe the nature of the adjustments needed to account for those errors, and give the procedures for the adjustments of networks surveyed using GNSS. An example problem is given to demonstrate the procedures.

17.2 GNSS OBSERVATIONS



Fundamentally, GNSS operate by observing distances (ranges) from receivers located on ground stations of unknown locations to orbiting GNSS satellites whose positions are precisely known. Thus, conceptually, GNSS surveying is like conventional resection, in that distances are observed with an EDM instrument from an unknown station to several control points. (The trilateration example in Section 14.3 illustrates this.) Of course, there are some major differences between GNSS position determination and trilateration example. Among them is GNSS provides three-dimensional positioning, major differences in how distances are observed, and the fact that the control stations used in GNSS work are moving satellites, which removes the need for intervisibility between stations.

Distances are determined in GNSS surveying by taking observations on the transmitted satellite signals. Two different observational procedures are used: positioning by *pseudoranging*, and positioning by *carrier phase-shift observations*. Pseudoranging involves determining distances (ranges) between satellites and receivers by precisely observing the time it takes transmitted signals to travel from satellites to ground receivers. With the velocity and travel times of the signals known, the pseudoranges can be computed. Finally, based on these ranges, the positions of the ground stations can be calculated. Because pseudoranging is based on observing pseudorandom noise (PRN) codes, this GNSS observation technique is also often referred to as the *code measurement* procedure.

In the carrier phase-shift procedure, the quantities observed are phase changes that occur because of the carrier wave traveling from the satellites to the receivers. The principle is like the phase-shift method employed by electronic distance-measuring instruments. However, a major difference is that the satellites are moving, and the signals are not returned to the transmitters for "true" phase-shift measurements. Instead, the phase shifts must be observed at the receivers. But to make true phase-shift observations, the clocks in the satellites and receivers would have to be perfectly synchronized, which, of course, cannot be achieved. To overcome this timing problem and to eliminate other errors in the system, *differencing* techniques (taking differences between phase observations) are used. Various differencing procedures can be applied. *Single differencing* is achieved by simultaneously observing one satellite with two receivers or two satellites with one receiver. Single differencing eliminates satellite clock biases. *Double differencing* (subtracting the results of single differences from two receivers) eliminates receiver clock biases and other systematic errors.

Another problem in making carrier-phase observations is that only the phase shift of the last cycle of the carrier wave is observed, and the number of full cycles in the travel distance is unknown. (In EDM work, this problem is overcome by transmitting longer wavelengths and observing their phase shifts.) Again, because the satellites are moving, this cannot be done in GNSS work. However, by extending the differencing technique to what is called *triple differencing*, this ambiguity in the number of cycles cancels out of the solution. Triple differencing consists of differencing the results of two double differences and thus involves making observations at two different times to two satellites from two stations.

Rather than performing triple differencing, a more common technique used in today's software to determine ambiguities is to develop probable ranges for each of the range ambiguities. This is known as the LAMBDA (Least-square AMBiguity Decorrelation Adjustment) method, which was developed by Teunissen in 1993. In this approach, a least squares adjustment is performed on the double-differenced ranges to determine the coordinate increments for the unknown station and the double-differenced ambiguities. The final objective being the determination of the most probable combination of ambiguities that provides the most probable solution for the position of the receiver.

In practice, when surveys are done by observing carrier phases, four or more satellites are observed simultaneously using two or more receivers located on ground stations. Also, the observations are repeated many times. This produces a very large number of redundant observations, from which many different combinations can be computed.

Of the two GNSS observing procedures, pseudoranging yields a somewhat lower order of accuracy, but it is preferred for navigation use because it gives instantaneous point positions of satisfactory accuracy. The carrier-phase technique produces a higher order of accuracy and is therefore the choice for high-precision

surveying applications. Adjustment of carrier-phase GNSS baseline vector observations is the subject of this chapter.

The differencing techniques used in carrier-phase observations, described briefly above, do not yield positions directly for the points occupied by receivers. Rather, *baselines* (vector distances between stations) are determined. These baselines are computed in terms of their coordinate difference components ΔX, ΔY, and ΔZ. These coordinate differences are reported in the reference three-dimensional rectangular coordinate system described in Section 17.4.

To use the GNSS carrier-phase procedure in surveying, at least two receivers located on separate stations must be operated simultaneously. For example, assume that two stations A and B were occupied for an observing session, that station A is a control point, and that station B is a point of unknown position. The session would yield coordinate differences ΔX_{AB}, ΔY_{AB}, and ΔZ_{AB} between stations A and B. The X, Y, Z coordinates of station B can then be obtained by adding the baseline components to the coordinates of A, as

$$\begin{aligned} X_B &= X_A + \Delta X_{AB} \\ Y_B &= Y_A + \Delta Y_{AB} \\ Z_B &= Z_A + \Delta Z_{AB} \end{aligned} \tag{17.1}$$

Because carrier-phase observations do not yield point positions directly, but rather give baseline components, this method of GNSS surveying is referred to as *relative positioning*. In practice, often more than two receivers are used simultaneously in relative positioning, which enables more than one baseline to be determined during each observing session. Also, after the first observing session, additional points are interconnected in the survey by moving the receivers to other nearby stations. In this procedure, at least one receiver is left on one of the previously occupied stations. By employing this technique, a network of interconnected points can be created. Figure 17.1 illustrates an example of a GNSS network. In this figure, stations A and B are control stations, and stations C, D, E, and F are points of unknown positions. Creation of such networks is a common procedure employed in GNSS relative positioning work.

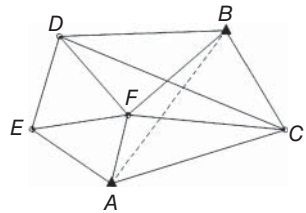


FIGURE 17.1 GNSS survey network.

17.3 GNSS ERRORS AND THE NEED FOR ADJUSTMENT

As in all types of surveying observations, GNSS observations contain errors. The principal sources of these errors are (1) orbital errors in the satellite, (2) signal transmission timing errors due to atmospheric conditions, (3) receiver errors, (4) multipath errors (signals being reflected so that they travel indirect routes from satellite to receiver), and (5) centering errors of the receiver antenna over the ground station, and in height measuring errors above the station. To account for these and other errors, and to increase the precision of point position, GNSS observations are very carefully made according to strict specifications, and redundant observations are taken. The fact that errors are present in the observations makes it necessary to analyze the observations for acceptance or rejection. Also, because redundant observations have been made, they must be adjusted so that all observed values are consistent.

In GNSS surveying work where the observations are made using carrier phases, there are two stages where least squares adjustment is typically applied. The first is in processing the redundant carrier phase-shift observations to obtain the adjusted baseline components (ΔX, ΔY, ΔZ), and the second is in adjusting networks of stations wherein the baseline components have been observed. The latter adjustment is discussed in more detail later in the chapter.

17.4 REFERENCE COORDINATE SYSTEMS FOR GNSS OBSERVATIONS

In GNSS surveys, three different reference coordinate systems are involved. First, the satellite positions at the instants of their observation are given in a space-related X_s, Y_s, Z_s three-dimensional rectangular coordinate system. This coordinate system is illustrated in Figure 17.2. In the figure, the elliptical orbit of a satellite is shown. It has one of its two foci at G, the Earth's center of gravity. Two points, *perigee* (the point where the satellite is closest to G) and *apogee* (the point where the satellite is farthest from G), define the *line of apsides*. This line, which also passes through the two foci, is the X_s axis of the satellite reference coordinate system. The origin of the system is at G, the Y_s axis is in the mean orbital plane, and Z_s is perpendicular to the X_s–Y_s plane. Since the instantaneous position of the satellites is used in computations, values of Z_s are assumed to be zero. For each specific instant of time that a given satellite is observed, its coordinates are calculated in its unique X_s, Y_s, Z_s system.

In processing GNSS observations, all X_s, Y_s, Z_s coordinates that were computed for satellite observations are converted to a common Earth-related X_e, Y_e, Z_e three-dimensional geocentric coordinate system. This Earth-centered, Earth-fixed (ECEF) coordinate system, illustrated in Figure 17.3, is also commonly called the *terrestrial geocentric* coordinate system, or simply the *geocentric* system. It is in this system that the baseline components are computed based on the differencing of

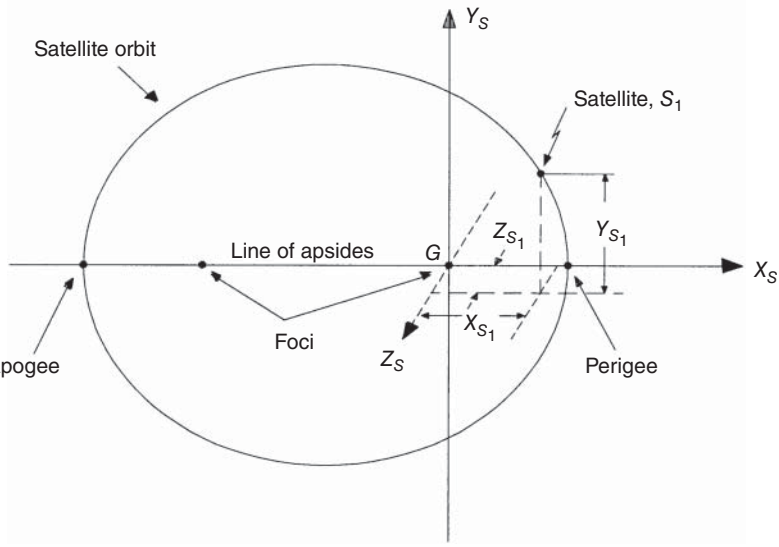


FIGURE 17.2 Satellite reference coordinate system.

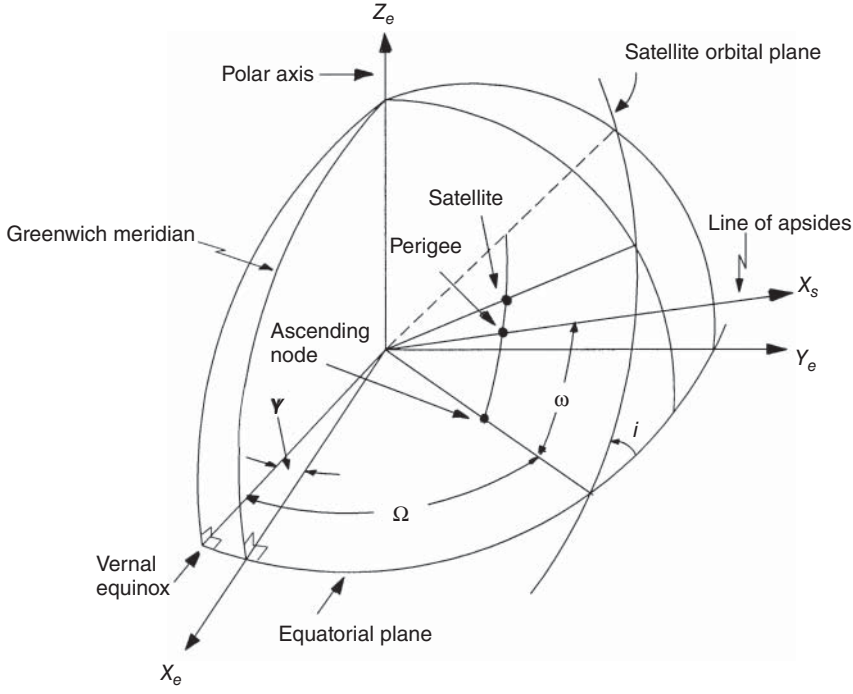


FIGURE 17.3 Earth-related, three-dimensional coordinate system used in GNSS carrier-phase differencing computations.

observed carrier-phase observations. The origin of this coordinate system is at the Earth's gravitational center. The Z_e axis coincides with the Earth's *Conventional Terrestrial Pole* (CTP) axis, the $X_e - Y_e$ plane is perpendicular to the Z_e axis, and the X_e axis passes through the Greenwich Meridian. To convert coordinates from the space-related (X_s, Y_s, Z_s) system to the Earth-centered, Earth-related (X_e, Y_e, Z_e) geocentric system, six parameters are needed at the time of transmission. These are (1) the *inclination angle i* (the angle between the orbital plane and the Earth's equatorial plane); (2) the *argument of perigee* ω (the angle observed in the orbital plane between the equator and the line of apsides); (3) *right ascension of the ascending node* Ω (the angle observed in the plane of the Earth's equator from the vernal equinox to the line of intersection between the orbital and equatorial planes); (4) the *Greenwich hour angle of the vernal equinox* γ (the angle observed in the equatorial plane from the Greenwich meridian to the vernal equinox); (5) the semimajor axis of the orbital ellipse, *a*; and (f) the eccentricity, *e*, of the orbital ellipse. The first four parameters are illustrated in Figure 17.3. For any satellite at any instant of time, these four parameters are available. Software provided by GNSS equipment manufacturers computes the X_s, Y_s, Z_s coordinates of satellites at the instance of transmission and transforms these coordinates into the X_e, Y_e, Z_e geocentric coordinate system used for computing the baseline components.

Oftentimes surveyors convert the baseline computations expressed in X_e, Y_e, Z_e coordinates into geodetic coordinates of latitude, longitude, and height. The geodetic coordinate system is illustrated in Figure 17.4, where the parameters are symbolized by φ, λ, and h, respectively. Geodetic coordinates are referenced to

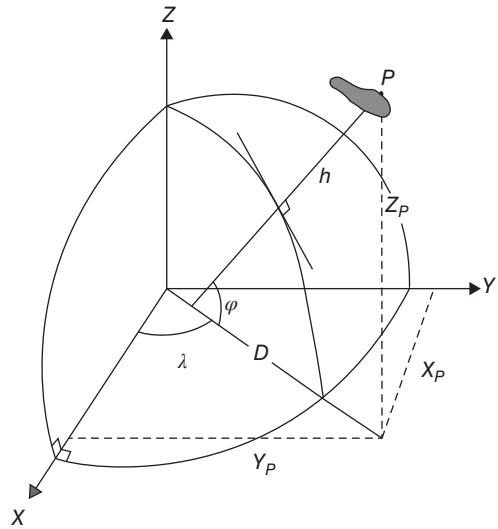


FIGURE 17.4 Geodetic coordinates with Earth-centered, Earth-fixed coordinates X_e, Y_e, and Z_e geocentric coordinate system superimposed.

the World Geodetic System of 1984, which employs the WGS84 ellipsoid and is defined by the ground coordinates of the tracking stations. The center of this ellipsoid is oriented at the Earth's gravitational center. From latitude and longitude, map projection coordinates, which are more convenient for use by local surveyors, can be computed.

It is important to note that geodetic heights are not *orthometric heights* (elevations referred to the geoid). To convert geodetic heights to orthometric heights, the *geoid heights* (vertical distances between the ellipsoid and geoid) must be subtracted from the geodetic heights. That is,

$$h = H + N \tag{17.2}$$

where h is the geodetic height of the point, H is the orthometric height (elevation) of the point, and N is the geoid height (separation between the geoid and ellipsoid) below the point.

17.5 CONVERTING BETWEEN THE TERRESTRIAL AND GEODETIC COORDINATE SYSTEMS

GNSS networks must include at least one control point, but more are preferable. The geodetic coordinates of these control points will normally be given from a previous GNSS survey or *Continuously Operating Reference Stations* (CORS). Prior to processing carrier-phase observations to obtain adjusted baselines for a network, the coordinates of the control stations in the network must be converted from their geodetic values into the X_e, Y_e, Z_e geocentric system. The equations for making these conversions are

$$X = (N + h)\ \cos(\varphi)\ \cos(\lambda) \tag{17.3}$$

$$Y = (N + h)\ \cos(\varphi)\ \sin(\lambda) \tag{17.4}$$

$$Z = [N(1 - e^2) + h]\ \sin(\varphi) \tag{17.5}$$

In the equations above, h is the geodetic height of the point, φ is the geodetic latitude of the point, and λ is the geodetic longitude of the point. Also, e is eccentricity for the ellipsoid, which is computed as

$$e^2 = 2f - f^2 \tag{17.6a}$$

or

$$e^2 = \frac{a^2 - b^2}{a^2} \tag{17.6b}$$

where f is the flattening factor of the ellipsoid; a and b are the semimajor and semiminor axes, respectively, of the ellipsoid[1]; and N is the normal to the ellipsoid at the point, which is computed as

$$N = \frac{a}{\sqrt{1 - e^2 \sin^2\varphi}} \tag{17.7}$$

Note that the normal at a point, N, in Equations (17.3) through (17.7) should not be confused with the geoid height mentioned previously in Equation (17.2).

Example 17.1 Control Stations A and B of the GNSS network of Figure 17.1 have the following NAD83 geodetic coordinates.

$$\varphi_A = 43°15'46.2890'' \quad \lambda_A = -89°59'42.1640'' \quad h_A = 1382.618\,\text{m}$$
$$\varphi_B = 43°23'46.3626'' \quad \lambda_B = -89°54'00.7570'' \quad h_B = 1235.457\,\text{m}$$

Compute their X_e, Y_e, Z_e geocentric coordinates.

SOLUTION *For Station A*: By Equation (17.6a),

$$e^2 = \frac{2}{298.257223563} - \left(\frac{1}{298.257223563}\right)^2 = 0.006694379990$$

By Equation (17.7),

$$N = \frac{6,378,137}{\sqrt{1 - e^2 \sin^2 43°15'46.2890''}} = 6,388,188.252\,\text{m}$$

By Equation (17.3),

$$X_A = (6{,}388{,}188.252 + 1382.618)\cos(43°15'46.2890'')\cos(-89°59'42.1640'')$$
$$= 402.3509\,\text{m}$$

By Equation (17.4),

$$Y_A = (6{,}388{,}188.252 + 1382.618)\cos(43°15'46.2890'')\sin(-89°59'42.1640'')$$
$$= -4{,}652{,}995.3011\,\text{m}$$

[1]The WGS84 ellipsoid is used, where a, b, and f values are 6,378,137.0 m, 6,356,752.3142 m, and 1/298.257223563, respectively.

By Equation (17.5),

$$Z_A = [6{,}388{,}188.252\,(1 - e^2) + 1382.618]\ \sin(43^\circ 15' 46.2890'')$$
$$= 4{,}349{,}760.7775\ \text{m}$$

For station B: Following the same procedure as above, the geocentric coordinates for station *B* are

$$X_B = 8{,}086.0318\ \text{m} \quad Y_B = -4{,}642{,}712.8474\ \text{m} \quad Z_B = 4{,}360{,}439.0833\ \text{m}$$

After completing the network adjustment, it is sometimes necessary to convert all X_e, Y_e, Y_e geocentric coordinates to their geodetic values for use by local surveyors. This conversion process follows these steps (refer to Figure 17.4):

Step 1: Determine the longitude, λ, from

$$\lambda = \tan^{-1}\left(\frac{Y}{X}\right) \tag{17.8}$$

Step 2: Compute D from

$$D = \sqrt{X^2 + Y^2} \tag{17.9}$$

Step 3: Calculate an approximate latitude value φ_0 from

$$\varphi_0 = \tan^{-1}\left[\frac{Z}{D(1 - e^2)}\right] \tag{17.10}$$

Step 4: Compute an approximate ellipsoid normal value N_0 from

$$N_0 = \frac{a}{\sqrt{1 - e^2\ \sin^2\varphi_0}} \tag{17.11}$$

Step 5: Calculate an improved value for latitude φ_0 from

$$\varphi_0 = \tan^{-1}\left(\frac{Z + e^2 N_0 \sin\varphi_0}{D}\right) \tag{17.12}$$

Step 6: Use the value of φ_0 computed in Step 5 and return to Step 4. Iterate Steps 4 and 5 until there is negligible change in φ_0. Using the values from the last iteration for N_0 and φ_0, the value for h can be computed[2] as

$$h = \frac{D}{\cos\varphi_0} - N_0 \tag{17.13}$$

[2]Equation (17.12) is numerically unstable for values of φ greater than 45°. For values of φ greater than 45°, use the equation $h = Z/\sin\varphi_0 - N_0(1 - e^2)$.

Example 17.2 Assume that the final adjusted coordinates for Station C of the network were

$$X_C = 12{,}046.5808\text{ m}, \quad Y_C = -4{,}649{,}394.0826\text{ m}, \quad Z_C = 4{,}353{,}160.0634\text{ m}.$$

Compute the geodetic coordinates for Station C.

SOLUTION By Equation (17.8),

$$\lambda = \tan^{-1}\left(\frac{-4,649,394.0826}{12,046.5808}\right) = -89°51'05.5691''$$

By Equation (17.9),

$$D = \sqrt{(12,046.5808)^2 + (-4,649,394.0826)^2} = 4,649,409.6889\text{ m}$$

Using Equation (17.10), the initial value for φ_0 is

$$\varphi_0 = \tan^{-1}\left(\frac{4,353,160.0634}{D(1-e^2)}\right) = 43°18'26.2228''$$

The first iteration for N_0 and φ_0 is

$$N_0 = \frac{6,378,137.0}{1 - e^2\sin^2 43°18'26.22280''} = 6,388,204.8545\text{ m}$$

The next iteration produced the final values as

$$N_0 = 6,388,204.8421 \quad \varphi_0 = 43°18'26.1030''$$

Using Equation (17.13), the elevation of Station C is

$$h = \frac{D}{\cos 43°18'26.1030''} - 6,388,204.8421 = 1103.101\text{ m}$$

A computer program included with the software package ADJUST, which is on the companion website for this book, will make these coordinate conversions; both from geodetic to geocentric and from geocentric to geodetic.

17.6 APPLICATION OF LEAST SQUARES IN PROCESSING GNSS DATA

As stated previously, a least squares adjustment is used at two different stages in processing GNSS carrier phase-shift observations. First, it is applied in the adjustment that yields baseline components between stations from the redundant carrier-phase observations. Recall that in this procedure, differencing techniques are employed to compensate for errors in the system and to resolve the cycle

ambiguities. In the solution, observation equations are written that contain the differences in coordinates between stations as parameters. The reference coordinate system for this adjustment is the X_e, Y_e, Z_e geocentric system. A highly redundant system of equations is obtained because, as described earlier, a minimum of four (and often more) satellites are simultaneously tracked using at least two (and often more) receivers. Furthermore, many repeat observations are taken. This system of equations is solved by least squares to obtain the most probable ΔX, ΔY, ΔZ components of the baseline vectors. The development of these observation equations is beyond the scope of this book, and thus their solution by least squares is also not covered herein.[3]

Software furnished by manufacturers of GNSS receivers will process observed phase changes to form the differencing observation equations, perform the least squares adjustment, and output the adjusted baseline vector components. The software will also output the covariance matrix, which expresses the correlation between the ΔX, ΔY, ΔZ components of each baseline. The software is proprietary and thus cannot be included herein.

The second stage, where least-squares adjustments are employed in processing GNSS observations, is in adjusting baseline vector components in networks. This adjustment is made after the least squares adjustment of the carrier-phase observations is completed. It is also done in the X_e, Y_e, Z_e geocentric coordinate system. In network adjustments, the goal is to make all X coordinates (and all X-coordinate differences) consistent throughout the figure. The same objective applies to all Y coordinates and to all Z coordinates. As an example, consider the GNSS network shown in Figure 17.1. It consists of two control stations and four stations whose coordinates are to be determined. A summary of the baseline observations obtained from the least squares adjustment of carrier phase-shift observations for this figure is given in Table 17.1. The covariance matrix elements, also listed in Table 17.1, are used for weighting the observations. These will be discussed in Section 17.8, and for the moment, they can be ignored.

A network adjustment of Figure 17.1 should yield adjusted X coordinates for the stations (and adjusted coordinate differences between stations) that are all mutually consistent. Specifically for this network, the adjusted X coordinate of station C should be obtained by adding ΔX_{AC} to the X coordinate of station A; and the same value should be obtained by adding ΔX_{BC} to the X coordinate of station B, or by adding ΔX_{DC} to the X coordinate of station D, and so on. Equivalent conditions exist for the Y and Z coordinates. Note that these conditions do not exist for the data of Table 17.1, which contains the unadjusted baseline observations. The procedure of adjusting GNSS networks is described in detail in Section 17.8, and an example is given.

[3]Readers interested in studying these observation equations should consult *GNSS*, by Hoffman-Wellenhof et al., or *GPS Satellite Surveying* by A. Leick. Complete references for these publications are given in the Bibliography at the end of the book.

TABLE 17.1 Observed Baseline Data for the Network of Figure 17.1

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
From	To	ΔX	ΔY	ΔZ	Covariance Matrix Elements					
A	*C*	11,644.2232	3,601.2165	3,399.2550	9.884E−4	−9.580E−6	9.520E−6	9.377E−4	−9.520E−6	9.827E−4
A	*E*	−5,321.7164	3,634.0754	3,173.6652	2.158E−4	−2.100E−6	2.160E−6	1.919E−4	−2.100E−6	2.005E−4
B	*C*	3,960.5442	−6,681.2467	−7,279.0148	2.305E−4	−2.230E−6	2.070E−6	2.546E−4	−2.230E−6	2.252E−4
B	*D*	−11,167.6076	−394.5204	−907.9593	2.700E−4	−2.750E−6	2.850E−6	2.721E−4	−2.720E−6	2.670E−4
D	*C*	15,128.1647	−6,286.7054	−6,371.0583	1.461E−4	−1.430E−6	1.340E−6	1.614E−4	−1.440E−6	1.308E−4
D	*E*	−1,837.7459	−6,253.8534	−6,596.6697	1.231E−4	−1.190E−6	1.220E−6	1.277E−4	−1.210E−6	1.283E−4
F	*A*	−1,116.4523	−4,596.1610	−4,355.9062	7.475E−5	−7.900E−7	8.800E−7	6.593E−5	−8.100E−7	7.616E−5
F	*C*	10,527.7852	−994.9377	−956.6246	2.567E−4	−2.250E−6	2.400E−6	2.163E−4	−2.270E−6	2.397E−4
F	*E*	−6,438.1364	−962.0694	−1,182.2305	9.442E−5	−9.200E−7	1.040E−6	9.959E−5	−8.900E−7	8.826E−5
F	*D*	−4,600.3787	5,291.7785	5,414.4311	9.330E−5	−9.900E−7	9.000E−7	9.875E−5	−9.900E−7	1.204E−4
F	*B*	6,567.2311	5,686.2926	6,322.3917	6.643E−5	−6.500E−7	6.900E−7	7.465E−5	−6.400E−7	6.048E−5
B	*F*	−6,567.2310	−5,686.3033	−6,322.3807	5.512E−5	−6.300E−7	6.100E−7	7.472E−5	−6.300E−7	6.629E−5
A	*F*	1,116.4577	4,596.1553	4,355.9141	6.619E−5	−8.000E−7	9.000E−7	8.108E−5	−8.200E−7	9.376E−5
A[a]	*B*	7,683.6883	10,282.4550	10,678.3008	7.2397E−4	−7.280E−6	7.520E−6	6.762E−4	−7.290E−6	7.310E−4

[a]Fixed baseline used only for checking, but not included in adjustment.

17.7 NETWORK PREADJUSTMENT DATA ANALYSIS

Prior to adjusting GNSS networks, a series of procedures should be followed to analyze the data for internal consistency and to eliminate possible blunders. No control points are needed for these analyses. Depending on the actual observations taken and the network geometry, these procedures may consist of analyzing (1) differences between fixed and observed baseline components, (2) differences between repeated observations of the same baseline components, and (3) loop closures. After making these analyses, a minimally constrained adjustment is usually performed that will help isolate any blunders that may have escaped the first set of analyses. Procedures for making these analyses are described in the following subsections.

17.7.1 Analysis of Fixed Baseline Measurements

GNSS job specifications often require that baseline observations be taken between fixed control stations. The benefit of making these observations is to verify the accuracy of both the GNSS observational system and the control being held fixed. Obviously, the smaller the discrepancies between observed and known baseline lengths, the more accurate the observations. If the discrepancies are too large to be tolerated, the conditions causing them must be investigated before proceeding further. Note that in the data of Table 17.1, one fixed baseline (between control points *A* and *B*) was observed. Table 17.2 gives the data for comparing the observed and fixed baseline components. The observed values are listed in column (2), and the fixed components are given in column (3). To compute the fixed values, X_e, Y_e, Z_e geocentric coordinates of the two control stations are first determined from their geodetic coordinates according to procedures discussed in Section 17.5. Then the ΔX, ΔY, ΔZ differences between the X_e, Y_e, Z_e coordinates for the two control stations are determined. Differences (in meters) between the observed and fixed baseline components are given in column (4). Finally, the differences, expressed in parts per million (ppm), are listed in column (5). These ppm values are obtained by

TABLE 17.2 Comparisons of Observed and Fixed Baseline Components

(1) Component	(2) Observed (m)	(3) Fixed (m)	(4) Difference (m)	(5) PPM[a]
ΔX	7,683.6883	7,683.6809	0.0074	0.44
ΔY	10,282.4550	10,282.4537	0.0013	0.08
ΔZ	10,678.3008	10,678.3058	0.0050	0.30

[a]The total baseline length used in computing these ppm values was 16,697 m, which was derived from the square root of the sum of the squares of ΔX, ΔY, and ΔZ values.

dividing the column (4) differences by their corresponding total baseline lengths and multiplying by 1,000,000. A determination of the acceptability of the computed ppm can be evaluated based on published standards such as the *Federal Geodetic Control Subcommittee's Geometric Geodetic Accuracy Standards and Specifications for Using GPS Relative Positioning Techniques*.

When the standards are inappropriate for the survey method being performed, the baselines can also be compared against the estimated error in the line based on the manufacturer's specifications for the particular survey method used. Since each receiver in a baseline adds an additional three-dimensional setup error σ_s, the error in any observed baseline can be estimated as

$$\sigma_{baseline} = \sqrt{2\sigma_s^2 + a^2 + (L \cdot \text{ppm})^2} \tag{17.14}$$

where a is the specified constant error for an observed baseline and L is the length of the baseline observed. For example, the typical specified accuracy for a baseline determined using static surveying procedures is 5 mm + 1 ppm, where a is 5 mm and 1 the ppm. Baseline AB is approximately 16,697 m in length. Based on the previously stated accuracies and assuming setup errors of 0.0015 m for each receiver, using Equation (17.14), the estimated error in the baseline is

$$\sigma_{baseline} = \sqrt{2(0.0015)^2 + 0.005^2 + \left(\frac{16,697}{1,000,000} \cdot 1\right)^2} = \pm 0.0176 \text{ m}$$

Since the baseline is determined from more than 30 individual epochs of data with multiple satellite observations, typically, the normal distribution multiplier can be used to raise this uncertainty to an appropriate percent probability. For example, at 95%, the estimated error in the baseline could be as much as 1.96(0.0176), which equals ±0.034 m. Since the actual misclosure between the observed and fixed baseline is only $\sqrt{0.0074^2 + 0.0013^2 + 0.0050^2} = 0.009$ m, it is well within the 95% range of the estimated misclosure.

17.7.2 Analysis of Repeat Baseline

Another procedure employed in evaluating the consistency of the observed data and in weeding out blunders is to make repeat observations of certain baselines. These repeat observations are taken in different sessions, and the results are compared. The repeat baselines provide checks on field and office procedures and aid in isolating poor practices. For example, in the data of Table 17.1, baselines AF and BF were repeated. Table 17.3 gives comparisons of these observations using the same procedure that was used in Table 17.2. Again, the ppm values listed in column (5) use the total baseline lengths in the denominator, which are computed

TABLE 17.3 Comparisons of Repeat Baseline Measurements

Component	First Observation	Second Observation	Difference (m)	PPM
ΔX_{AF}	1116.4577	−1116.4523	0.0054	0.84
ΔY_{AF}	4596.1553	−4596.1610	0.0057	0.88
ΔZ_{AF}	4355.9141	−4355.9062	0.0079	1.23
ΔX_{BF}	−6567.2310	6567.2311	0.0001	0.01
ΔY_{BF}	−5686.3033	5686.2926	0.0107	1.00
ΔZ_{BF}	−6322.3807	6322.3918	0.0110	1.02

from the square root of the sum of the squares of the measured baseline components. It is wise to perform repeat observations at the end of each day to check the repeatability of the software, hardware, and field procedures.

The Federal Geodetic Control Subcommittee (FGCS) has developed a document entitled "Geometric Geodetic Accuracy Standards and Specifications for Using GPS Relative Positioning Techniques." It is intended to serve as a guideline for planning, executing, and classifying geodetic surveys performed by GNSS relative positioning methods. This document may be consulted to determine whether or not the ppm values of column (5) are acceptable for the required order of accuracy for the survey. Besides ppm requirements, the FGCS guidelines specify other criteria that must be met for the different orders of accuracy in connection with repeat baseline observations. Again the acceptability of the line can be computed based on the FGCS guidelines or by comparing the baseline against the manufacturer's specified accuracies.

Developing a statistical test for repeat baselines may be impossible since it must be remembered that we are comparing the difference between two observations. The proper statistical test is demonstrated in Section 17.18. However, this test requires knowledge of each baseline's computed error, the number of redundant observations, and the total number of observations. While the root-mean-square (RMS) error is typically reported by software, the number of redundant and total observations is seldom provided in a baseline adjustment report. Thus, another approach needs to be considered. One method might simply be to see if the linear difference between the two baseline observations is greater than the estimated error for a baseline of this length. In this case, Equation (17.14) will yield the estimated error in the length of the line, which can be multiplied by an appropriate level of probability. Then, similar to the procedures used for fixed baseline observations, the actual error can be compared against the estimated value. Although this procedure lacks the rigor of a proper statistical test, it should provide enough information to isolate baseline observations that have obvious problems. This method is demonstrated in Example 17.3 with repeat baselines *FA* and *AF*.

Example 17.3 The manufacturer's specified accuracy of their GNSS receivers during a static survey is 3 mm + 0.5 ppm. The user estimates their setup errors to be within ±1.5 mm. Analyze repeat baselines *FA* and *AF* from Table 17.1 at a 99% level of confidence.

SOLUTION For simplicity, a table like Table 17.3 is created.

Component	1st Obs	2nd Obs	Difference (m)	ppm
ΔX	1116.4577	−1116.4523	0.005	0.84
ΔY	4596.1553	−4596.1610	−0.006	0.89
ΔZ	4355.9141	−4355.9062	0.008	1.23
Length	6430.016	6430.014	0.0111	1.73

The estimated error for this length of a baseline is

$$\sigma = \sqrt{2(1.5^2) + 3^2 + \left[\frac{0.5(6,430,014)}{1,000,000}\right]^2} = \pm 4.88 \text{ mm}$$

Since the number of observations used to generate this baseline observation during a static survey is typically greater than 30, a normal distribution multiplier can be used to obtain the $E_{99\%}$ value for this error. From Table 3.2, we see the appropriate multiplier is 2.576.

$$E_{99} = 2.576(4.88) = \pm 12.58 \text{ mm}$$

The actual linear difference between the two observations was 11.1 mm. Since 11.1 mm < 12.58 mm, the difference in the two observations is within the 99% range. It should be noted that if either of these baselines are a blunder then more advanced statistical techniques discussed in Chapter 22 may catch the observation that has a blunder during the GNSS network adjustment discussed in Section 17.8. These early tests are to ensure that no large blunders are allowed to enter into the baseline network adjustment.

17.7.3 Analysis of Loop Closures

GNSS networks will typically consist of many interconnected closed loops. For example, in the network of Figure 17.1, a closed loop is formed by points *ACBDEA*. Similarly, *ACFA*, *CFBC*, *BDFB*, and so on, are other closed loops. For each closed loop, the algebraic sum of the ΔX components should equal

zero. The same condition should exist for the ΔY and ΔZ components. These loop misclosure conditions are very similar to the leveling loop misclosures imposed in differential leveling and latitude and departure misclosures imposed in closed-polygon traverses. An unusually large misclosure within any loop will indicate that either a blunder or a large random error exists in one (or more) of the baselines of the loop.

To compute loop misclosures, the baseline components are added algebraically for the chosen loop. For example, the misclosure in X components for loop *ACB-DEA* would be computed as

$$cx = \Delta X_{AC} + \Delta X_{CB} + \Delta X_{BD} + \Delta X_{DE} + \Delta X_{EA} \tag{17.15}$$

where cx is the loop misclosure in X coordinates. Similar equations apply for computing misclosures in Y and Z coordinates.

Substituting numerical values into Equation (17.15), the misclosure in X coordinates for loop *ACBDEA* is

$$\begin{aligned} cx &= 11{,}644.2232 - 3960.5442 - 11{,}167.6076 - 1837.7459 + 5321.7164 \\ &= 0.0419 \text{ m} \end{aligned}$$

Similarly, misclosures in Y and Z coordinates for that loop are

$$\begin{aligned} cy &= 3601.2165 + 6681.2467 - 394.5204 - 6253.8534 - 3634.0754 \\ &= 0.0140 \text{ m} \\ cz &= 3399.2550 + 7279.0148 - 907.9593 - 6596.6697 - 3173.6652 \\ &= -0.0244 \text{ m} \end{aligned}$$

For evaluation purposes, loop misclosures are expressed in terms of the ratios of resultant misclosures to the total loop lengths. They are given in ppm. For any loop, the resultant misclosure is the square root of the sum of the squares of its cx, cy, and cz values, and for loop *ACBDEA,* the resultant is 0.0505 m. The total length of a loop is computed by summing its legs, each leg being computed from the square root of the sum of the squares of its observed ΔX, ΔY, and ΔZ values. For loop *ACBDEA*, the total loop length is 50,967 m, and the misclosure ppm ratio is therefore (0.0505/50,967)(1,000,000) = 0.99 ppm. Again, these ppm ratios can be compared against values given in the FGCS guidelines to determine if they are acceptable for the order of accuracy of the survey. As was the case with repeat baseline observations, the FGCS guidelines also specify other criteria that must be met in loop analyses besides the ppm values.

These misclosures can also be checked against manufacturer specifications. In this case, Equation (17.14) can be modified as

$$\sigma_{loop} = \sqrt{2n\sigma_s^2 + na^2 + (L \cdot \text{ppm})^2} \tag{17.16}$$

where n is the number of baselines in the loop, L is the overall length of the loop, and the other terms are as defined in Equation (17.14).

Assuming, as before, an accuracy of 5 mm + 1 ppm and setup errors of 0.0015 m, the estimated error in the loop is

$$\sqrt{2(5)0.0015^2 + 5(0.005)^2 + \left(\frac{50,967}{1,000,000} \cdot 1\right)^2} = \pm 0.052 \text{ m}$$

Again, the actual misclosure is within the estimated error of the baseline at a 68% probability. As with Equation (17.14), multipliers derived from the normal distribution can be used to estimate errors at any other level of probability.

For any network, enough loop closures should be computed so that every baseline is included within at least one loop. This should expose any large blunders that exist. If a blunder does exist, its location can often be determined through additional loop-closure analyses. For example, assume that the misclosure of loop *ACDEA* discloses the presence of a blunder. By also computing the misclosures of loops *AFCA*, *CFDC*, *DFED*, and *EFAE*, the baseline containing the blunder can often be detected. In this example, if a large misclosure was found in loop *DFED* and all other loops appeared to be blunder-free, the blunder could be in line *DE*.

A computer program included within the software package ADJUST can determine these loop-closure computations. Figure 17.5 shows the resultant loop misclosures. The "LC" column lists the actual misclosure of each loop. The "s" column lists the estimated error in the loop based on user-supplied setup errors and estimated errors in the observed GNSS baseline vectors. This column is at 68% and must be multiplied by the appropriate multiplier to raise

Loop Distance	Misclosure dx	dy	dz	LC	s	ppm	Loop				
12,860.030	0.005	-0.006	0.008	0.011	0.015	0.9	7	13			
50,966.697	0.042	0.014	-0.024	0.050	0.052	1.0	1	3	4	6	2
29,701.415	0.020	0.001	0.034	0.040	0.031	1.3	13	8	1		
37,054.577	0.001	0.011	-0.003	0.011	0.038	0.3	8	10	5		
24,748.975	-0.012	0.005	0.008	0.015	0.026	0.6	10	9	6		
20,229.388	-0.038	-0.011	-0.018	0.043	0.022	2.1	9	13	2		

FIGURE 17.5 Results of loop misclosure computations for the example in Section 17.7.3.

```
14
1 A C 11644.2232 3601.2165 3399.2550
2 A E -5321.7164 3634.0754 3173.6652
3 B C 3960.5442 -6681.2467 -7279.0148
4 B D -11167.6076 -394.5204 -907.9593
5 D C 15128.1647 -6286.7054 -6371.0583
6 D E -1837.7459 -6253.8534 -6596.6697
7 F A -1116.4523 -4596.1610 -4355.9062
8 F C 10527.7852 -994.9377 -956.6246
9 F E -6438.1364 -962.0694 -1182.2305
10 F D -4600.3787 5291.7785 5414.4311
11 F B 6567.2311 5686.2926 6322.3917
12 B F -6567.2310 -5686.3033 -6322.3807
13 A F 1116.4577 4596.1553 4355.9141
14 A B 7683.6620 10282.4234 10678.2972
7 13
1 3 4 6 2
13 8 1
8 10 5
10 9 6
9 13 2
```

FIGURE 17.6 ADJUST data file for computing loop misclosure as discussed in Section 17.7.3.

it to another level of probability. Finally, the "ppm" column lists the computed parts-per-million error for each loop. The help file that accompanies the software explains the format of the data file shown in Figure 17.6. This file lists the number of baseline vectors (14) followed by each baseline vector with a consecutive number. These numbers are used to determine which baseline vectors are included in any loop. This file lists the six loops discussed previously. For example, loop *ACBDEA* is defined as "1 3 4 6 2," where "1" is baseline *AC*, "3" is *BC*, and so on. As can be seen with baseline *EA*, the software will automatically reverse the signs of the baseline vector components when the letters are shown in the reverse order.

17.7.4 Minimally Constrained Adjustment

Prior to making the final adjustment of baseline observations in a network, a "minimally constrained" least squares adjustment is usually performed. In this adjustment, sometimes called a "free" adjustment, any station in the network may be held fixed with arbitrary coordinates. All other stations in the network are therefore free to adjust as necessary to accommodate the baseline observations and network geometry. The residuals that result from this adjustment

are strictly related to the baseline observations and not to faulty control coordinates. These residuals are examined and, from them, blunders that may have gone undetected through the first set of analyses can be found and eliminated. Often these adjustments use statistical blunder detection methods discussed in Chapter 21.

17.8 LEAST SQUARES ADJUSTMENT OF GNSS NETWORKS



As noted earlier, because GNSS networks contain redundant observations, they must be adjusted to make all coordinate differences consistent. In applying least squares to the problem of adjusting baselines in GNSS networks, observation equations are written that relate station coordinates to the observed coordinate differences and their residual errors. To illustrate this procedure, consider the example of Figure 17.1. For line AC of this figure, an observation equation can be written for each observed baseline component as

$$\begin{aligned} X_C &= X_A + \Delta X_{AC} + v_{X_{AC}} \\ Y_C &= Y_A + \Delta Y_{AC} + v_{Y_{AC}} \\ Z_C &= Z_A + \Delta Z_{AC} + v_{Z_{AC}} \end{aligned} \tag{17.17}$$

Similarly, the observation equations for the baseline components of line CD are

$$\begin{aligned} X_D &= X_C + \Delta X_{CD} + v_{X_{CD}} \\ Y_D &= Y_C + \Delta Y_{CD} + v_{Y_{CD}} \\ Z_D &= Z_C + \Delta Z_{CD} + v_{Z_{CD}} \end{aligned} \tag{17.18}$$

Observation equations of the form above are written for all observed baselines in any figure. For Figure 17.1 there were a total of 13 observed baselines, so the number of observation equations that can be developed is 39. Also, each of stations C, D, E, and F has three unknown coordinates, for a total of 12 unknowns in the problem. Thus, there are $39 - 12 = 27$ redundant observations in the network. The 39 observation equations can be expressed in matrix form as

$$AX = L + V \tag{17.19}$$

If the observation equations for adjusting the network of Figure 17.1 are written in the same order that the observations are listed in Table 17.1, the A, X, L, and V matrices are

$$A = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
 & & & & & \vdots & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\quad X = \begin{bmatrix} X_C \\ Y_C \\ Z_C \\ X_E \\ Y_E \\ Z_E \\ X_D \\ Y_D \\ Z_D \\ X_F \\ Y_F \\ Z_F \end{bmatrix}$$

$$L = \begin{bmatrix}
12,046.5741 \\
-4,649,394.0846 \\
4,353,160.0325 \\
-4919.3655 \\
-4,649,361.2257 \\
4,352,934.4427 \\
12,046.5760 \\
-4,649,394.0941 \\
4,353,160.0685 \\
-3081.5758 \\
-46,443,107.3678 \\
4,359,531.1240 \\
15,128.1647 \\
-6286.7054 \\
-6371.0583 \\
-1837.7459 \\
-6253.8534 \\
-6596.6697 \\
-1518.8032 \\
4,648,399.1401 \\
-4,354,116.6737 \\
10,527.7852 \\
-994.9377 \\
-956.6246 \\
-6438.1364 \\
-962.0694 \\
-1182.2305 \\
\vdots \\
1518.8086 \\
-4,648,399.1458 \\
4,354,116.6916
\end{bmatrix}
\quad V = \begin{bmatrix}
v_{X_{AC}} \\ v_{Y_{AC}} \\ v_{Z_{AC}} \\ v_{X_{AE}} \\ v_{Y_{AE}} \\ v_{Z_{AE}} \\ v_{X_{BC}} \\ v_{Y_{BC}} \\ v_{Z_{BC}} \\ v_{X_{BD}} \\ v_{Y_{BD}} \\ v_{Z_{BD}} \\ v_{X_{DC}} \\ v_{Y_{DC}} \\ v_{Z_{DC}} \\ v_{X_{DE}} \\ v_{Y_{DE}} \\ v_{Z_{DE}} \\ v_{X_{FA}} \\ v_{Y_{FA}} \\ v_{Z_{FA}} \\ v_{X_{FC}} \\ v_{Y_{FC}} \\ v_{Z_{FC}} \\ v_{X_{FE}} \\ v_{Y_{FE}} \\ v_{Z_{FE}} \\ \vdots \\ v_{X_{AF}} \\ v_{Y_{AF}} \\ v_{Z_{AF}}
\end{bmatrix}$$

The numerical values of the elements of the L matrix are determined by rearranging the observation equations. Its first three elements are for the ΔX, ΔY, ΔZ baseline components of line AC, respectively. Those elements are calculated as follows:

$$\begin{aligned} L_X &= X_A + \Delta X_{AC} \\ L_Y &= Y_A + \Delta Y_{AC} \\ L_Z &= Z_A + \Delta Z_{AC} \end{aligned} \tag{17.20}$$

The other elements of the L matrix are formed in the same manner as demonstrated for baseline AC. However, before numerical values for the L matrix elements can be obtained, the X_e, Y_e, Z_e geocentric coordinates of all control points in the network must be computed. This is done by following the procedures described in Section 17.5 and demonstrated in Example 17.1. That example problem provided the X_e, Y_e, Z_e coordinates of control points A and B of Figure 17.1, which are used to compute the elements of the L matrix given above.

Note that the observation equations for GNSS network adjustment are linear and that the only nonzero elements of the A matrix are either 1 or -1. This is the same type of matrix that was developed in adjusting differential leveling networks by least squares. In fact, GNSS network adjustments are performed in the very same manner, with the exception of the stochastic model. In relative positioning, the three observed baseline components are correlated. Therefore, a covariance matrix of dimensions 3×3 is derived for each baseline as a product of the least squares adjustment of the carrier-phase observations. This covariance matrix is used to weight the observations in the network adjustment in accordance with Equation (11.4). The weight matrix for any GNSS network is therefore a block-diagonal type, with an individual 3×3 matrix for each observed baseline on the diagonal. When more than two receivers are used, additional 3×3 matrices are created in the off-diagonal region of the matrix to provide the correlation that exists between the simultaneously observed baselines. All other elements of the matrix are zeros. Since commercial software computes individual baseline vectors, the correlation (off-diagonal 3×3 matrices) between simultaneously observed baseline vectors using three or more receivers is not determined. This fact results in a stochastic model that is incorrect theoretically and can cause problems in post-adjustment statistical analysis.

The covariances for the observations in Table 17.1 are given in columns (6) through (11). Only the six lower-triangular elements of the 3×3 covariance matrix for each observation are listed. However, because the covariance matrix is symmetrical, this gives complete weighting information. Columns 6 through 11 list $\sigma_x^2, \sigma_{xy}, \sigma_{xz}, \sigma_y^2, \sigma_{yz}$, and σ_z^2, respectively. Thus, the full 3×3 covariance matrix for baseline AC is

$$\Sigma_{AC} = \begin{bmatrix} 9.884\text{E}{-4} & -9.580\text{E}{-6} & 9.520\text{E}{-6} \\ -9.580\text{E}{-6} & 9.377\text{E}{-4} & -9.520\text{E}{-6} \\ 9.520\text{E}{-6} & -9.520\text{E}{-6} & 9.827\text{E}{-4} \end{bmatrix}$$

The complete weight matrix for the example network of Figure 17.1 has dimensions of 39×39. After inverting the full matrix and multiplying by the a priori estimate for the reference variance, S_0^2, in accordance with Equation (10.4), the weight matrix for the network of Figure 17.1 is as follows. (Note that S_0^2 is taken as 1.0 for this computation, and that no correlation between separate baselines is included.)

$$W = \begin{bmatrix}
1011.8 & 10.2 & -9.7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
10.2 & 1066.6 & 10.2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-9.7 & 10.2 & 1017.7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4634.5 & 50.2 & -49.4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 50.2 & 5209.7 & 54.0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -49.4 & 54.0 & 4988.1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4339.1 & 37.7 & -39.5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 37.7 & 3927.8 & 38.5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -39.5 & 38.5 & 4441.0 & 0 & 0 & 0 \\
 & & & & & & & & & \ddots & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 15,111.8 & 147.7 & -143.8 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 147.7 & 12,336.0 & 106.5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -143.8 & 106.5 & 10,667.8
\end{bmatrix}$$

The system of observation Equations (17.8) is solved by least squares using Equation (11.35). This yields the most probable values for the coordinates of the unknown stations. The ADJUST data file is shown in Figure 17.7. As with other ADJUST data files, the first line is a description of the contents of the file. This line is followed with the number of control stations, number of unknown stations, and number of baseline vectors. The control stations are listed with a point identifier that cannot contain a space, comma, or tab since these characters are used as text delimiters. The point identifier is followed by the point's geocentric coordinates in the order of X, Y, and Z. Following the control station entries is a listing of each baseline vector. These lines have the format of the primary station identifier, secondary station identifier, baseline vector components of ΔX, ΔY, and ΔZ. The baseline vector components are followed by the covariance elements for the vector given in the same order as listed in Table 17.1. The ADJUST data file *Example from Section 17-8.dat* is available on the companion website for this book. For those wishing to program in a higher-level language, the Mathcad® worksheet *C17.xmcd* is available on the companion website. A listing of the adjustment of the data in Figure 17.1 obtained using the program ADJUST follows.

```
Example from Section 17.8
2 4 13
A  402.35087  -4652995.30109   4349760.77753
B  8086.03178  -4642712.84739   4360439.08326
A C 11644.2232 3601.2165 3399.2550 0.00098848 -0.00000958 0.00000952 0.00093777 -0.00000952 0.00098279
A E -5321.7164 3634.0754 3173.6652 0.00021582 -0.00000210 0.00000216 0.00019199 -0.00000210 0.00020052
B C 3960.5442 -6681.2467 -7279.0148 0.00023050 -0.00000223 0.00000207 0.00025464 -0.00000223 0.00022521
B D -11167.6076 -394.5204 -907.9593 0.00027006 -0.00000275 0.00000285 0.00027210 -0.00000272 0.00026707
D C 15128.1647 -6286.7054 -6371.0583 0.00014610 -0.00000143 0.00000134 0.00016145 -0.00000144 0.00013086
D E -1837.7459 -6253.8534 -6596.6697 0.00012311 -0.00000119 0.00000122 0.00012770 -0.00000121 0.00012830
F A -1116.4523 -4596.1610 -4355.9062 0.00007475 -0.00000079 0.00000088 0.00006593 -0.00000081 0.00007616
F C 10527.7852 -994.9377 -956.6246 0.00025671 -0.00000225 0.00000240 0.00021639 -0.00000227 0.00023977
F E -6438.1364 -962.0694 -1182.2305 0.00009442 -0.00000092 0.00000104 0.00009959 -0.00000089 0.00008826
F D -4600.3787 5291.7785 5414.4311 0.00009330 -0.00000099 0.00000090 0.00009875 -0.00000099 0.00012045
F B 6567.2311 5686.2926 6322.3917 0.00006643 -0.00000065 0.00000069 0.00007465 -0.00000064 0.00006048
B F -6567.2310 -5686.3033 -6322.3807 0.00005512 -0.00000063 0.00000061 0.00007472 -0.00000063 0.00006629
A F 1116.4577 4596.1553 4355.9141 0.00006619 -0.00000080 0.00000090 0.00008108 -0.00000082 0.00009376
A B 7683.6620 10282.4234 10678.2972 0.00072397 -0.00000728 0.00000752 0.00067618 -0.00000729 0.00073104
```

FIGURE 17.7 ADJUST data file, for example, in Section 17.8.

Control stations

Station	X	Y	Z
A	402.35087	-4652995.30109	4349760.77753
B	8086.03178	-4642712.84739	4360439.08326

Distance Vectors

From	To	ΔX	ΔY	ΔZ	Covariance matrix elements					
A	C	11644.2232	3601.2165	3399.2550	9.884E-4	-9.580E-6	9.520E-6	9.377E-4	-9.520E-6	9.827E-4
A	E	-5321.7164	3634.0754	3173.6652	2.158E-4	-2.100E-6	2.160E-6	1.919E-4	-2.100E-6	2.005E-4
B	C	3960.5442	-6681.2467	-7279.0148	2.305E-4	-2.230E-6	2.070E-6	2.546E-4	-2.230E-6	2.252E-4
B	D	-11167.6076	-394.5204	-907.9593	2.700E-4	-2.750E-6	2.850E-6	2.721E-4	-2.720E-6	2.670E-4
D	C	15128.1647	-6286.7054	-6371.0583	1.461E-4	-1.430E-6	1.340E-6	1.614E-4	-1.440E-6	1.308E-4
D	E	-1837.7459	-6253.8534	-6596.6697	1.231E-4	-1.190E-6	1.220E-6	1.277E-4	-1.210E-6	1.283E-4
F	A	-1116.4523	-4596.1610	-4355.8962	7.475E-5	-7.900E-7	8.800E-7	6.593E-5	-8.100E-7	7.616E-5
F	C	10527.7852	-994.9377	-956.6246	2.567E-4	-2.250E-6	2.400E-6	2.163E-4	-2.270E-6	2.397E-4
F	E	-6438.1364	-962.0694	-1182.2305	9.442E-5	-9.200E-7	1.040E-6	9.959E-5	-8.900E-7	8.826E-5
F	D	-4600.3787	5291.7785	5414.4311	9.330E-5	-9.900E-7	9.000E-7	9.875E-5	-9.900E-7	1.204E-4
F	B	6567.2311	5686.2926	6322.3917	6.643E-5	-6.500E-7	6.900E-7	7.465E-5	-6.400E-7	6.048E-5
B	F	-6567.2310	-5686.3033	-6322.3807	5.512E-5	-6.300E-7	6.100E-7	7.472E-5	-6.300E-7	6.629E-5
A	F	1116.4577	4596.1553	4355.9141	6.619E-5	-8.000E-7	9.000E-7	8.108E-5	-8.200E-7	9.376E-5

Normal Matrix

16093.0	148.0	-157.3	0	0	0	-6845.9	-60.0	69.4	-3896.2	-40.1	38.6
148.0	15811.5	159.7	0	0	0	-60.0	-6195.0	-67.6	-40.1	-4622.2	-43.4
157.3	159.7	17273.4	0	0	0	69.4	-67.6	-7643.2	38.6	-43.4	-4171.5
0	0	0	23352.1	221.9	-249.8	-8124.3	-75.0	76.5	-10593.3	-96.8	123.8
0	0	0	221.9	23084.9	227.3	-75.0	-7832.2	-73.2	-96.8	-10043.0	-100.1
0	0	0	-249.8	227.3	24116.4	76.5	-73.2	-7795.7	123.8	-100.1	-11332.6
6845.9	-60.0	69.4	-8124.3	-75.0	76.5	29393.8	278.7	-264.4	-10720.0	-106.7	79.2
60.0	-6195.0	-67.6	-75.0	-7832.2	-73.2	278.7	27831.6	260.2	-106.7	-10128.5	-82.5
69.4	-67.6	-7643.2	76.5	-73.2	-7795.7	-264.4	260.2	27487.5	79.2	-82.5	-8303.5
3896.2	-40.1	38.6	-10593.3	-96.8	123.8	-10720.0	-106.7	79.2	86904.9	830.9	-874.3
40.1	-4622.2	-43.4	-96.8	-10043.0	-100.1	-106.7	-10128.5	-82.5	830.9	79084.9	758.1
38.6	-43.4	-4171.5	123.8	-100.1	-11332.6	79.2	-82.5	-8303.5	-874.3	758.1	79234.9

Constant Matrix

```
227790228.2336
23050461170.3104
23480815458.7631
554038059.5699
24047087640.5196
21397654262.6187
491968929.7795
16764436256.9406
16302821193.7660
5314817963.4907
250088821081.7488
238833986695.9468
```

X Matrix

```
12046.5808
4649394.0826
4353160.0634
4919.3391
```

4649361.2199
4352934.4534
3081.5831
4643107.3692
4359531.1220
1518.8012
4648399.1453
4354116.6894

Degrees of Freedom= 27
Reference Variance= 0.6135
Reference So= ±0.78

Adjusted Distance Vectors

From	To	ΔX	ΔY	ΔZ	Vx	Vy	Vz
A	C	11644.2232	3601.2165	3399.2550	0.00669	0.00203	0.03082
A	E	-5321.7164	3634.0754	3173.6652	0.02645	0.00582	0.01068
B	C	3960.5442	-6681.2467	-7279.0148	0.00478	0.01153	-0.00511
B	D	-11167.6076	-394.5204	-907.9593	-0.00731	-0.00136	-0.00194
D	C	15128.1647	-6286.7054	-6371.0583	-0.00081	-0.00801	-0.00037
D	E	-1837.7459	-6253.8534	-6596.6697	-0.01005	0.00268	0.00109
F	A	-1116.4523	-4596.1610	-4355.8962	0.00198	0.00524	-0.01563
F	C	10527.7852	-994.9377	-956.6246	-0.00563	0.00047	-0.00140
F	E	-6438.1364	-962.0694	-1182.2305	-0.00387	-0.00514	-0.00545
F	D	-4600.3787	5291.7785	5414.4311	-0.00561	-0.00232	0.00156
F	B	6567.2311	5686.2926	6322.3917	-0.00051	0.00534	0.00220
B	F	-6567.2310	-5686.3033	-6322.3807	0.00041	0.00536	-0.01320
A	F	1116.4577	4596.1553	4355.9141	-0.00738	0.00046	-0.00227

Advanced Statistical Values

From	To	±S	Slope Dist	Prec
A	C	0.0116	12,653.537	1,089,000
A	E	0.0100	7,183.255	717,000
B	C	0.0116	10,644.669	916,000
B	D	0.0097	11,211.408	1,158,000
D	C	0.0118	17,577.670	1,484,000
D	E	0.0107	9,273.836	868,000
F	A	0.0053	6,430.014	1,214,000
F	C	0.0115	10,617.871	921,000
F	E	0.0095	6,616.111	696,000
F	D	0.0092	8,859.036	964,000
F	B	0.0053	10,744.076	2,029,000
B	F	0.0053	10,744.076	2,029,000
A	F	0.0053	6,430.014	1,214,000

Adjusted Coordinates

Station	X	Y	Z	Sx	Sy	Sz
A	402.35087	-4,652,995.30109	4,349,760.77753			
B	8,086.03178	-4,642,712.84739	4,360,439.08326			
C	12,046.58076	-4,649,394.08256	4,353,160.06335	0.0067	0.0068	0.0066
E	-4,919.33908	-4,649,361.21987	4,352,934.45341	0.0058	0.0058	0.0057
D	-3,081.58313	-4,643,107.36915	4,359,531.12202	0.0055	0.0056	0.0057
F	1,518.80119	-4,648,399.14533	4,354,116.68936	0.0030	0.0031	0.0031

PROBLEMS

Note: For problems requiring least squares adjustment, if a computer program is not distinctly specified for use in the problem, it is expected that the least squares algorithm will be solved using the program MATRIX, which is on the companion website for this book. Partial solutions to problems marked with an asterisk are given in Appendix G.

17.1 Using WGS 84 ellipsoid parameters convert the following geodetic coordinates to geocentric coordinates for these points.

***(a)** φ: 39°23′15.2034″ N, λ: 79°05′53.2957″ W, h: 264.248 m

(b) φ: 64°50′01.7345″ N, λ: 147°43′58.6447″ W, h: 134.722 m

(c) φ: 29°38′46.8500″ N, λ: 82°20′53.3500″ W, h: 59.131 m

(d) φ: 44°33′27.0313″ N, λ: 123°16′32.0695″ W, h: 65.532 m

17.2 Using the WGS 84 ellipsoid parameters convert the following geocentric coordinates (in meters) to geodetic coordinates for these points.

***(a)** $X = -2{,}269{,}821.297$; $Y = -3{,}613{,}410.945$; $Z = 4{,}724{,}567.664$

(b) $X = -1{,}555{,}730.285$; $Y = -5{,}169{,}649.824$; $Z = 3{,}387{,}199.192$

(c) $X = -720{,}398.857$; $Y = -5{,}604{,}531.679$; $Z = 2{,}948{,}315.250$

(d) $X = 1{,}646{,}067.646$; $Y = -4{,}215{,}403.464$; $Z = 4{,}479{,}584.256$

17.3 Given the following GNSS observations and geocentric control station coordinates to Figure P17.3, identify the following:

***(a)** The most probable coordinates and their standard deviations for stations B and C?

(b) The adjusted baseline vector components and their residuals.

(c) The standard deviation of unit weight.

(d) The adjusted geodetic coordinates for stations B and C.

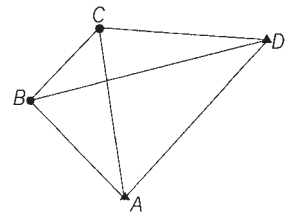


FIGURE P17.3

Control Stations

ID	X (m)	Y (m)	Z (m)
A	1,646,067.646	−4,215,403.464	4,479,584.256
D	1,645,917.137	−4,215,094.021	4,479,928.411

All the data were collected with only two receivers, and the vector covariance matrices for the ΔX, ΔY, and ΔZ values (in meters) given are as follows. For baseline AB:

$\Delta X = 553.861$	3.75E−6	1.89E−7	1.46E−6
$\Delta Y = 216.818$		3.94E−6	6.14E−8
$\Delta Z = 6.945$			3.86E−6

For baseline BC:

$\Delta X = -234.082$	3.85E−6	2.05E−7	1.67E−7
$\Delta Y = 336.957$		3.89E−6	1.13E−7
$\Delta Z = 407.283$			3.88E−6

For baseline CD:

$\Delta X = -470.286$	3.80E−6	1.06E−7	8.29E−8
$\Delta Y = -244.332$		3.86E−6	8.96E−8
$\Delta Z = -70.074$			3.89E−6

For baseline AC:

$\Delta X = 319.780$	3.87E−6	1.27E−7	1.12E−7
$\Delta Y = 553.775$		4.03E−6	2.22E−7
$\Delta Z = 414.229$			3.89E−6

For baseline BD:

$\Delta X = -704.365$	3.94E−6	7.77E−8	6.99E−8
$\Delta Y = 92.623$		3.86E−6	1.29E−7
$\Delta Z = 337.213$			3.94E−6

For baseline DA: (Do not use in adjustment.)

$\Delta X = 150.507$
$\Delta Y = -309.440$
$\Delta Z = -344.157$

17.4 Using the data in Problem 17.3 and static survey specifications of 3 mm + 0.5 ppm with setup errors of ±1.5 mm, analyze the following at a 99.7% level of confidence.

***(a)** The check baseline *DA*.

(b) The loop *ABCA*.

(c) Perform a χ^2 test on the adjustment in Problem 17.3 at a 0.05 level of significance.

17.5 Repeat Problem 17.3 with the following set of data.

Control Stations

ID	*X* (m)	*Y* (m)	*Z* (m)
A	45,570.035	−4,669,039.169	4,330,798.308
D	46,339.014	−4,665,365.079	4,334,701.770

All the data were collected with only two receivers, and the vector covariance matrices for the ΔX, ΔY, and ΔZ values (in meters) given are as follows. For baseline *AB*:

$\Delta X = 3237.273$	5.79E−6	2.04E−7	2.55E−6
$\Delta Y = 2621.034$		6.01E−6	5.90E−8
$\Delta Z = 2742.896$			5.79E−6

For baseline *BC*:

$\Delta X = 2218.856$	4.61E−6	2.94E−6	−9.37E−9
$\Delta Y = 1537.756$		4.60E−6	1.24E−7
$\Delta Z = 1618.385$			4.73E−6

For baseline *CD*:

$\Delta X = -4687.158$	5.56E−6	1.24E−7	7.01E−8
$\Delta Y = -484.708$		9.23E−6	3.88E−7
$\Delta Z = -457.818$			5.78E−6

For baseline *AC*:

$\Delta X = 5456.137$	9.14E−6	1.22E−7	4.32E−8
$\Delta Y = 4158.795$		9.23E−6	3.88E−7
$\Delta Z = 4361.280$			9.33E−6

For baseline *BD*:

$\Delta X = -2468.300$	4.54E−6	4.92E−8	1.27E−7
$\Delta Y = 1053.053$		4.42E−6	7.89E−8
$\Delta Z = 1160.568$			4.53E−6

For baseline *DA*: (Do not use in adjustment.)

$\Delta X = -768.977$
$\Delta Y = -3674.086$
$\Delta Z = -3903.462$

17.6 Using the data in Problem 17.5 and static survey specifications of 3 mm + 0.5 ppm with setup errors of ±1.5 mm, analyze the following at a 99.7% level of confidence.

(a) The check baseline *DA*.

(b) The loops *ABCA*.

(c) Perform a χ^2 test on the adjustment in Problem 17.5 at a 0.05 level of significance.

17.7 Given the following GNSS observations and geocentric control station coordinates to Figure P17.7, what are:

(a) the most probable coordinates and their standard deviations for the adjusted station *E*?

(b) the adjusted baseline vector components and their residuals?

(c) the standard deviation of unit weight?

(d) the adjusted geodetic coordinates for the stations?

Control Station

ID	*X* (m)	*Y* (m)	*Z* (m)
A	−1,555,653.508	−5,169,774.359	3,387,052.832
B	−1,555,482.937	−5,169,744.338	3,387,180.730
C	−1,555,658.816	−5,169,603.978	3,387,301.300
D	−1,555,879.386	−5,169,572.170	3,387,242.042

All the data were collected with only two receivers, and the vector covariance matrices for the ΔX, ΔY, and ΔZ values (in meters) given are as follows. For baseline *AE*:

$\Delta X = -93.226$	3.02	1.06E−1	−3.51E−1
$\Delta Y = 122.018$		3.10	1.05E−1
$\Delta Z = 137.721$			2.95

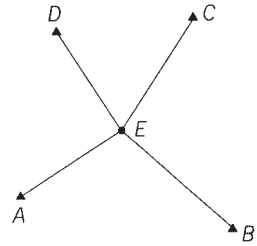


FIGURE P17.7

For baseline *BE*:

$\Delta X = -267.444$	3.03	8.67E−3	9.71E−2
$\Delta Y = 92.352$		3.11	1.61E−1
$\Delta Z = 10.644$			3.01

For baseline *CE*:

$\Delta X = -91.675$	3.09	−2.87E−1	1.21E−1
$\Delta Y = -48.005$		3.14	−9.06E−2
$\Delta Z = -110.961$			3.07

For baseline *DE*:

$\Delta X = 136.126$	4.29	9.99E−2	1.20E−1
$\Delta Y = -80.832$		3.01	6.90E−2
$\Delta Z = -53.311$			3.10

17.8 A parallelogram figure is observed using only control station *A*. The results of the survey are as follows. Adjusted the GNSS network.

(a) the most probable coordinates and their standard deviations for the adjusted stations *B*, *C*, and *D*?

(b) the adjusted baseline vector components and their residuals?

(c) the standard deviation of unit weight?

(d) the adjusted geodetic coordinates for the stations?

Control Station

ID	*X* (m)	*Y* (m)	*Z* (m)
A	2,681.756	−4,652,569.792	4,348,467.482

All the data were collected with only two receivers and the vector covariance matrices for the ΔX, ΔY, and ΔZ values (in meters) given are as follows. For baseline *AB*:

$\Delta X = 3449.378$	1.42E−5	4.99E−7	6.23E−6
$\Delta Y = 4841.932$		1.47E−5	1.44E−7
$\Delta Z = 5152.811$			1.42E−5

For baseline *BC*:

$\Delta X = -3561.818$	1.07E−5	6.81E−7	−2.17E−8
$\Delta Y = 1659.155$		1.07E−5	2.89E−7
$\Delta Z = 1711.806$			1.10E−5

For baseline *CD*:

$\Delta X = -6760.347$	1.85E−5	4.12E−7	2.33E−7
$\Delta Y = -5722.930$		1.86E−5	4.89E−7
$\Delta Z = -6071.922$			1.92E−5

For baseline *DA*:

$\Delta X = 6872.776$	1.30E−5	1.73E−7	6.13E−8
$\Delta Y = -778.171$		1.31E−5	5.51E−7
$\Delta Z = -792.695$			1.32E−5

For baseline *AC*:

$\Delta X = -112.432$	1.68E−5	1.82E−7	4.71E−7
$\Delta Y = 6501.102$		1.64E−5	2.92E−7
$\Delta Z = 6864.615$			1.68E−5

17.9 Use the data in Problem 17.8 and static survey specifications of 5 mm + 0.5 ppm with setup errors of ±1.5 mm:

(a) Analyze the loops *ADCA* and *ACBA* at a 99.7% level of confidence.

(b) Perform a χ^2 test on the adjustment in Problem 17.8 at a 0.05 level of significance.

17.10 Given the following GNSS observations and geocentric control station coordinates to Figure P17.10, what are:

(a) the most probable coordinates and their standard deviations for the adjusted stations *B*, *D*, and *E*?

(b) the adjusted baseline vector components and their residuals?
(c) the standard deviation of unit weight?
(d) the adjusted geodetic coordinates for the station?

Control Stations

ID	X (m)	Y (m)	Z (m)
A	−1,786,918.107	−4,329,502.702	4,316,902.134
C	−1,786,752.692	−4,329,190.002	4,317,237.271

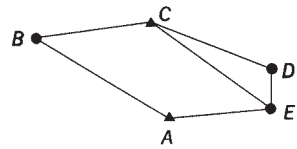


FIGURE P17.10

The vector covariance matrices for the ΔX, ΔY, and ΔZ values (in meters) given are as follows. For baseline AE:

$\Delta X = 362.316$	3.78E−6	5.79E−8	5.32E−8
$\Delta Y = -29.809$		3.77E−6	8.31E−8
$\Delta Z = 125.53$			3.75E−6

For baseline AB:

$\Delta X = -182.539$	3.80E−6	3.16E−7	1.75E−6
$\Delta Y = 401.073$		3.61E−6	8.98E−8
$\Delta Z = 281.895$			3.76E−6

For baseline BC:

$\Delta X = 347.955$	3.71E−6	2.39E−7	5.77E−10
$\Delta Y = -88.371$		3.80E−6	1.78E−8
$\Delta Z = 53.244$			3.72E−6

For baseline *CE*:

$\Delta X = 196.899$	3.77E−6	1.77E−7	8.79E−8
$\Delta Y = -342.507$		3.73E−6	−9.85E−8
$\Delta Z = -209.605$			3.64E−6

For baseline *CD*:

$\Delta X = 222.286$	3.67E−6	8.30E−8	1.05E−7
$\Delta Y = -243.485$		3.74E−6	1.11E−7
$\Delta Z = -115.310$			3.76E−6

For baseline *DE*:

$\Delta X = -25.389$	3.74E−6	1.44E−7	−5.00E−9
$\Delta Y = -99.024$		3.82E−6	1.97E−7
$\Delta Z = -94.290$			3.70E−6

17.11 Use the data in Problem 17.10 and static survey specifications of 3 mm + 0.5 ppm with setup errors of ±1.5 mm:

(a) Analyze the loops *ABCEA* and *CDEC* at a 99.7% level of confidence.

(b) Perform a χ^2 test on the adjustment in Problem 17.10 at a 0.05 level of significance.

17.12 Given the additional GNSS observations for Problem 17.10, what are:

(a) the most probable coordinates and their standard deviations for the adjusted stations *B*, *D*, and *E*?

(b) the adjusted baseline vector components and their residuals?

(c) the standard deviation of unit weight?

The vector covariance matrices for the ΔX, ΔY, and ΔZ values (in meters) given are as follows. For baseline *ED*:

$\Delta X = 25.388$	3.81E−6	1.06E−7	−6.06E−7
$\Delta Y = 99.021$		3.81E−6	1.32E−7
$\Delta Z = 94.287$			3.73E−6

For baseline *BA*:

$\Delta X = 182.537$	3.74E−6	7.35E−8	7.83E−8
$\Delta Y = -401.070$		3.72E−6	2.54E−7
$\Delta Z = -281.894$			3.81E−6

For baseline *CB*:

$\Delta X = -347.953$	2.45E−6	9.44E−7	7.51E−8
$\Delta Y = 88.370$		3.90E−6	6.88E−8
$\Delta Z = -53.244$			3.81E−6

For baseline *EC*:

$\Delta X = -196.898$	2.09E−6	3.94E−8	4.92E−8
$\Delta Y = 342.508$		3.77E−6	2.07E−7
$\Delta Z = 209.602$			3.72E−6

17.13 Assuming a setup error of ±1 mm and a manufacturer's specified accuracy for a static survey of 3 mm + 0.5 ppm, analyze the following repeat baselines from Problems 17.10 and 17.12 at a 0.01 level of significance.

***(a)** Baselines *ED* and *DE*.

(b) Baselines *AB* and *BA*.

(c) Baselines *BC* and *CB*.

(d) Baselines *CE* and *EC*.

17.14 Repeat Problem 17.5 with the following set of data.

Control Stations

ID	*X* (m)	*Y* (m)	*Z* (m)
A	594,115.232	−4,856,468.191	4,078,370.661
D	592,453.775	−4,855,933.893	4,079,280.982

All the data were collected with only two receivers, and the vector covariance matrices for the ΔX, ΔY, and ΔZ values (in meters) given are as follows. For baseline *AB*:

$\Delta X = 299.562$	3.88E−6	1.37E−7	1.71E−6
$\Delta Y = 869.562$		4.02E−6	3.95E−8
$\Delta Z = 984.493$			3.88E−6

For baseline *BC*:

$\Delta X = -838.595$	3.84E−6	2.44E−7	−7.81E−9
$\Delta Y = 3.527$		3.83E−6	1.04E−7
$\Delta Z = 135.888$			3.94E−6

For baseline *CD*:

$\Delta X = -1122.431$	3.83E−6	8.53E−8	4.83E−8
$\Delta Y = -338.798$		3.86E−6	1.01E−7
$\Delta Z = -210.060$			3.98E−6

For baseline *AC*:

$\Delta X = -539.027$	3.90E−6	5.19E−8	1.84E−8
$\Delta Y = 873.094$		3.93E−6	1.65E−7
$\Delta Z = 1120.381$			3.98E−6

For baseline *BD*:

$\Delta X = -1961.024$	4.15E−6	4.50E−8	1.16E−7
$\Delta Y = -335.266$		4.04E−6	7.21E−8
$\Delta Z = -74.171$			4.14E−6

17.15 Given the additional GNSS observations for Problem 17.14, what are:

(a) the most probable coordinates and their standard deviations for the adjusted stations *B* and *C*?

(b) the adjusted baseline vector components and their residuals?

(c) the standard deviation of unit weight?

The vector covariance matrices for the ΔX, ΔY, and ΔZ values (in meters) given are as follows. For baseline *BA*:

$\Delta X = -299.564$	3.98E−6	7.29E−8	−3.64E−7
$\Delta Y = -869.560$		3.78E−6	1.64E−7
$\Delta Z = -984.494$			3.91E−6

For baseline *BD*:

$\Delta X = -1961.024$	4.15E−6	4.50E−8	1.16E−7
$\Delta Y = -335.266$		4.04E−6	7.21E−8
$\Delta Z = -74.171$			4.14E−6

For baseline *CB*:

$\Delta X = 838.592$	3.85E−6	6.78E−8	9.35E−8
$\Delta Y = -3.53$		3.86E−6	−4.95E−9
$\Delta Z = -135.893$			3.81E−6

For baseline *DC*:

$\Delta X = 1122.427$	5.25E−6	1.08E−7	6.84E−8
$\Delta Y = 338.795$		3.84E−6	5.64E−9
$\Delta Z = 210.057$			3.91E−6

17.16 Assuming a setup error of ±1.5 mm and a manufacturer's specified accuracy for a static survey of 3 mm + 0.5 ppm, analyze the following repeat baselines from Problems 17.14 and 17.15 at a 0.01 level of significance.

(a) Baselines *AB* and *BA*.

(b) Baselines *BD* and *DB*.

(c) Baselines *BC* and *CB*.

(d) Baselines *DC* and *CD*.

17.17 Using the data from Problems 17.11 with an estimated setup error of = ±1.5 mm and a manufacturer's estimated static survey accuracy of 3 mm + 0.5 ppm, analyze the following observations at a 99% level of confidence.

(a) Loop *ABCDEA*.

(b) Loop *ABCEA*.

(c) Loop *CDEC*.

17.18 Given the following GNSS observations and geocentric control station coordinates to Figure P17.18, what are:

(a) the most probable coordinates and their standard deviations for the adjusted stations?

(b) the adjusted baseline vector components and their residuals?

(c) the standard deviation of unit weight?

(d) the adjusted geodetic coordinates for the stations?

Control Stations

ID	*X* (m)	*Y* (m)	*Z* (m)
A	−2,408,843.951	−3,766,414.748	4,534,152.298
C	−2,409,355.162	−3,765,292.295	4,534,760.702

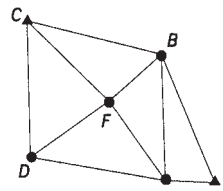


FIGURE P17.18

All the data were collected with only two receivers and the vector covariance matrices for the ΔX, ΔY, and ΔZ values (in meters) given are as follows. For baseline *AB*:

$\Delta X =$ 62.600	1.01E−5	1.01E−6	4.31E−6
$\Delta Y =$ 590.153		9.86E−6	2.08E−7
$\Delta Z =$ 35.955			9.99E−6

For baseline *BC*:

$\Delta X =$ −573.806	1.02E−5	1.75E−7	5.00E−7
$\Delta Y =$ 532.297		1.00E−5	3.23E−7
$\Delta Z =$ 572.451			9.98E−6

For baseline *CD*:

$\Delta X =$ −281.658	9.98E−6	4.87E−8	2.79E−7
$\Delta Y =$ −507.897		1.01E−5	4.62E−7
$\Delta Z =$ −572.446			1.02E−5

For baseline *DE*:

$\Delta X =$ 792.973	1.00E−5	2.01E−7	2.76E−7
$\Delta Y =$ −555.113		9.56E−6	3.28E−7
$\Delta Z =$ −33.420			9.78E−6

For baseline *EA*:

$\Delta X =$ −0.100	1.00E−5	2.99E−7	−1.16E−7
$\Delta Y =$ −59.446		9.71E−6	2.15E−7
$\Delta Z =$ −2.541			9.85E−6

For baseline *FB*:

$\Delta X =$ 445.908	1.01E−5	5.06E−7	−1.08E−7
$\Delta Y =$ −204.416		1.01E−5	5.84E−7
$\Delta Z =$ −377.022			9.92E−6

For baseline *FC*:

$\Delta X =$ −127.899	9.55E−6	1.14E−8	−1.29E−6
$\Delta Y =$ 327.881		9.85E−6	4.20E−7
$\Delta Z =$ 195.429			9.99E−6

For base line *FD*:

$\Delta X = -409.554$	1.00E−5	4.13E−7	4.52E−7
$\Delta Y = -180.018$		1.01E−5	7.26E−7
$\Delta Z = -377.014$			1.00E−5

For baseline *FE*:

$\Delta X = 383.417$	1.07E−5	1.61E−7	1.27E−7
$\Delta Y = -735.120$		1.00E−5	4.46E−7
$\Delta Z = -410.433$			1.02E−5

For baseline *EB*:

$\Delta X = 62.501$	1.53E−5	4.96E−7	−2.91E−7
$\Delta Y = 530.708$		1.02E-5	−1.57E−7
$\Delta Z = 33.415$			1.01E−5

For baseline *DE*:

$\Delta X = 792.961$	1.02E−5	3.03E−7	2.37E−6
$\Delta Y = -555.107$		1.03E-5	4.73E−7
$\Delta Z = -33.418$			1.01E−5

For baseline *DC*:

$\Delta X = 281.655$	1.00E−5	4.99E−8	−2.20E−7
$\Delta Y = 507.904$		1.00E-5	3.91E−8
$\Delta Z = 572.446$			9.81E−6

17.19 The baselines in Problem 17.18 were observed with GNSS instruments having manufacturer's specified accuracies for static surveys of 3 mm + 0.5 ppm. The user estimated the setup error as ±1.5 mm. Analyze the following loops at a 99.7% level of confidence.

(a) Loop *DEFD*.

(b) Loop *DFCD*.

(c) Loop *BCFB*.

(d) Loop *BFEB*.

(e) Loop *ABEA*.

Use the ADJUST program to solve the following problems.

17.20 Problem 17.5.

17.21 Problem 17.14.

17.22 Problem 17.15.

17.23 Problem 17.18.

17.24 Problem 17.19.

PROGRAMMING PROBLEMS

17.25 Write a computational package that reads a file of station coordinates and GNSS baselines and then

(a) Writes the data to a file in a formatted fashion.

(b) Computes the A, L, and W matrices.

(c) Writes the matrices to a file that is compatible with the MATRIX program.

(d) Demonstrate this program with Problem 17.19.

17.26 Write a computational package that reads a file of station coordinates and GNSS baselines and then

(a) Writes the data to a file in a formatted fashion.

(b) Computes the A, L, and W matrices.

(c) Performs a weighted least squares adjustment.

(d) Writes the matrices used in computations in a formatted fashion to a file.

(e) Computes the final station coordinates, their estimated errors, the adjusted baseline vectors, their residuals, and their estimated errors, and writes them to a file in a formatted fashion.

(f) Demonstrate this program with Problem 17.19.

CHAPTER 18

COORDINATE TRANSFORMATIONS

18.1 INTRODUCTION

The transformation of points from one coordinate system to another is a common problem encountered in surveying and mapping. For instance, a surveyor who works initially in an assumed coordinate system on a project may find it necessary to transform the coordinates to the state plane coordinate system. In GNSS surveying and in photogrammetry, coordinate transformations are used extensively. Since the inception of the North American Datum of 1983 (NAD 83), many land surveyors, management agencies, state departments of transportation, and others struggled with the problem of converting their multitudes of stations defined in the 1927 datum (NAD 27) to the 1983 datum. Although several mathematical models have been developed to make these conversions, all involve some form of coordinate transformation. This chapter covers the introductory procedures of using least squares to compute several well-known and commonly used transformations. More rigorous procedures, which employ the *general least-squares* procedure, are given in Chapter 23.

18.2 THE TWO-DIMENSIONAL CONFORMAL COORDINATE

The two-dimensional conformal coordinate transformation, also known as the *four-parameter similarity transformation*, has the characteristic that true shape is retained after transformation. It is used typically in surveying when converting separate surveys into a common reference coordinate system. An example of this is the procedure used by several commercial software packages to convert satellite-derived coordinates into a local reference frame. For example, this procedure can be used to convert GNSS-derived WGS84 coordinates into one of the realizations in the NAD83 coordinate system. This process, known as *localization*, is discussed further in Section 24.5. The two-dimensional conformal coordinate transformation can be thought of as a three-step process:

1. *Scaling* to create equal dimensions in the two coordinate systems.
2. *Rotation* to make the reference axes of the two systems parallel.
3. *Translation*s to create a common origin for the two coordinate systems.

The scaling and rotation are each defined by one parameter. The translations involve two parameters. Thus, there are a total of four parameters in this transformation. The transformation requires a minimum of two points, called *control points*, which are common to both systems. With the minimum of two points, the four parameters of the transformation can be determined uniquely. If more than two control points are available, a least-squares adjustment is possible. After determining values for the transformation parameters, any point in the original system can be transformed into the second system.

18.3 EQUATION DEVELOPMENT

Figure 18.1(*a*) and (*b*) illustrates two independent coordinate systems. In these systems, three common control points, *A*, *B*, and *C*, exist (i.e., their coordinates

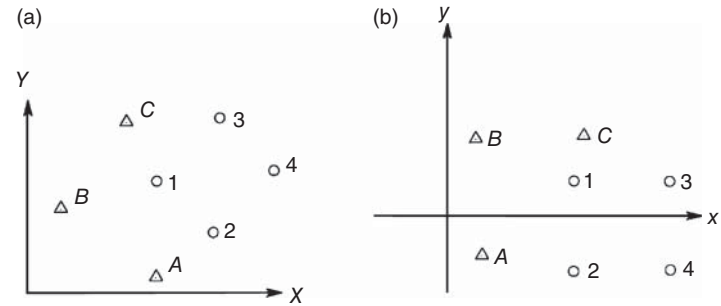


FIGURE 18.1 Two-dimensional coordinate systems.

are known in both systems). Points 1 through 4 have coordinates known only in the *xy* system of Figure 18.1(*b*). The problem is to determine *XY* coordinates in the system of Figure 18.1(*a*). The necessary equations are developed as follows:

Step 1: Scaling. To make line lengths as defined by the *xy* coordinate system equal to their lengths in the *XY* system, it is necessary to multiply *xy* coordinates by a scale factor, *S*. Thus, the scaled coordinates x' and y' are

$$x' = Sx$$
$$y' = Sy \tag{18.1}$$

Step 2: Rotation. In Figure 18.2, the *XY* coordinate system has been superimposed on the scaled $x'y'$ system. The rotation angle, θ, is shown between the y'- and *Y*-axes. To analyze the effects of this rotation, an $X'Y'$ system is constructed parallel to the *XY* system such that its origin is common with that of the $x'y'$ system. Expressions that give the (X', Y') rotated coordinates for any point (such as point 4 shown) in terms of its $x'y'$ coordinates are

$$X' = x' \cos\theta - y' \sin\theta$$
$$Y' = x' \sin\theta + y' \cos\theta \tag{18.2}$$

Step 3: Translation. To finally arrive at *XY* coordinates for a point, it is necessary to translate the origin of the $X'Y'$ system to the origin of the *XY* system. Referring to Figure 18.2, this translation is accomplished by adding translation factors as follows:

$$X = X' + T_X \text{ and } Y = Y' + T_Y \tag{18.3}$$

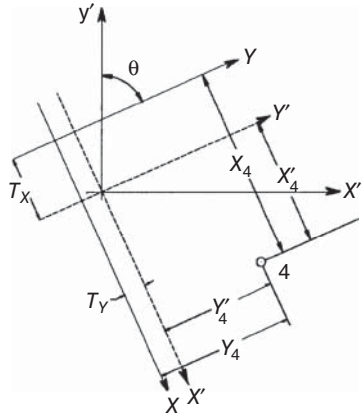


FIGURE 18.2 Super-imposed coordinate systems.

If Equations (18.1), (18.2), and (18.3) are combined, a single set of equations results that transform the points of Figure 18.1(b) directly into Figure 18.1(a) as

$$\begin{aligned} X &= (S\cos\theta)x - (S\sin\theta)y + T_X \\ Y &= (S\sin\theta)x + (S\cos\theta)y + T_Y \end{aligned} \tag{18.4}$$

Now letting $S\cos\theta = a$, $S\sin\theta = b$, $T_X = c$, and $T_Y = d$ and adding residuals to make redundant equations consistent, the resulting equation can be written as

$$\begin{aligned} ax - by + c &= X + v_X \\ bx + ay + d &= Y + v_Y \end{aligned} \tag{18.5}$$

18.4 APPLICATION OF LEAST SQUARES

Equations (18.5) represent the basic observation equations for a two-dimensional conformal coordinate transformation that have four unknowns: a, b, c, and d. The four unknowns embody the transformation parameters S, θ, T_X, and T_Y. Since two equations can be written for every control point, only two control points are needed for a unique solution. When more than two are present, a redundant system exists for which a least-squares solution can be found. As an example, consider the equations that could be written for the situation illustrated in Figure 18.1. There are three control points, A, B, and C, and thus the following six equations can be written as

$$\begin{aligned} ax_a - by_a + c &= X_A + v_{X_A} \\ bx_a + ay_a + d &= Y_A + v_{Y_A} \\ ax_b - by_b + c &= X_B + v_{X_B} \\ bx_b + ay_b + d &= Y_B + v_{Y_B} \\ ax_c - by_c + c &= X_C + v_{X_C} \\ bx_c + ay_c + d &= Y_C + v_{Y_C} \end{aligned} \tag{18.6}$$

Equations (18.6) can be expressed in matrix form as

$$AX = L + V \tag{18.7}$$

where

$$A = \begin{bmatrix} x_a & -y_a & 1 & 0 \\ y_a & x_a & 0 & 1 \\ x_b & -y_b & 1 & 0 \\ y_b & x_b & 0 & 1 \\ x_c & -y_c & 1 & 0 \\ y_c & x_c & 0 & 1 \end{bmatrix} \quad X = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \quad L = \begin{bmatrix} X_A \\ Y_A \\ X_B \\ Y_B \\ X_C \\ Y_C \end{bmatrix} \quad V = \begin{bmatrix} v_{X_A} \\ v_{Y_A} \\ v_{X_B} \\ v_{Y_B} \\ v_{X_C} \\ v_{Y_C} \end{bmatrix}$$

The redundant system of equations is solved using Equation (11.32). Having obtained the most probable values for the coefficients from the least-squares solution, the *XY* coordinates for any additional points whose coordinates are known in the *xy* system are obtained by applying Equations (18.5) (where the residuals are now considered to be zeros).

After the adjustment, the scale factor S and rotation angle θ are computed with the following equations:

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$
$$S = \frac{a}{\cos\theta} \tag{18.8}$$

Example 18.1 A survey conducted in an arbitrary *xy* coordinate system produced station coordinates for *A*, *B*, and *C* as well as for stations 1 through 4. Stations *A*, *B*, and *C* also have known state plane coordinates, labeled *E* and *N*. It is required to derive the state plane coordinates of stations 1 through 4. Table 18.1 has a tabulation of the arbitrary coordinates and state plane coordinates.

SOLUTION The observation equations are

$$\begin{bmatrix} 121.622 & 128.066 & 1 & 0 \\ -128.066 & 121.622 & 0 & 1 \\ 141.228 & -187.718 & 1 & 0 \\ 187.718 & 141.228 & 0 & 1 \\ 175.802 & -135.728 & 1 & 0 \\ 135.728 & 175.802 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1049422.40 \\ 51089.20 \\ 1049413.95 \\ 49659.30 \\ 1049244.95 \\ 49884.950 \end{bmatrix} + \begin{bmatrix} v_{X_A} \\ v_{Y_A} \\ v_{X_B} \\ v_{Y_B} \\ v_{X_C} \\ v_{Y_C} \end{bmatrix}$$

The least-squares solution is obtained by applying Equation (11.32) to these observation equations. A computer listing from the program ADJUST is given below. The data file shown in Figure 18.3 can be found on the companion website for this book as *Example 18-1.dat*. The first line of the file is an explanatory description of the file. Following this line is the number of common points in the

TABLE 18.1 Data for Example 18.1

Point	*E*	*N*	*x*	*y*
A	1,049,422.40	51,089.20	121.622	−128.066
B	1,049,413.95	49,659.30	141.228	187.718
C	1,049,244.95	49,884.95	175.802	135.728
1			174.148	−120.262
2			513.520	−192.130
3			754.444	−67.706
4			972.788	120.994

```
Survey transformation
3
A 1049422.40 51089.20 0.05 0.05
B 1049413.95 49659.30 0.05 0.05
C 1049244.95 49884.95 0.05 0.05
A 121.622 -128.066 0.005 0.005
B 141.228 187.718 0.005 0.005
C 175.802 135.728 0.005 0.005
1 174.148 -120.262 0.005 0.005
2 513.52 -192.13 0.005 0.005
3 754.444 -67.706
4 972.788 120.994
5 1068.118 161.624
```

FIGURE 18.3 ADJUST data file for Example 18.1.

file. This line is followed by the control points, which in this file are the E and N coordinates. The control points are followed by the points in the arbitrary file. All the lines with coordinates have the format of point identifiers, x, y, S_x, S_y, where the standard deviations S_x and S_y are optional, and the point identifiers are limited to 10 characters. As with all ADJUST data files, the identifiers cannot contain the data delimiters of a space, comma, or tab. The identifiers entered as *pt A* will be read as two separate point identifiers of *pt* and *A*. The software will notify the user of this as a reading error.

The output shown in Figure 18.4 includes the input data, the coordinates of transformed points, the transformation coefficients, and their estimated errors. After obtaining the solution using Equation (11.32), the program solved Equation (18.8) to obtain the rotation angle and scale factor of the transformation. Furthermore,

```
Two-Dimensional Conformal Coordinate Transformation
---------------------------------------------------
ax - by + Tx = X + VX
bx + ay + Ty = Y + VY

Transformed Control Points
     POINT          X             Y          VX         VY
-------------------------------------------------------------
         A 1,049,422.400   51,089.200     0.004     -0.029
         B 1,049,413.950   49,659.300     0.101     -0.077
         C 1,049,244.950   49,884.950    -0.105      0.106

Transformation Parameters:
  a =      -4.51249 ±       0.00058
  b =      -0.25371 ±       0.00058
 Tx =  1050003.715 ±         0.123
 Ty =    50542.131 ±         0.123

Rotation = 183° 13' 05.0"
Scale = 4.51962
Adjustment's Reference Variance = 0.0195
Transformed Points
     POINT          X             Y         ±Sx        ±Sy
-------------------------------------------------------------
         1 1,049,187.361   51,040.629     0.135      0.135
         2 1,047,637.713   51,278.829     0.271      0.271
         3 1,046,582.113   50,656.241     0.368      0.368
         4 1,045,644.713   49,749.336     0.484      0.484
```

FIGURE 18.4 Results from ADJUST for Example 18.1.

this adjustment is also performed in the spreadsheet file *Chapter 18.xlsx*, which is available on the companion website. The Mathcad file *C18.xmcd* is available on the companion website for those wishing to program this problem in a higher-level computer language.

18.5 TWO-DIMENSIONAL AFFINE COORDINATE TRANSFORMATION



The two-dimensional affine coordinate transformation is also known as the *six-parameter transformation*. It is a slight variation from the two-dimensional conformal transformation. In the affine transformation, there is the additional allowance for two different scale factors: one in the x direction and the other in the y direction. This transformation is commonly used in photogrammetry for interior orientation of the image. That is, it is used to transform photo coordinates from an arbitrary measurement photo coordinate system to the camera fiducial coordinate system and thus account for the differential shrinkages that can occur in the x and y directions. As in the conformal transformation, the affine transformation also applies two translations of the origin, and a rotation about the origin, plus a small nonorthogonality correction between the x- and y-axes. This results in a total of six unknowns. The observation equations for the affine transformation are

$$\begin{aligned} ax + by + c &= X + v_X \\ dx + ey + f &= Y + v_Y \end{aligned} \tag{18.9}$$

These equations are linear and can be solved uniquely when three control points exist, since each control point results in an equation set in the form of Equations (18.9). Thus, three points yield six equations involving six unknowns yielding a unique solution. If more than three control points are available, a least-squares solution can be obtained. Assume, for example, that four common points (1, 2, 3, and 4) exist. Then, the equation system is

$$\begin{aligned} ax_1 + by_1 + c &= X_1 + v_{X_1} \\ dx_1 + ey_1 + f &= Y_1 + v_{Y_1} \\ ax_2 + by_2 + c &= X_2 + v_{X_2} \\ dx_2 + ey_2 + f &= Y_2 + v_{Y_2} \\ ax_3 + by_3 + c &= X_3 + v_{X_3} \\ dx_3 + ey_3 + f &= Y_3 + v_{Y_3} \\ ax_4 + by_4 + c &= X_4 + v_{X_4} \\ dx_4 + ey_4 + f &= Y_4 + v_{Y_4} \end{aligned} \tag{18.10}$$

In matrix notation, Equations (18.10) are expressed as $AX = L + V$, where

$$\begin{bmatrix} x_1 & y_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1 & y_1 & 1 \\ x_2 & y_2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_2 & y_2 & 1 \\ x_3 & y_3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_3 & y_3 & 1 \\ x_4 & y_4 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x_4 & y_4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} X_1 \\ Y_1 \\ X_2 \\ Y_2 \\ X_3 \\ Y_3 \\ X_4 \\ Y_4 \end{bmatrix} + \begin{bmatrix} v_{X_1} \\ v_{Y_1} \\ v_{X_2} \\ v_{Y_2} \\ v_{X_3} \\ v_{Y_3} \\ v_{X_4} \\ v_{Y_4} \end{bmatrix} \tag{18.11}$$

The most probable values for the unknown parameters are computed using least-squares Equation (11.32). They are then used to transfer the remaining points from the xy coordinate system to the XY coordinate system. The two different rotations and scale factors can be computed from these parameters as follows:

Rotation of the measurement system, θ: $\theta = \tan^{-1}\left(\dfrac{-b}{e}\right)$

Let t be: $t = \tan^{-1}\left(\dfrac{-d}{a}\right)$

Nonorthogonality of the measurement axes: $\alpha = \theta + t$

Scale in x: $S_x = a\dfrac{\cos\alpha}{\cos t}$

Scale in y: $S_y = e\dfrac{\cos\alpha}{\cos\theta}$

Example 18.2 Photo coordinates, which have been measured using a digitizer, must be transformed into the camera's fiducial coordinate system. The four fiducial points and the additional points were observed in the digitizer's xy coordinate system and are listed in Table 18.2, together with the known camera XY fiducial coordinates.

TABLE 18.2 Coordinates of Points for Example 18.2

Point	X	Y	x	y	S_x	S_y
1	−113.000	0.003	0.764	5.960	0.104	0.112
3	0.001	112.993	5.062	10.541	0.096	0.120
5	112.998	0.003	9.663	6.243	0.112	0.088
7	0.001	−112.999	5.350	1.654	0.096	0.104
306			1.746	9.354		
307			5.329	9.463		

SOLUTION The observation equations in matrix form are

$$\begin{bmatrix} 0.764 & 5.960 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.764 & 5.960 & 1 \\ 5.062 & 10.541 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5.062 & 10.541 & 1 \\ 9.663 & 6.243 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 9.663 & 6.243 & 1 \\ 5.350 & 1.654 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5.350 & 1.654 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \end{bmatrix} = \begin{bmatrix} -113.000 \\ 0.003 \\ 0.001 \\ 112.993 \\ 112.998 \\ 0.003 \\ 0.001 \\ 112.999 \end{bmatrix} + \begin{bmatrix} v_{X_1} \\ v_{Y_1} \\ v_{X_2} \\ v_{Y_2} \\ v_{X_3} \\ v_{Y_3} \\ v_{X_4} \\ v_{Y_4} \end{bmatrix}$$

The observation equations are then solved using Equation (11.32). A self-explanatory computer solution from the program ADJUST that yields the least-squares solution for an affine transformation is shown in Figure 18.5. The data file *Example 18-2.dat* that generated this listing can be found on the companion website for this book. This file follows the same format as described in the solution of Example 18.1. Furthermore, this adjustment is also performed in the spreadsheet file *Chapter 18.xls* also available on the companion website. The Mathcad file *C18.xmcd* is available on the companion website for those wishing to program this problem in a higher-level computer language.

```
Two-Dimensional Affine Coordinate Transformation
------------------------------------------------
ax + by + c = X + VX
dx + ey + f = Y + VY

Transformed Control Points
     POINT          X             Y          VX          VY
-------------------------------------------------------------
         1    -113.000         0.003      0.101       0.049
         3       0.001       112.993     -0.086      -0.057
         5     112.998         0.003      0.117       0.030
         7       0.001      -112.999     -0.086      -0.043

Transformation Parameters:
 a =     25.37152 ± 0.02532
 b =      0.82220 ± 0.02256
 c =     -137.183 ± 0.203
 d =     -0.80994 ± 0.02335
 e =     25.40166 ± 0.02622
 f =     -150.723 ± 0.216

Adjustment's Reference Variance = 2.1828
Transformed Points
     POINT          X             Y         ±?x         ±?y
-------------------------------------------------------------
         1    -112.899         0.052      0.132       0.141
         3      -0.085       112.936      0.125       0.147
         5     113.115         0.033      0.139       0.118
         7      -0.085      -113.042      0.125       0.134
       306     -85.193        85.470      0.134       0.154
```

FIGURE 18.5 Results from ADJUST for Example 18.2.

18.6 THE TWO-DIMENSIONAL PROJECTIVE COORDINATE TRANSFORMATION



The two-dimensional projective coordinate transformation is also known as the *eight-parameter transformation*. It is appropriate to use when one two-dimensional coordinate system is projected onto another nonparallel system. This transformation is commonly used in photogrammetry and can also be used to transform NAD 27 coordinates into the NAD 83 system. In their final form, the two-dimensional projective coordinate observation equations are

$$X = \frac{a_1 x + b_1 y + c_1}{a_3 x + b_3 y + 1}$$
$$Y = \frac{a_2 x + b_2 y + c_2}{a_3 x + b_3 y + 1} \tag{18.12}$$

Upon inspection, these equations are similar to the affine transformation. In fact, if a_3 and b_3 are equal to zero, these equations are the affine transformation. With eight unknowns, this transformation requires a minimum of four control points. If there are more than four control points, the least-squares solution may be used. Since these are nonlinear equations, they must be linearized and solved using Equation (11.37) or (11.39). The linearized form of these equations is

$$\begin{bmatrix} \left(\frac{\partial X}{\partial a_1}\right)_0 & \left(\frac{\partial X}{\partial b_1}\right)_0 & \left(\frac{\partial X}{\partial c_1}\right)_0 & 0 & 0 & 0 & \left(\frac{\partial X}{\partial a_3}\right)_0 & \left(\frac{\partial X}{\partial b_3}\right)_0 \\ 0 & 0 & 0 & \left(\frac{\partial X}{\partial a_2}\right)_0 & \left(\frac{\partial X}{\partial b_2}\right)_0 & \left(\frac{\partial X}{\partial c_2}\right)_0 & \left(\frac{\partial X}{\partial a_3}\right)_0 & \left(\frac{\partial X}{\partial b_3}\right)_0 \end{bmatrix} \begin{bmatrix} da_1 \\ db_1 \\ dc_1 \\ da_2 \\ db_2 \\ dc_2 \\ da_3 \\ db_3 \end{bmatrix}$$
$$= \begin{bmatrix} X - X_0 \\ Y - Y_0 \end{bmatrix} \tag{18.13}$$

where

$$\frac{\partial X}{\partial a_1} = \frac{x}{a_3 x + b_3 + 1} \quad \frac{\partial X}{\partial b_1} = \frac{y}{a_3 x + b_3 + 1} \quad \frac{\partial X}{\partial c_1} = \frac{1}{a_3 x + b_3 + 1}$$
$$\frac{\partial Y}{\partial a_2} = \frac{x}{a_3 x + b_3 + 1} \quad \frac{\partial Y}{\partial b_2} = \frac{y}{a_3 x + b_3 + 1} \quad \frac{\partial Y}{\partial c_2} = \frac{1}{a_3 x + b_3 + 1}$$
$$\frac{\partial X}{\partial a_3} = -\frac{a_1 x + b_1 y + c_1}{(a_3 x + b_3 + 1)^2} x \quad \frac{\partial X}{\partial b_3} = -\frac{a_1 x + b_1 y + c_1}{(a_3 x + b_3 + 1)^2} y$$
$$\frac{\partial Y}{\partial a_3} = -\frac{a_2 x + b_2 y + c_2}{(a_3 x + b_3 + 1)^2} x \quad \frac{\partial Y}{\partial b_3} = -\frac{a_2 x + b_2 y + c_2}{(a_3 x + b_3 + 1)^2} y$$

For each control point, a set of equations of the form in Equation (18.13) can be written. A redundant system of equations can be solved by least squares to yield the eight unknown parameters. With these values, the remaining points in the xy coordinate system are transformed into the XY system using Equation (18.12).

Example 18.3 Given the data in Table 18.3, determine the best-fit projective transformation parameters and use them to transform the remaining points into the XY coordinate system.

The approximate J and K matrices for the first iteration are

$$J = \begin{bmatrix}
58.445 & 58.445 & 0.649 & 0 & 0 & 0 & -83000.064 & -83000.064 \\
0 & 0 & 0 & 58.445 & 58.445 & 0.649 & -52334.927 & -52334.927 \\
39.064 & 31.251 & 0.781 & 0 & 0 & 0 & -35012.162 & -28009.729 \\
0 & 0 & 0 & 39.064 & 31.251 & 0.781 & -13719.422 & -10975.538 \\
-32.608 & 21.739 & 1.087 & 0 & 0 & 0 & -30791.646 & 20527.764 \\
0 & 0 & 0 & -32.608 & 21.739 & 1.087 & 20924.176 & -13949.451 \\
44.644 & -35.715 & 0.893 & 0 & 0 & 0 & -43186.235 & 34548.988 \\
0 & 0 & 0 & 44.644 & -35.715 & 0.893 & 61787.575 & -49430.060 \\
85.941 & -62.502 & 0.781 & 0 & 0 & 0 & -171318.968 & 124595.613 \\
0 & 0 & 0 & 85.941 & -62.502 & 0.781 & 203472.866 & -147980.266 \\
-131.572 & 105.258 & 1.316 & 0 & 0 & 0 & -445062.971 & 356050.376 \\
0 & 0 & 0 & -131.572 & 105.258 & 1.316 & 458870.304 & -367096.243
\end{bmatrix}$$

$$K = \begin{bmatrix}
0.158 \\ -0.088 \\ -0.388 \\ 0.195 \\ -0.636 \\ -0.250 \\ 0.729 \\ -0.120 \\ -0.198 \\ 0.090 \\ 0.362 \\ 0.175
\end{bmatrix}$$

Program ADJUST was used to solve this problem with the results shown in Figure 18.6. Its data file follows the same format as described in the solution of

TABLE 18.3 Data for Example 18.3

Point	X	Y	x	y	S_x	S_y
1	1420.407	895.362	90.0	90.0	0.3	0.3
2	895.887	351.398	50.0	40.0	0.3	0.3
3	−944.926	641.434	−30.0	20.0	0.3	0.3
4	968.084	−1384.138	50.0	−40.0	0.3	0.3
5	1993.262	−2367.511	110.0	−80.0	0.3	0.3
6	−3382.284	3487.762	−100.0	80.0	0.3	0.3
7			−60.0	20.0	0.3	0.3
8			−100.0	−100.0	0.3	0.3

```
Two-Dimensional Projective Coordinate Transformation of File
a1x + b1y + c1
--------------  = X + VX
a3x + b3y + 1

a2x + b2y + c2
--------------  = Y + VY
a3x + b3y + 1

Transformation Parameters:
--------------------------
a1 =     25.00274 ± 0.01538
b1 =      0.80064 ± 0.01996
c1 =     -134.715 ± 0.377
a2 =     -8.00771 ± 0.00954
b2 =     24.99811 ± 0.01350
c2 =     -149.815 ± 0.398
a3 =      0.00400 ± 0.00001
b3 =      0.00200 ± 0.00001

Adjustment's Reference Variance = 3.8888
Number of Iterations = 2
Transformed Control Points
     POINT         X            Y         VX        VY
-----------------------------------------------------------
         1    1,420.165      895.444   -0.242     0.082
         2      896.316      351.296    0.429    -0.102
         3     -944.323      641.710    0.603     0.276
         4      967.345   -1,384.079   -0.739     0.059
         5    1,993.461   -2,367.676    0.199    -0.165
         6   -3,382.534    3,487.612   -0.250    -0.150

Transformed Points
     POINT         X            Y         ±?x       ±?y
-----------------------------------------------------------
         1    1,420.165      895.444    0.511     0.549
         2      896.316      351.296    0.465     0.458
         3     -944.323      641.710    0.439     0.438
         4      967.345   -1,384.079    0.360     0.388
         5    1,993.461   -2,367.676    0.482     0.494
         6   -3,382.534    3,487.612    0.558     0.563
         7   -2,023.678    1,038.310    1.717     0.602
         8   -6,794.740   -4,626.976   51.225    34.647
```

FIGURE 18.6 Results from ADJUST for Example 18.3.

Example 18.1. The data file *Example 18-3.dat* that generated this listing can be found on the companion website for this book (http://www.wiley.com/college/ghilani). Furthermore, this adjustment is also performed in the spreadsheet file *Chapter 18.xls*, which is also available on the companion website. The Mathcad file *C18.xmcd* is available on the companion website for those wishing to program this problem in a higher-level computer language.

18.7 THREE-DIMENSIONAL CONFORMAL COORDINATE TRANSFORMATION



The three-dimensional conformal coordinate transformation is also known as the *seven-parameter similarity transformation*. It transfers points from one three-dimensional coordinate system to another. It is applied in the process of

reducing data from GNSS surveys and is also used extensively in photogrammetry and laser scanning. The three-dimensional conformal coordinate transformation involves seven parameters, three rotations, three translations, and one scale factor. The rotation matrix is developed from three consecutive two-dimensional rotations about the *x*-, *y*-, and *z*-axes, respectively. Given in sequence, these are as follows.

In Figure 18.7, the rotation θ_1 about the *x*-axis expressed in matrix form is

$$X_1 = R_1 X_0 \tag{18.14}$$

where

$$X_1 = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} \quad R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & \sin\theta_1 \\ 0 & -\sin\theta_1 & \cos\theta_1 \end{bmatrix} \quad X_0 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

In Figure 18.8, the rotation θ_2 about the *y*-axis expressed in matrix form is

$$X_2 = R_2 X_1 \tag{18.15}$$

where

$$X_2 = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} \quad \text{and} \quad R_2 = \begin{bmatrix} \cos\theta_2 & 0 & -\sin\theta_2 \\ 0 & 1 & 0 \\ \sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix}$$

In Figure 18.9, the rotation θ_3 about the *z*-axis expressed in matrix form is

$$X = R_3 X_2 \tag{18.16}$$

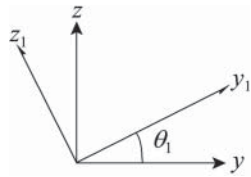


FIGURE 18.7 θ_1 rotation.

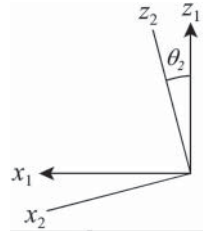


FIGURE 18.8 θ_2 rotation.

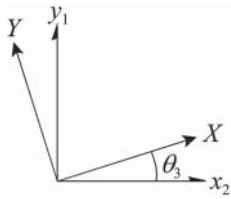


FIGURE 18.9 θ_3 rotation.

where

$$X = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \quad \text{and} \quad R_3 = \begin{bmatrix} \cos\theta_3 & \sin\theta_3 & 0 \\ -\sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Substituting (*a*) into (*b*) and in turn into (*c*) yields

$$X = R_3 R_2 R_1 X_0 = R X_0 \tag{18.17}$$

The three matrices R_3, R_2, and R_1 in (*d*), when multiplied together develop a single rotation matrix R for the transformation whose individual elements are

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \tag{18.18}$$

where

$$r_{11} = \cos\theta_2 \cos\theta_3$$

$$r_{12} = \sin\theta_1 \sin\theta_2 \cos\theta_3 + \cos\theta_1 \sin\theta_3$$

$$r_{13} = -\cos\theta_1 \sin\theta_2 \cos\theta_3 + \sin\theta_1 \sin\theta_3$$

$$r_{21} = -\cos\theta_2 \sin\theta_3$$

$$r_{22} = -\sin\theta_1 \sin\theta_2 \sin\theta_3 + \cos\theta_1 \cos\theta_3$$

$$r_{23} = \cos\theta_1 \sin\theta_2 \sin\theta_3 + \sin\theta_1 \cos\theta_3$$

$$r_{31} = \sin\theta_2$$

$$r_{32} = -\sin\theta_1 \cos\theta_2$$

$$r_{33} = \cos\theta_1 \cos\theta_2$$

Since the rotation matrix is orthogonal, it has the property that its inverse is equal to its transpose. Using this property and multiplying the terms of the

matrix X by a scale factor, S, and adding translations factors T_x, T_y, and T_z to translate to a common origin yields the following mathematical model for the transformation

$$\begin{aligned} X &= S(r_{11}x + r_{21}y + r_{31}z) + Tx \\ Y &= S(r_{12}x + r_{22}y + r_{32}z) + Ty \\ Y &= S(r_{13}x + r_{23}y + r_{33}z) + Tz \end{aligned} \tag{18.19}$$

Equations (18.15) involve seven unknowns (S, θ_1, θ_2, θ_3, Tx, Ty, and Tz). For a unique solution, seven equations must be written. This requires a minimum of two control stations with known XY coordinates and xy coordinates, plus three stations with known Z and z coordinates. If there is more than the minimum number of control points, a least-squares solution can be used. Equations (18.15) are nonlinear in their unknowns and thus must be linearized for a solution. The following linearized observation equations can be written for each point as

$$\begin{bmatrix} \left(\frac{\partial X}{\partial S}\right)_0 & 0 & \left(\frac{\partial X}{\partial \theta_2}\right)_0 & \left(\frac{\partial X}{\partial \theta_3}\right)_0 & 1 & 0 & 0 \\ \left(\frac{\partial Y}{\partial S}\right)_0 & \left(\frac{\partial Y}{\partial \theta_1}\right)_0 & \left(\frac{\partial Y}{\partial \theta_2}\right)_0 & \left(\frac{\partial Y}{\partial \theta_3}\right)_0 & 0 & 1 & 0 \\ \left(\frac{\partial Z}{\partial S}\right)_0 & \left(\frac{\partial Z}{\partial \theta_1}\right)_0 & \left(\frac{\partial Z}{\partial \theta_2}\right)_0 & \left(\frac{\partial Z}{\partial \theta_3}\right)_0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} dS \\ d\theta_1 \\ d\theta_2 \\ d\theta_3 \\ dTx \\ dTy \\ dTz \end{bmatrix} = \begin{bmatrix} X - X_0 \\ Y - Y_0 \\ Z - Z_0 \end{bmatrix} \tag{18.20}$$

where

$$\frac{\partial X}{\partial S} = r_{11}x + r_{21}y + r_{31}z \quad \frac{\partial Y}{\partial S} = r_{12}x + r_{22}y + r_{32}z \quad \frac{\partial Z}{\partial S} = r_{13}x + r_{23}y + r_{33}z$$

$$\frac{\partial Y}{\partial \theta_1} = -S[r_{13}x + r_{23}y + r_{33}z] \quad \frac{\partial Z}{\partial \theta_1} = S[r_{12}x + r_{22}y + r_{32}z]$$

$$\frac{\partial X}{\partial \theta_2} = S(-x \sin\theta_2 \cos\theta_3 + y \sin\theta_2 \sin\theta_3 + z \cos\theta_2)$$

$$\frac{\partial Y}{\partial \theta_2} = S(x \sin\theta_1 \cos\theta_2 \cos\theta_3 - y \sin\theta_1 \cos\theta_2 \sin\theta_3 + z \sin\theta_1 \sin\theta_2)$$

$$\frac{\partial Z}{\partial \theta_2} = S(-x \cos\theta_1 \cos\theta_2 \cos\theta_3 + y \cos\theta_1 \cos\theta_2 \sin\theta_3 - z \cos\theta_1 \sin\theta_2)$$

$$\frac{\partial X}{\partial \theta_3} = S(r_{21}x - r_{11}y) \quad \frac{\partial Y}{\partial \theta_3} = S(r_{22}x - r_{12}y) \quad \frac{\partial Z}{\partial \theta_3} = S(r_{23}x - r_{13}y)$$

Example 18.4 The three-dimensional *xyz* coordinates were observed for six points. Four of these points (1, 2, 3, and 4) were control points whose coordinates were also known in the *XYZ* control system. The data are shown in Table 18.4. Compute the parameters of a three-dimensional conformal coordinate transformation and use them to transform points 5 and 6 in the *XYZ* system.

SOLUTION The first iteration approximate values for the J and K matrices are

$$J = \begin{bmatrix}
0 & 104.471 & 1286.313 & 1 & 0 & 0 & -207.156 \\
-104.471 & 0 & 197.065 & 0 & 1 & 0 & -1352.181 \\
-1286.313 & 197.065 & 0 & 0 & 0 & 1 & 109.821 \\
0 & 111.952 & 1421.257 & 1 & 0 & 0 & 759.998 \\
-111.952 & 0 & 722.976 & 0 & 1 & 0 & -1494.036 \\
-1421.257 & -722.976 & 0 & 0 & 0 & 1 & 117.685 \\
0 & 144.013 & 1706.244 & 1 & 0 & 0 & -1531.494 \\
-144.013 & 0 & -1456.891 & 0 & 1 & 0 & -1793.616 \\
-1706.244 & 1456.891 & 0 & 0 & 0 & 1 & 151.387 \\
0 & 204.775 & 1845.520 & 1 & 0 & 0 & -51.274 \\
-204.775 & 0 & -48.777 & 0 & 1 & 0 & -1940.024 \\
-1845.520 & 48.777 & 0 & 0 & 0 & 1 & 215.261
\end{bmatrix}$$

$$K = \begin{bmatrix}
244.782 \\
0.879 \\
11.626 \\
243.611 \\
1.904 \\
15.106 \\
246.878 \\
-0.989 \\
-17.335 \\
244.484 \\
-1.794 \\
-9.397
\end{bmatrix}$$

TABLE 18.4 Data for a Three-Dimensional Conformal Coordinate Transformation

Point	X	Y	Z	$x \pm S_x$	$y \pm S_y$	$z \pm S_z$
1	10,037.81	5262.09	772.04	1094.883 ± 0.007	820.085 ± 0.008	109.821 ± 0.005
2	10,956.68	5128.17	783.00	503.891 ± 0.011	1598.698 ± 0.008	117.685 ± 0.009
3	8780.08	4840.29	782.62	2349.343 ± 0.006	207.658 ± 0.005	151.387 ± 0.007
4	10,185.80	4700.21	851.32	1395.320 ± 0.005	1348.853 ± 0.008	215.261 ± 0.009
5				265.346 ± 0.005	1003.470 ± 0.007	78.609 ± 0.003
6				784.081 ± 0.006	512.683 ± 0.008	139.551 ± 0.008

The observation equations are solved using Equation (11.37). The results from the program ADJUST are shown in Figure 18.10. The data file that generated this listing is shown in Figure 18.11. It can be found on the companion website for this book with the name *Example 18-4.dat*. Similar to the other files described in this chapter, the first line of the file is a description of its contents. This is followed by a line containing the number of horizontal and vertical control points. Since horizontal and vertical control points can be different, the horizontal control points are listed immediately following the second line. Similar to the other files described in this chapter, these lines contain the point identifier, x, and y coordinates. Their coordinates may also be followed by their standard deviations, but standard deviations for these points are optional since they are not used in this adjustment; however, they will be used in the method of general least squares described in Chapter 22. The horizontal control points are followed by the vertical control points, which are then followed by the common points in the arbitrary

Three-Dimensional Coordinate Transformation

MEASURED POINTS

NAME	X	Y	Z	Sx	Sy	Sz
1	1094.883	820.085	109.821	0.007	0.008	0.005
2	503.891	1598.698	117.685	0.011	0.008	0.009
3	2349.343	207.658	151.387	0.006	0.005	0.007
4	1395.320	1348.853	215.261	0.005	0.008	0.009

CONTROL POINTS

NAME	X	Y	Z
1	10037.810	5262.090	772.040
2	10956.680	5128.170	783.000
3	8780.080	4840.290	782.620
4	10185.800	4700.210	851.320

Transformation Coefficients

Scale = 0.9499595773 +/- 0.0000367175
x-rot = 2°17'05.3" +/- 0°00'30.1"
y-rot = -0°33'02.8" +/- 0°00'09.7"
z-rot = 224°32'10.9" +/- 0°00'06.9"
Tx = 10233.858 +/- 0.0649
Ty = 6549.981 +/- 0.0709
Tz = 720.897 +/- 0.2128

Standard Deviation of Unit Weight >> 8.663
Degrees of Freedom: 5
Iterations: 4

CONTROL COORDINATES

NAME	X	Vx	Y	Vy	Z	Vz
1	10037.810	0.064	5262.090	0.037	772.040	0.001
2	10956.680	0.025	5128.170	-0.057	783.000	0.011
3	8780.080	-0.007	4840.290	-0.028	782.620	0.007
4	10185.800	-0.033	4700.210	0.091	851.320	-0.024

TRANSFORM COORDINATES

NAME	X	Y	Z	Sx	Sy	Sz
1	10037.874	5262.127	772.041	0.032	0.034	0.040
2	10956.705	5128.113	783.011	0.053	0.052	0.056
3	8780.073	4840.262	782.627	0.049	0.041	0.057
4	10185.767	4700.301	851.296	0.032	0.037	0.067
5	10722.020	5691.221	766.068	0.053	0.053	0.088
6	10043.246	5675.898	816.867	0.040	0.042	0.092

FIGURE 18.10 Results from ADJUST for Example 18.4.

```
Example 18.4
4 4
1 10037.81 5262.09
2 10956.68 5128.17
3 8780.08 4840.29
4 10185.80 4700.21
1 772.04
2 783.00
3 782.52
4 851.32
1 1094.883 820.085 109.821 0.007 0.008 0.005
2 503.891 1598.698 117.685 0.011 0.008 0.009
3 2349.343 207.658 151.387 0.006 0.005 0.007
4 1395.320 1348.853 215.261 0.005 0.008 0.009
5 265.346 1003.470 78.609 0.005 0.007 0.003
6 784.081 512.683 139.551 0.006 0.008 0.008
```

FIGURE 18.11 ADJUST data file for Example 18.4.

coordinate system. The points in the arbitrary coordinate system have the format of point identifiers, x, y, z, S_x, S_y, and S_z, where the standard deviations are optional. Furthermore, the Mathcad file *C18.xmcd* is available on the companion website for those wishing to program this problem in a higher-level computer language.

Note that in this adjustment, with four control points available having X, Y, and Z coordinates, 12 equations could be written, three for each point. With seven unknown parameters, this gave $12 - 7 = 5$ degrees of freedom in the solution.

18.8 STATISTICALLY VALID PARAMETERS

Beside the coordinate transformations described in preceding sections, it is possible to develop numerous others. For example, polynomial equations of various degrees could be used to transform data. As additional terms are added to a polynomial, the resulting equation will force better fits on any given set of data. However, caution should be exercised when doing this since the resulting transformation parameters may not be statistically significant.

It is important to use the coordinate transformation that matches the physical conditions of the problem. For example, if a survey is performed in arbitrary two-dimensional coordinate system and then transformed into some map projection coordinate system, it is unlikely that scales would differ in the x- and y-axes. That is, if the two measurement systems differ in units of feet and meters, then the scale factor should be close to the unit conversion between the two units' systems. It would be highly unlikely that the scale factor in the x-axis differs from that in the y-axis. However, when doing a photogrammetric interior orientation with a traditional film camera, it is likely that the scales in the two axes will differ, and thus a two-dimensional affine coordinate transformation would be appropriate. However, this same situation may not be appropriate for a digital

camera image. In this case, a projective transformation may be more appropriate. Thus, when deciding on which two-dimensional coordinate transformation to use, first consider the physical conditions that exist.

Additionally, when using a two-dimensional conformal coordinate transformation with a data set having four control points, nonzero residuals would be expected. However, if a projective transformation were used, this data set would yield a unique solution, and thus the residuals would be zero. Is the projective a more appropriate transformation for this data set? Is this truly a better fit? Guidance in the answers to these questions can be obtained by checking the statistical validity of the parameters.

The adjusted parameters divided by their standard deviations represent a t statistic with υ degrees of freedom. If a parameter is to be judged as statistically different from zero, and thus significant, the computed t value (the test statistic) must be greater than $t_{\alpha/2,\upsilon}$. Simply stated, the test statistic is

$$t = \frac{|\text{Parameter}|}{S} \tag{18.21}$$

For example, in the adjustment in Example 18.2, the following computed *t values* are found

Parameter	S	t-value
$a =$ 25.37152	±0.02532	1002
$b =$ 0.82220	±0.02256	36.4
$c = -137.183$	±0.203	675.8
$d =$ −0.80994	±0.02335	34.7
$e =$ 25.40166	±0.02622	968.8
$f = -150.723$	±0.216	697.8

In this problem, there were eight equations involving six unknowns and thus two degrees of freedom. From the t distribution table (Table D.3), $t_{0.025,2} = 4.303$. Because all computed t values are greater than 4.303, each parameter is significantly different from zero at a 95% level of confidence. From the adjustment results of Example 18.3, the computed t values are listed below.

Parameters	Value	S	t Value
a_1	25.00274	0.01538	1626
b_1	0.80064	0.01896	42.3
c_1	−134.715	0.377	357.3
a_2	−8.00771	0.00954	839.4
b_2	24.99811	0.01350	1851.7
c_2	−149.815	0.398	376.4
a_3	0.00400	0.00001	400
b_3	0.00200	0.00002	100

This adjustment has 8 unknown parameters and 12 observations. From the t-distribution table (Table D.3), $t_{0.025,4} = 2.776$. By comparing the tabular t value against each computed value, all parameters are significantly different from zero at a 95% confidence level. This is true for a_3 and b_3 even though they seem relatively small at 0.004 and 0.002, respectively.

Using this statistical technique, a check can be made to determine whether the projective transformation is appropriate for a particular set of data since it will default to an affine transformation when a_3 and b_3 are both statistically equal to zero. Similarly, if the confidence intervals for the means of a and b at a selected probability level of the two-dimensional conformal coordinate transformation contain two of the parameters from the affine transformation, then the computed values of the affine transformation are statistically equal to those from the conformal transformation. Thus, if the interval for the population mean of a from the conformal transformation contains both a and e parameters from the affine transformation, then there is no statistical difference between these parameters. This must also be true for the population mean of b from the conformal transformation when compared to the absolute values of b and d from the affine transformation. Note that a negative sign is part of the mathematical model for the conformal coordinate transformation, and thus b and d parameters are opposite in signs typically. If both of these conditions exist, then conformal transformation is the more appropriate adjustment to use for the given data since it involves fewer parameters and provides more redundant observations. *One should always use the minimum number of unknown parameters to solve any problem*.

A statistical test can be performed to check the parameters individually against their respective conformal-transformation parameters. This test assumes that the sample data come from populations having the same variance. Since, in this example, the same data are being used, this assumption is valid. A two-tailed test should be used. The test setup is

$$\mathrm{H}_0\colon (\mu_1 - \mu_2) = 0$$

$$\mathrm{H}_a\colon (\mu_1 - \mu_2) \neq 0$$

$$\text{Test statistic: } t = \frac{\bar{y}_1 - \bar{y}_2}{\sqrt{S^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}}, \text{ where } S^2 = \frac{\upsilon_1 S_1^2 + \upsilon_2 S_2^2}{\upsilon_1 + \upsilon_2}$$

$$\text{Rejection region: } |t| > t_{\alpha/2,\upsilon}, \text{ where } \upsilon = \upsilon_1 + \upsilon_2$$

Example 18.5 A two-dimensional conformal coordinate transformation was run on the data from Example 18.2. Compare the parameters from the two-dimensional conformal coordinate transformation against their equivalent parameters from the two-dimensional affine coordinate transformation to determine the most appropriate transformation for this set of data.

SOLUTION The results of the two-dimensional conformal coordinate transformation are presented here:

Parameter	Value	S
A	25.36443	0.01435
B	−0.81531	0.1892
Tx	−137.185	0.180
Ty	−150.477	0.165

These results are compared with the results of the affine coordinate transformation, which was listed earlier in this section. If any of the parameters from the affine transformation are statistically different from the conformal transformation, then the affine transformation is the appropriate mathematical model for the data. However, if none of the parameters are different, then the conformal coordinate transformation mathematical model is the more appropriate mathematical model for the set of data.

Method 1: The 95% confidence interval is developed for the population mean based on the values of a and b from the conformal coordinate transformation. In this approach, if the equivalent pair of parameters from the affine coordinate transformation is contained inside the range for the population mean, then the conformal coordinate transformation is the appropriate mathematical model. If any of the parameters from the affine transformation fall outside their respective ranges, then the affine transformation should be used. Using Equation (4.7), at 95% level, the population ranges are

$$\text{For } a\text{: } 25.36443 \pm t_{0.025,4}\frac{S}{\sqrt{8}} = (25.3503, 25.3785)$$

$$\text{For } b\text{: } -0.81531 \pm t_{0.025,4}\frac{0.01892}{\sqrt{8}} = (-0.83388, -0.79674)$$

When comparing the affine transformation parameters of a and e, we see that e with a value of 25.40166 is outside the 95% confidence interval for the population mean of a from the conformal coordinate transformation. Thus, since one parameter is outside the interval, the affine transformation is the most appropriate mathematical model for the data. This is true even though b and d from the affine transformation are inside their respective range. It should be remembered that the mathematical model for the conformal coordinate transformation incorporates the sign differences in these parameters.

Method 2: In this method, each of the four parameters from the affine transformation must be tested for a statistical difference from the mean of the respective conformal coordinate parameters. It should be noted that the conformal coordinate transformation for this problem has four redundant observations (8 − 4), the affine

coordinate transformation only has two redundant observations (8 – 6) since both adjustments use the same eight observations. To check *e* from the affine transformation against *a* from the conformal coordinate transformation, the following test is constructed.

H_0: $(\mu_1 - \mu_2) = 0$

H_a: $(\mu_1 - \mu_2) \neq 0$

Test statistic: $S^2 = \dfrac{4(0.01435^2) + 2(0.02622^2)}{4 + 2} = 0.00036644$

$$t = \frac{|25.36443 - 25.40166|}{\sqrt{0.00036644\left(\frac{1}{8} + \frac{1}{8}\right)}} = 3.89$$

Rejection region: $3.89 > 2.45 = t_{0.025,6}$ (true)

Since the rejection region is true, we can reject the hypothesis that the two means are the same. Thus, the affine coordinate transformation is the most appropriate mathematical model for this set of data. This test should be performed on the other three parameters. However, there is no further need to do this since *e* has already determined that the affine transformation is appropriate for this set of data.

PROBLEMS

Note: For problems requiring least-squares adjustment, if a computer program is not distinctly specified for use in the problem, it is expected that the least-squares algorithm will be solved using the program MATRIX, which is available on the book's companion website. Partial answers to problems marked with an asterisk are given in Appendix G.

18.1 Points *A*, *B*, *C*, and *D* have their coordinates known in both an *EN* system and an *XY* system. Points *E*, *F*, and *G* have their coordinates known only in the *XY* system. These coordinates are shown in the table below. Using a two-dimensional conformal coordinate transformation, determine the

***(a)** Transformation parameters and their standard deviations.

(b) Most probable coordinates and their standard deviations for *E*, *F*, and *G* in the *EN* coordinate system.

***(c)** Rotation angle and scale factor.

Station	Easting (m)	Northing (m)	X (ft)	Y (ft)	S_x	S_y
A	682,200.766	324,762.383	1221.41	1032.09	0.016	0.015
B	692,248.627	329,498.579	4607.18	1046.11	0.015	0.015
C	688,403.585	334,584.608	4200.14	2946.41	0.017	0.017
D	689,997.153	329,419.473	3974.99	1314.30	0.015	0.018
E			3585.74	2114.27	0.016	0.017
F			2767.87	1621.42	0.015	0.015
G			2596.42	2692.89	0.016	0.015

18.2 Points A, B, C, and D have their coordinates known in both a (X, Y) coordinate camera fiducial system and an arbitrary (x, y) system from a digitizing tablet. Points E, F, and G have their coordinates known only in the xy system. These coordinates are shown in the following table. Using a two-dimensional conformal coordinate transformation to determine the

(a) Transformation parameters and their standard deviations.

(b) Most probable coordinates and their standard deviations for E, F, and G in the XY coordinate system.

(c) Rotation angle and scale factor.

Point	X (mm)	Y (mm)	x (in.)	y (in.)	S_x	S_y
A	−112.126	−0.097	0.386	5.000	0.005	0.005
B	0.066	0.021	4.800	4.850	0.005	0.004
C	112.136	0.090	9.210	4.699	0.006	0.005
D	−0.006	−112.089	4.644	0.441	0.004	0.005
E			1.567	2.827	0.006	0.005
F			7.213	5.692	0.006	0.005
G			6.068	2.089	0.005	0.005

18.3 Repeat Problem 18.1, parts *a* and *b*, using a two-dimensional affine coordinate transformation. Determine if all the parameters are statistically significant at a 5% level of significance and use Method 1 to determine if the affine transformation is significantly different from the conformal transformation used in Problem 18.1 at a 0.05 level of significance.

18.4 Repeat Problem 18.3 with the data from Problem 18.2 but use Method 2 to determine if the transformations are statistically different.

18.5 Using a two-dimensional projective coordinate transformation and the data listed below, determine the

***(a)** Transformation parameters.

(b) Most probable coordinates for 9 and 10 in the XY coordinate system.

Point	Observed		Control	
	x	y	X	Y
1	177.562 ± 0.060	-39.647 ± 0.048	-106.005	-105.902
2	138.835 ± 0.040	35.597 ± 0.040	105.995	106.154
3	-69.851 ± 0.040	-531.386 ± 0.040	-105.965	105.940
4	44.499 ± 0.020	-108.903 ± 0.020	105.696	-105.990
5	90.823 ± 0.042	-201.408 ± 0.040	-112.005	-0.026
6	97.382 ± 0.044	-29.917 ± 0.040	111.940	-0.110
7	217.344 ± 0.040	242.207 ± 0.040	0.065	112.090
8	-198.417 ± 0.040	-254.362 ± 0.036	-0.006	-111.993
9	168.454 ± 0.042	132.710 ± 0.036		
10	225.533 ± 0.040	-347.645 ± 0.048		

18.6 Do Problem 18.5 using a two-dimensional affine coordinate transformation. Determine the

(a) Transformation parameters.

(b) Most probable coordinates for 9 and 10 in the XY coordinate system.

18.7 Do Problem 18.5 using a two-dimensional conformal coordinate transformation. Determine the

***(a)** Transformation parameters.

(b) Most probable coordinates for 9 and 10 in the XY coordinate system.

18.8 Determine the appropriate two-dimensional transformation for Problems 18.5 through 18.7 at a 0.01 level of significance.

18.9 Using a two-dimensional affine coordinate transformation and the following data, determine the

(a) Transformation parameters and their standard deviations.

(b) Most probable XY coordinates and their standard deviations for points 9 and 10.

Station	Control Points		Observed Points			
	X	Y	x	y	S_x	S_y
1	-106.008	-105.997	$-24{,}868.061$	7,295.026	0.003	0.003
2	105.993	106.006	25,085.596	$-7{,}079.976$	0.003	0.003
3	-106.009	106.012	10,711.269	$-7{,}080.038$	0.003	0.003
4	105.998	-105.990	$-10{,}492.926$	7,294.817	0.003	0.003
5	-112.004	0.013	$-7{,}484.631$	107.298	0.003	0.003
6	111.991	0.003	7,703.107	107.348	0.003	0.003
7	-0.004	112.003	18,905.503	$-7{,}486.847$	0.003	0.003
8	-0.005	-111.994	$-18{,}687.631$	7,701.763	0.003	0.003
9			12,181.515	$-5{,}725.191$	0.003	0.003
10			10,870.476	$-2{,}653.425$	0.003	0.003

18.10 Using a two-dimensional projective coordinate transformation, do Problem 18.9.

18.11 For the data of Problem 18.9, which two-dimensional transformation is most appropriate and why? Use a 0.01 level of significance.

18.12 A survey is connected to a map projection through four stations yielding the following data.

(a) Using Method 2, determine whether the two-dimensional transformation should be conformal or affine at a 0.01 level of significance.

(b) Using the appropriate transformation, determine the map projection coordinates for stations 5, 6, and 7.

Station	E (m)	N (m)	X (ft)	Y (ft)	S_X (ft)	S_Y (ft)
1	295,102.571	450,794.430	913.98	55.37	0.043	0.039
2	296,801.650	454,383.894	2124.48	48.38	0.041	0.044
3	291,555.574	452,353.594	887.83	1236.11	0.037	0.039
4	294,846.148	450,416.597	776.55	77.57	0.037	0.040
5			1675.67	834.88	0.039	0.038
6			1675.31	834.09	0.051	0.038
7			1216.89	913.66	0.043	0.041

18.13 A survey is connected to a map projection through four stations yielding the following data.

(a) Using Method 2, determine whether the two-dimensional transformation should be conformal or affine at a 0.01 level of significance.

(b) Using the appropriate transformation, determine the map projection coordinates for stations 5, 6, and 7.

Station	E (m)	N (m)	x (mm)	y (mm)	S_X	S_Y
1	338,691.180	441,572.850	89.746	91.003	0.021	0.013
2	440,608.658	422,353.428	39.974	0.012	0.012	0.009
3	459,245.191	364,922.702	−20.405	0.034	0.015	0.015
4	475,632.492	444,440.889	50.158	−40.123	0.016	0.011
5			109.617	−80.322	0.012	0.016
6			−100.971	79.814	0.015	0.016
7			80.356	23.845	0.014	0.011

18.14 Using a level of significance of 0.01, which two-dimensional adjustment is appropriate for the following set of data?

Station	X	Y	x	y	S_x	S_y
1	112.630	110.274	1.481	1.519	0.0047	0.0051
2	123.381	108.377	10.550	5.932	0.0054	0.0052
3	118.499	118.087	1.642	10.554	0.0048	0.0044
4	122.273	115.233	6.006	10.468	0.0055	0.0053
5	126.232	112.253	10.581	10.386	0.0049	0.0054
6	123.298	108.396	10.483	5.915	0.0052	0.0038
7	120.448	104.641	10.396	1.562	0.0051	0.0052

18.15 Repeat Problem 18.14 with the following set of data.

Point	X	Y	x	y	S_x	S_y
1	215.83	62.48	88.998	92.996	0.008	0.007
2	127.71	32.27	56.011	41.997	0.010	0.009
3	9.62	97.75	−30.991	28.005	0.008	0.008
4	43.92	-69.54	53.995	−43.997	0.008	0.007
5	64.60	−187.12	112.014	−96.009	0.007	0.008
6	−21.61	239.23	−103.999	87.992	0.007	0.008
7	−168.60	−0.11	-83.004	−93.987	0.008	0.010

18.16 A survey was performed with GNSS in survey feet units using SPCS 1983 coordinates in 2020. Four points were relocated with GNSS in international feet units with SPCS 2022 coordinates. Using a weighted three-dimensional conformal coordinate transformation, determine the transformation parameters, their standard deviations, and the SPCS 2022 coordinates for points E and F.

Control Points

Station	E (sft)	N (sft)	H (sft)
A	6,343,127.40	2,180,089.14	354.04
B	6,344,248.34	2,180,351.49	330.10
C	6,344,356.61	2,179,315.05	327.99
D	6,343,368.49	2,179,406.64	325.87

Measured Points

Station	E (ft)	N (ft)	H (ft)	S_x	S_y	S_z
A	156,438.45	2,026,366.31	249.84	0.008	0.007	0.01
B	157,559.49	2,026,628.73	225.96	0.003	0.017	0.008
C	157,667.81	2,025,592.19	223.83	0.005	0.008	0.015
D	156,679.59	2,025,683.75	221.67	0.013	0.016	0.006
E	156,284.86	2,025,960.90	221.91			
F	157,028.50	2,026,214.09	221.60			

18.17 Repeat Problem 18.16 using the following data.

Control Stations

Station	E (ft)	N (ft)	h (ft)
A	300,502.09	1,256,749.61	1153.02
B	302,480.35	1,256,808.07	1184.00
C	304,007.91	1,255,585.22	1178.98
D	301,438.29	1,257,289.14	1192.02

Measured Stations

Station	E (ft)	N (ft)	H (ft)	S_E	S_N	S_H
A	2,444,186.39	420,407.99	1256.85	0.030	0.028	0.037
B	2,446,142.95	420,511.97	1287.72	0.029	0.027	0.024
C	2,447,683.30	419,337.30	1283.04	0.019	0.030	0.030
D	2,445,100.38	420,963.81	1295.02	0.034	0.020	0.035
E	2,445,461.93	420,214.87	1298.67			
F	2,444,664.40	419,750.44	1301.80			

18.18 Repeat Problem 18.17 with the following data but also list all the transformed coordinates.

Control Stations

Station	E (m)	N (m)	h (m)
1	648,310.393	452,394.453	170.062
2			131.926
3	645,324.766	452,557.056	
4	647,117.161	453,846.222	161.601
5			167.270
6	645,324.746	452,556.980	

Measured Stations

Station	x (ft)	y (ft)	H (ft)	S_x (ft)	S_y (ft)	S_H (ft)
1	6,019.68	5,163.17	210.33	±0.033	±0.024	±0.033
2	5,901.12	11,192.94	85.11	±0.029	±0.037	±0.030
3	12,591.53	12,446.33	120.16	±0.033	±0.031	±0.030
4	12,184.77	5,214.17	182.42	±0.031	±0.026	±0.029
5	5,901.06	11,192.99	201.02	±0.030	±0.034	±0.032
6	12,591.29	12,446.47	153.12	±0.020	±0.030	±0.029
7	12,184.99	5,214.33	95.35	±0.027	±0.027	±0.030
8	9,728.88	7,873.23	132.99	±0.027	±0.030	±0.029

Use the program ADJUST to solve each problem.

18.19 Problem 18.4.

18.20 Problem 18.14.

18.21 Problem 18.17.

PROGRAMMING PROBLEMS

Develop a computational program that calculates the coefficient and constants matrix for each transformation.

18.22 A two-dimensional conformal coordinate transformation.

18.23 A two-dimensional affine coordinate transformation.

18.24 A two-dimensional projective coordinate transformation.

18.25 A three-dimensional conformal coordinate transformation.

CHAPTER 19

ERROR ELLIPSE

19.1 INTRODUCTION



As discussed previously, after completing a least-squares adjustment, the estimated standard deviations in the coordinates of an adjusted station can be calculated from covariance matrix elements. These standard deviations provide error estimates in the reference axes directions. In graphical representation, they are half the dimensions of a *standard error rectangle* centered on each station. The standard error rectangle has dimensions of $2S_x$ by $2S_y$ as illustrated for station B in Figure 19.1, but this is not a true representation of the error present at the station.

Simple deductive reasoning can be used to show the basic problem. Assume in Figure 19.1 that the XY coordinates of station A have been computed from the observations of distance AB and azimuth Az_{AB}, which is approximately 30°. Further assume that the observed azimuth has no error at all, but that the distance has a large error, say ±2 ft. From Figure 19.1, it should then be readily apparent that the largest uncertainty in the station's position would not lie in either cardinal direction. That is, neither S_x nor S_y represents the largest positional uncertainty for the station. Rather, the largest uncertainty would be collinear with line AB and approximately equal to the estimated error in the distance. In fact, this is what happens.

In the usual case, the position of a station is uncertain in both direction and distance, and the estimated error of the adjusted station involves the errors of two jointly distributed variables, the x and y coordinates. Thus, the positional error at a station follows a *bivariate normal distribution*. The general shape

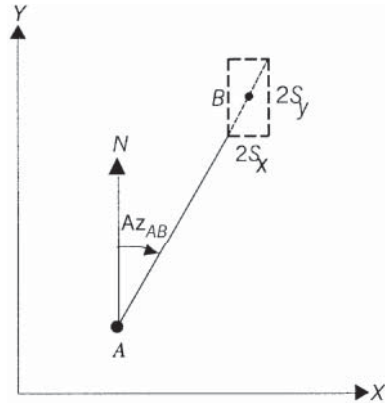


FIGURE 19.1 Standard error rectangle.

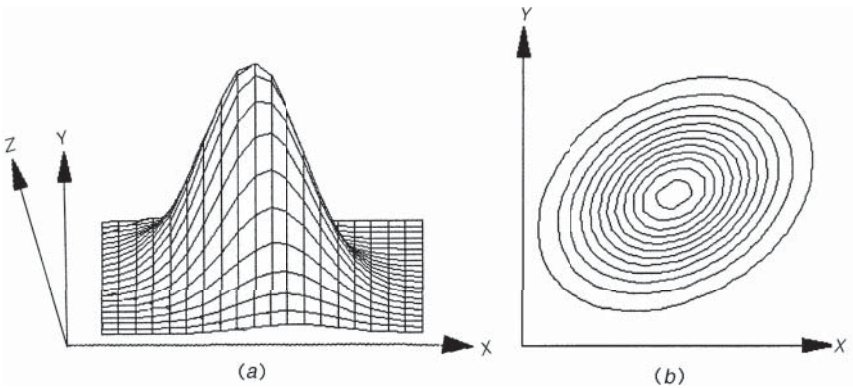


FIGURE 19.2 (*a*) Three-dimensional view and (*b*) contour plot of a bivariate distribution.

of this distribution for a station is shown in Figure 19.2. In this figure, the three-dimensional *surface plot* [Figure 19.2(*a*)] of a bivariate normal distribution curve is shown along with its *contour plot* [Figure 19.2(*b*)]. Note that the ellipses shown in the *xy* plane of Figure 19.2(*b*) can be obtained by passing planes of varying probability levels through Figure 19.2(*a*) parallel to the *xy* plane. The volume of the region inside the intersection of any plane with the surface of Figure 19.2(*a*) represents the probability level of the ellipse. The orthogonal projection of the surface plot of Figure 19.2(*a*) onto the *xz* plane would give the normal distribution curve of the *x* coordinate, from which S_x is obtained. Likewise, its orthogonal projection onto the *yz* plane would give the normal distribution in the *y* coordinate from which S_y is obtained.

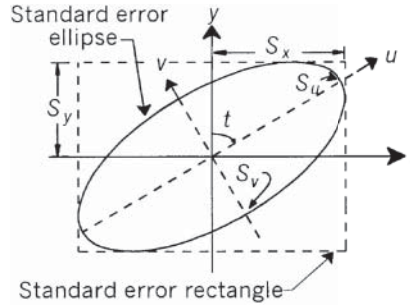


FIGURE 19.3 Standard error ellipse.

To fully describe the estimated error of a station, it is only necessary to show the orientation and lengths of the semiaxes of the error ellipse. A detailed diagram of an error ellipse is shown in Figure 19.3. In this figure, the *standard error ellipse* of a station is shown (i.e., one whose arcs are tangent to the sides of the standard error rectangle). The orientation of the ellipse depends on the t angle, which fixes the directions of the auxiliary, orthogonal uv axes along which the ellipse axes lie. The u-axis defines the weakest direction in which the station's adjusted position is known. In other words, it lies in the direction of maximum estimated error in the station's coordinates. The v-axis is orthogonal to u and defines the strongest direction in which the station's position is known or the direction of minimum error. For any station, the value of t that orients the ellipse to provide these maximum and minimum values can be determined after the adjustment from the elements of the covariance matrix.

The exact probability level of the standard error ellipse is dependent on the number of degrees of freedom in the adjustment. This standard error ellipse can be modified in dimensions using F critical values to represent an error probability at any selected percentage. It will be shown later that the percent probability within the boundary of the standard error ellipse for a simple closed traverse is only 35%. Surveyors often use the 95% probability since this affords a higher level of confidence.

19.2 COMPUTATION OF ELLIPSE ORIENTATION AND SEMIAXES

As shown in Figure 19.4, the method for calculating the orientation angle t that yields maximum and minimum semiaxes involves a two-dimensional coordinate rotation. Notice that the t angle is defined as a clockwise angle from the y-axis to the u-axis. To propagate the errors in a point I from the xy system into an orthogonal uv system, the generalized law of the propagation of variances, discussed in Chapter 6, is used. The specific value for t that yields the maximum error along the u-axis must be determined. The following steps accomplish this task.

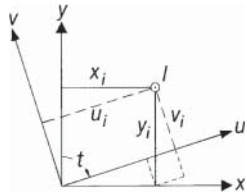


FIGURE 19.4 Two-dimensional rotation.

Step 1: Any point I in the uv system can be represented with respect to its XY coordinates as

$$\begin{aligned} u_i &= x_i \sin t \quad + y_i \cos t \\ v_i &= -x_i \cos t + y_i \sin t \end{aligned} \tag{19.1}$$

Rewriting Equations (19.1) in matrix form yields

$$\begin{bmatrix} u_i \\ v_i \end{bmatrix} = \begin{bmatrix} \sin t & \cos t \\ -\cos t & \sin t \end{bmatrix} \begin{bmatrix} x_i \\ y_i \end{bmatrix} \tag{19.2}$$

or in simplified matrix notation,

$$Z = RX \tag{19.3}$$

Step 2: Assume that for the adjustment problem in which I appears, there is a Q_{xx} matrix for the xy coordinate system. The problem is to develop from the Q_{xx} matrix, a new covariance matrix Q_{zz} for the uv coordinate system. This can be done using the generalized law for the propagation of variances given in Chapter 6 as

$$\Sigma_{zz} = S_0^2 R Q_{xx} R^{\mathrm{T}} \tag{19.4}$$

where $R = \begin{bmatrix} \sin t & \cos t \\ -\cos t & \sin t \end{bmatrix}$.

Since S_0^2 is a scalar, it can temporarily be dropped and recalled again after the derivation. Thus,

$$Q_{zz} = R Q_{xx} R^T = \begin{bmatrix} q_{uu} & q_{uv} \\ q_{uv} & q_{vv} \end{bmatrix} \tag{19.5}$$

where $Q_{xx} = \begin{bmatrix} q_{xx} & q_{xy} \\ q_{xy} & q_{yy} \end{bmatrix}$.

Step 3: Expanding Equation (19.5), the elements of the Q_{zz} matrix are

$$Q_{zz} = \begin{bmatrix} \begin{pmatrix} q_{xx}\sin^2 t + q_{xy}\cos t \sin t \\ +q_{xy}\sin t \cos t + q_{yy}\cos^2 t \end{pmatrix} & \begin{pmatrix} -q_{xx}\sin t \cos t - q_{xy}\cos^2 t \\ +q_{xy}\sin^2 t + q_{yy}\cos t \sin t \end{pmatrix} \\ \begin{pmatrix} -q_{xx}\cos t \sin t + q_{xy}\sin^2 t \\ -q_{xy}\cos^2 t + q_{yy}\sin t \cos t \end{pmatrix} & \begin{pmatrix} q_{xx}\cos^2 t - q_{xy}\sin t \cos t \\ -q_{xy}\cos t \sin t + q_{yy}\sin^2 t \end{pmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} q_{uu} & q_{uv} \\ q_{uv} & q_{vv} \end{bmatrix} \tag{19.6}$$

Step 4: The q_{uu} element of Equation (19.6) can be rewritten as

$$q_{uu} = q_{xx}\sin^2 t + 2q_{xy}\cos t \sin t + q_{yy}\cos^2 t \tag{19.7}$$

The following trigonometric identities are useful in developing the equation for t.

$$\sin 2t = 2\sin t \cos t \tag{a}$$

$$\cos 2t = \cos^2 t - \sin^2 t \tag{b}$$

$$\cos^2 t + \sin^2 t = 1 \tag{c}$$

Substituting identity (a) into Equation (19.7) yields

$$q_{uu} = q_{xx}\sin^2 t + q_{yy}\cos^2 t + 2q_{xy}\frac{\sin 2t}{2} \tag{19.8}$$

Regrouping the first two terms and adding the necessary terms to Equation (19.8) to maintain equality yields

$$q_{uu} = \frac{q_{xx} + q_{yy}}{2}(\sin^2 t + \cos^2 t) + \frac{q_{xx}\sin^2 t}{2} + \frac{q_{yy}\cos^2 t}{2} - \frac{q_{yy}\sin^2 t}{2} - \frac{q_{xx}\cos^2 t}{2} + q_{xy}\sin 2t \tag{19.9}$$

Substituting identity (c) and regrouping Equation (19.9) results in

$$q_{uu} = \frac{q_{xx} + q_{yy}}{2} + \frac{q_{yy}}{2}(\cos^2 t - \sin^2 t) - \frac{q_{xx}}{2}(\cos^2 t - \sin^2 t) + q_{xy}\sin 2t \tag{19.10}$$

Finally, substituting identity (b) into Equation (19.10) yields

$$q_{uu} = \frac{q_{xx} + q_{yy}}{2} + \frac{q_{yy} - q_{xx}}{2}\cos 2t + q_{xy}\sin 2t \tag{19.11}$$

TABLE 19.1 Selection of the Proper Quadrant for $2t^a$

Algebraic Sign of		
$\sin 2t$	$\cos 2t$	Quadrant
+	+	1
+	−	2
−	−	3
−	+	4

[a] When calculating for t, always remember to select the proper quadrant of $2t$ before dividing by 2.

To find the value of t that maximizes q_{uu}, differentiate q_{uu} in Equation (19.8) with respect to t and set the results equal to zero. This yields

$$\frac{dq_{uu}}{dt} = -\frac{q_{yy} - q_{xx}}{2} 2 \sin 2t + 2q_{xy} \cos 2t = 0 \tag{19.12}$$

Reducing Equation (19.12) yields

$$(q_{yy} - q_{xx}) \sin 2t = 2q_{xy} \cos 2t \tag{19.13}$$

Finally, rearranging Equation (19.13) yields

$$\frac{\sin 2t}{\cos 2t} = \tan 2t = \frac{2q_{xy}}{q_{yy} - q_{xx}} \tag{19.14a}$$

Equation (19.14a) is used to compute $2t$, and hence the desired angle t that yields the maximum value of q_{uu}. Note that the correct quadrant of $2t$ must be determined by noting the sign of the numerator and denominator in Equation (19.14a) before dividing by 2 to obtain t. Table 19.1 shows the proper quadrant for the different possible sign combinations of the numerator and denominator.

Table 19.1 can be avoided by using the atan2 function available in most software packages. This function returns a value between $-180° \leq 2t \leq 180°$. If the value returned is negative, the correct value for $2t$ is obtained by adding 360°. The correct usage of the atan2 function in this instance is

$$2t = \text{atan2}(q_{yy} - q_{xx}, 2q_{xy}) \tag{19.14b}$$

The use of Equation (19.14b) is demonstrated in the Mathcad® worksheet *c19.xmcd* on the companion website.

Correlation between the latitude and departure of a station was discussed in Chapter 8. Similarly, the adjusted coordinates of a station are also

correlated. Computing the value of t that yields the maximum and minimum values for the semiaxes is equivalent to rotating the covariance matrix until the off diagonals are nonzero. Thus, the u and v coordinate values will be uncorrelated, which is equivalent to setting the q_{uv} element of Equation (19.6) equal to zero. Using the trigonometric identities previously noted, the element q_{uv} from Equation (19.6) can be written as

$$q_{uv} = \frac{q_{xx} - q_{yy}}{2} \sin 2t + q_{xy} \cos 2t \tag{19.15}$$

Setting q_{uv} equal to zero and solving for t

$$\frac{\sin 2t}{\cos 2t} = \tan 2t = \frac{2q_{xy}}{q_{yy} - q_{xx}} \tag{19.16}$$

Note that this yields the same result as Equation (19.14*a*), which verifies the removal of the correlation.

Step 5: In a fashion like that demonstrated in Step 4, the q_{vv} element of Equation (19.6) can be rewritten as

$$q_{vv} = q_{xx}\cos^2 t - 2q_{xy} \cos t \sin t + q_{yy}\sin^2 t \tag{19.17}$$

In summary, the t angle, semimajor axis (q_{uu}), and semiminor axis (q_{vv}) of the error ellipse are calculated using Equations (19.14*b*), (19.7), and (19.17), respectively. These equations are repeated here, in order, for convenience. Note that these equations use only elements from the covariance matrix particular to the station, which will be demonstrated by example.

$$\tan 2t = \frac{2q_{xy}}{q_{yy} - q_{xx}} \tag{19.18}$$

$$q_{uu} = q_{xx}\sin^2 t + 2q_{xy} \cos t \sin t + q_{yy}\cos^2 t \tag{19.19}$$

$$q_{vv} = q_{xx}\cos^2 t - 2q_{xy} \cos t \sin t + q_{yy}\sin^2 t \tag{19.20}$$

Equation (19.18) gives the t angle that the u-axis makes with the y-axis. Equation (19.19) yields the numerical value for q_{uu}, which when multiplied by the reference variance S_0^2 gives the variance along the u-axis. The square root of the variance is the *semimajor axis* of the standard error ellipse. Equation (19.20) yields the numerical value for q_{vv}, which when multiplied by S_0^2 gives the variance along the v-axis. The square root of this variance is the *semiminor axis* of the standard error ellipse. Thus, the semimajor and semiminor axes are

$$S_u = S_0\sqrt{q_{uu}} \text{ and } S_v = S_0\sqrt{q_{vv}} \tag{19.21}$$

19.3 EXAMPLE PROBLEM OF STANDARD ERROR ELLIPSE CALCULATIONS



In this section, the error ellipse data for the trilateration example in Section 14.5 will be calculated. From the computer listing given for the solution of that problem, the following values are recalled:

1. $S_0 = \pm 0.136$ ft
2. The unknown X and covariance Q_{xx} matrices were

$$X = \begin{bmatrix} \boldsymbol{dX_{Wisconsin}} \\ \boldsymbol{dY_{Wisconsin}} \\ dX_{Campus} \\ dY_{Campus} \end{bmatrix} \quad Q_{xx} = \begin{bmatrix} \mathbf{1.198574} & \mathbf{-1.160249} & -0.099772 & -1.402250 \\ \mathbf{-1.160249} & \mathbf{2.634937} & 0.193956 & 2.725964 \\ -0.099772 & 0.193956 & 0.583150 & 0.460480 \\ -1.402250 & 2.725964 & 0.460480 & 3.962823 \end{bmatrix}$$

19.3.1 Error Ellipse for Station Wisconsin

Since the coordinates for station Wisconsin are in the first two rows of the X matrix, only the corresponding 2×2 block diagonal elements of the Q_{xx} matrix (bold values in Q_{xx}) are used in the computations. From these elements, the tangent of $2t$ is

$$\tan 2t = \frac{2(-1.160249)}{2.634937 - 1.198574} = -1.6155$$

Notice that the sign of the numerator is negative and the denominator is positive. Thus, from Table 19.1, angle $2t$ is in the fourth quadrant and 360° must be added to the computed angle. Hence,

$$2t = \tan^{-1}(-1.6155) = -58°14.5' + 360° = 301°45.5'$$

and

$$t = 150°53'$$

Substituting the appropriate values into Equation (19.21), S_u is

$$S_u = \pm 0.136\sqrt{1.198574\sin^2 t + 2(-1.160249)\cos t \sin t + 2.634937\cos^2 t}$$
$$= \pm 0.25 \text{ ft}$$

Similarly, substituting the appropriate values into Equation (19.21), S_v is

$$S_v = \pm 0.136\sqrt{1.198574\cos^2 t - 2(-1.160249)\cos t \sin t + 2.634937\sin^2 t}$$
$$= \pm 0.10 \text{ ft}$$

The standard deviations in the coordinates, as computed by Equation (13.24) are

$$S_x = S_0\sqrt{q_{xx}} = \pm 0.136\sqrt{1.198574} = \pm 0.15 \text{ ft}$$

$$S_y = S_0\sqrt{q_{yy}} = \pm 0.136\sqrt{2.634937} = \pm 0.22 \text{ ft}$$

Since deriving the components of an error ellipse is an orthogonalization of the 2×2 block diagonal matrix for a station, this process can also be performed using eigenvectors and eigenvalues. This process is demonstrated in the Mathcad® worksheet *C19.xmcd* on the companion website. For example, the eigenvalues of the 2×2 block diagonal matrix for station Wisconsin are 0.55222 and 3.28129, respectively. Thus, $S_{u\text{-}WIS}$ is $0.136\sqrt{3.28129} = \pm 0.25$ ft and $S_{v\text{-}WIS}$ is $0.136\sqrt{0.55222} = \pm 0.101$ ft. Similarly, the eigenvector associated with the u-axis can be used to compute the t angle. Using Station Wisconsin as an example, the eigenvector associated with the eigenvalue of 3.28129 is $[-0.48666\ 0.87359]^T$. The t angle is computed as $t = \text{atan}(-0.487/0.874) + 180° = 150.8787°$, which is the same value that was computed using Equation (19.18).

19.3.2 Error Ellipse for Station Campus

Using the same procedures as in Section 19.3.1, the error ellipse data for station Campus are

The 2×2 block diagonal submatrix is $Qxx = \begin{bmatrix} 0.583150 & 0.460480 \\ 0.460480 & 3.96823 \end{bmatrix}$

$$2t = \tan^{-1}\left(\frac{2(0.460480)}{3.962823 - 0.583150}\right) = 15°14'$$

$$t = 7°37'$$

$$S_u = \pm 0.136\sqrt{0.583150\sin^2 t + 2(0.460480)\cos t \sin t + 3.962823\cos^2 t}$$

$$= \pm 0.27 \text{ ft}$$

$$S_v = \pm 0.136\sqrt{0.583150\cos^2 t - 2(0.460480)\cos t \sin t + 3.962823\sin^2 t}$$

$$= \pm 0.10 \text{ ft}$$

$$S_x = S_0\sqrt{q_{xx}} = \pm 0.136\sqrt{0.583150}$$

$$= \pm 0.10 \text{ ft}$$

$$S_y = S_0\sqrt{q_{yy}} = \pm 0.136\sqrt{3.962823}$$

$$= \pm 0.27 \text{ ft}$$

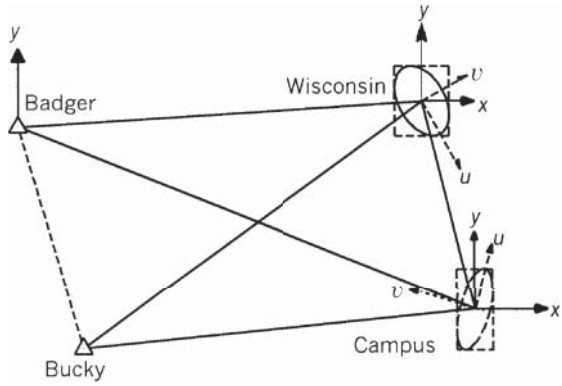


FIGURE 19.5 Graphical representation of error ellipses.

19.3.3 Drawing the Standard Error

To draw the error ellipses for stations Wisconsin and Campus of Figure 19.5, the error rectangle is first constructed by laying out the values of S_x and S_y using a convenient scale along the x and y axes, respectively. For this example, an ellipse scale of 4800 times the map scale was selected. The t angle is laid off clockwise from the positive y-axis to construct the u-axis. The v-axis is drawn 90° counter-clockwise from u to form a right-handed coordinate system. The values of S_u and S_v are laid off along the U and V axes, respectively, to locate the semi-axis points. Finally, as shown in Figure 19.3, the ellipse is constructed so that it is tangent to the error rectangle and passes through the semi-axes' points.

19.4 ANOTHER EXAMPLE PROBLEM

In this section, the standard error ellipse for station U in the example of Section 16.4 is calculated. For the adjustment, $S_0 = \pm 1.82$ ft, and the X and Q_{xx} matrices are

$$X = \begin{bmatrix} dx_u \\ dy_u \end{bmatrix} \quad Q_{xx} = \begin{bmatrix} q_{xx} & q_{xy} \\ q_{xy} & q_{yy} \end{bmatrix} = \begin{bmatrix} 0.000532 & 0.000602 \\ 0.000602 & 0.000838 \end{bmatrix}$$

Error ellipse calculations are

$$2t = \tan^{-1}\left[\frac{2(0.000602)}{0.000838 - 0.000532}\right] + 0° = 75°44'$$

$$t = 37°52'$$

Notice in this example that the reference variance passed the χ^2 test, and thus the sample variance can be simply an estimate of the population variance, which

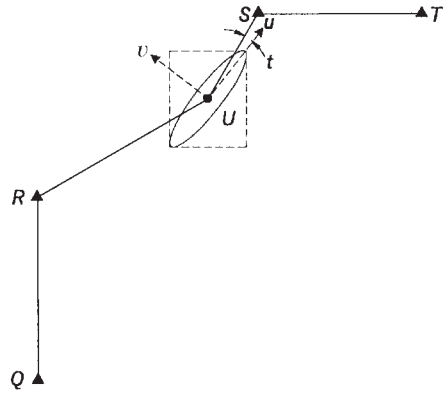


FIGURE 19.6 Graphical representation of error ellipse.

has a value of 1. Thus, the population variance can be used to compute the semiaxes of the ellipse. This is done in the following computations.

$$S_u = \pm 1\sqrt{0.000532\sin^2 t + 2(0.000602)\cos t \sin t + 0.000838\cos^2 t}$$
$$= \pm 0.036 \text{ ft}$$
$$S_v = \pm 1\sqrt{0.000532\cos^2 t - 2(0.000602)\cos t \sin t + 0.000838\sin^2 t}$$
$$= \pm 0.008 \text{ ft}$$
$$S_x = \pm 1\sqrt{0.000532} = \pm 0.023 \text{ ft}$$
$$S_y = \pm 1\sqrt{0.000838} = \pm 0.029 \text{ ft}$$

Figure 19.6 shows the plotted standard error ellipse and its error rectangle.

19.5 THE ERROR ELLIPSE CONFIDENCE LEVEL

The calculations in Sections 19.3 and 19.4 produce *standard error ellipses*. These ellipses can be modified to produce error ellipses at any confidence level α by using an F statistic with two numerator degrees of freedom and the degrees of freedom for the adjustment in the denominator. Since the F statistic represents a ratio of two variances having different degrees of freedom, it can be expected that with increases in the number of degrees of freedom, there will be corresponding increases in precision. The critical value for $F_{(\alpha,2,\ degrees\ of\ freedom)}$ modifies the probability of the error ellipse for various confidence levels, as listed in Table 19.2. Notice that as the degrees of freedom increase, the F-statistic modifiers decrease

TABLE 19.2 $F_{\alpha,2,\,degrees\,of\,freedom}$ Statistics for Selected Probability Levels

	Probability		
Degrees of Freedom	90%	95%	99%
1	49.50	199.5	4999.50
2	9.00	19.00	99.00
3	5.46	9.55	30.82
4	4.32	6.94	18.00
5	3.78	5.79	13.27
10	2.92	4.10	7.56
15	2.70	3.68	6.36
20	2.59	3.49	5.85
30	2.49	3.32	5.39
60	2.39	3.15	4.98

rapidly and begin to stabilize for larger degrees of freedom. The confidence level of an error ellipse can be increased to any level by using the multiplier

$$c = \sqrt{2F_{(\alpha,2,degrees\,of\,freedom)}} \tag{19.22}$$

Using the following equations, the percent probability for the semimajor and semiminor axes can be computed as

$$S_{u\%} = S_u c = S_u \sqrt{2F_{(\alpha,2,degrees\,of\,freedom)}}$$
$$S_{v\%} = S_v c = S_v \sqrt{2F_{(\alpha,2,degrees\,of\,freedom)}} \tag{19.23}$$

From the foregoing, it should be apparent that as the number of degrees of freedom (redundancies) increases, precision increases, and the size of error ellipse decreases. Using the techniques discussed in Chapter 4, the values listed in Table 19.2 are for the F distribution at 90% ($\alpha = 0.10$), 95% ($\alpha = 0.05$), and 99% ($\alpha = 0.01$) probability. These probabilities are those most used. Values from this table can be substituted into Equation (19.22) to determine the value of c for the probability selected and the given number of redundancies in the adjustment. This table is for convenience only and does not contain the values necessary for all situations that might arise.

Example 19.1 Calculate the 95% error ellipse for station Wisconsin of Section 19.3.1.

SOLUTION Using Equations (19.23) yields

$$S_{u_{95\%}} = \pm 0.25\sqrt{2(199.50)} = \pm 4.99 \text{ ft}$$
$$S_{v_{95\%}} = \pm 0.10\sqrt{2(199.50)} = \pm 2.00 \text{ ft}$$

$$S_{x_{95\%}} = \sqrt{2(199.50)}S_x = \pm(19.97 \times 0.15) \pm 3.00 \text{ ft}$$

$$S_{y_{95\%}} = \sqrt{2(199.50)}S_y = \pm(19.97 \times 0.22) \pm 4.39 \text{ ft}$$

The probability of the standard error ellipse can be found by setting the multiplier $2F_{\alpha,v1,v2}$ equal to 1 so that $F_{\alpha,v1,v2} = 0.5$. For a simple closed traverse with three degrees of freedom, this means that $F_{\alpha,2,3} = 0.5$. The value of α that satisfies this condition is 0.65, which was found by trial-and-error procedures using the program STATS. Thus, the percent probability of the standard error ellipse in a simple closed traverse is $(1 - 0.65) = 0.35$ or 35%. The reader is encouraged to verify this result using program STATS. It is also left as an exercise for the reader to show that the percent probability for the standard error ellipse ranges from 35% to only 39% for horizontal surveys having less than 100 degrees of freedom.

19.6 ERROR ELLIPSE ADVANTAGES

Besides providing critical information regarding the precision of an adjusted station position, a major advantage of error ellipses is that they offer a method of making a visual comparison of the relative precisions between any two stations. By viewing the shapes, sizes, and orientations of error ellipses, various surveys can be compared rapidly and meaningfully.

19.6.1 Survey Network Design

The sizes, shapes, and orientations of error ellipses are dependent on (1) the control used to constrain the adjustment, (2) the observational precisions, and (3) the geometry of the survey. The last two of these three elements are variables that can be altered in the design of a survey to produce optimal results. In designing surveys that involve the traditional observations of distances and angles, estimated precisions can be computed for observations made with differing combinations of equipment and field procedures. Also, trial variations in station placement, which creates the network geometry, can be made. Then, these varying combinations of observations and geometry can be processed through least-squares adjustments and the resulting station error ellipses computed, plotted, and checked against the desired results. Once acceptable goals are achieved in this process, the observational equipment, field procedures, and network geometry that provide these results can be adopted. This overall process is called *network design*. It enables surveyors to select the equipment and field techniques, and to decide on the number and locations of stations that provide the highest precision at lowest cost. This can be done in the office using simulation software before bidding on the contract or entering the field.

In designing networks to be surveyed using the traditional observations of distance, angle, and directions, it is important to understand the relationships of those

observations to the resulting positional uncertainties of the stations. The following relationships apply:

1. Distance observations strengthen the positions of stations in directions collinear with the observed lines.
2. Angle and direction observations strengthen the positions of stations in directions perpendicular to the lines of sight.

A simple analysis made with reference to Figure 19.1 should clarify the two relationships. Assume first that the length of line *AB* was precisely measured but its direction was not observed. Then, the positional uncertainty of station *B* should be held within close tolerances by the observed distance, but it would only be held in the direction collinear with *AB*. The distance observation would do nothing to keep line *AB* from rotating, and in fact the position of *B* would be weak perpendicular to *AB*. On the other hand, if the direction of *AB* had been observed precisely but its length had not been measured, the positional strength of station *B* would be strongest in the direction perpendicular to *AB*. But an angle observation alone does nothing to fix distances between observed stations, and thus the position of station *B* would be weak along line *AB*. If both the length and direction *AB* were observed with equal precision, a positional uncertainty for station *B* that is more uniform in all directions would be expected. In a survey network that consists of many interconnected stations, analyzing the effects of observations is not quite as simple as was just demonstrated for the single line *AB*. Nevertheless, the two basic relationships stated still apply.

Uniform positional strength in all directions for all stations is the desired goal in survey network design. This would be achieved if, following the least-squares adjustment, all error ellipses were circular in shape and of equal size. Although this goal is seldom possible completely by analyzing various combinations of geometric figures together with different combinations of observations and precisions diligently, this goal can be approached. Sometimes, however, other overriding factors such as station accessibility, terrain, and vegetation preclude the actual use of an optimal design.

The network design process discussed above is aided significantly using aerial photos, topographic maps, and/or online mapping software. These products enable layout of trial station locations and permits analysis of the accessibility and intervisibility of these stations to be investigated. However, a field reconnaissance should be made before adopting a final design.

Global navigation satellite systems (GNSS) have brought about dramatic changes in all areas of surveying and network design is not an exception. Although GNSS does require overhead visibility at each receiver station for tracking satellites, problems of intervisibility between ground stations are eliminated. Thus, networks having uniform geometry can normally be laid out. Because each station in the network is occupied in a GNSS survey, the *XYZ* coordinates of the stations can be determined. This simplifies the problem of designing networks to attain

error ellipses of uniform shapes and sizes. Additionally, recent photogrammetric images in a project area are often available over the Internet, which make checking satellite visibility problems possible. However, the geometric configuration of satellites plays a more important role in final station precisions. The *positional dilution of precision* (PDOP) can be a guide to the geometric strength of the observed satellites. In this case, the lower the PDOP, the stronger the satellite geometry. For more discussion on designing GNSS surveys, readers are referred to books devoted to the subject of GNSS surveying.

19.6.2 Example Network

Figure 19.7 shows error ellipses for two survey networks. Figure 19.7(*a*) illustrates the error ellipses from a trilateration survey with the nine stations, two of which (Red and Bug) were control stations. The survey includes 19 distance observations and 5 degrees of freedom. Figure 19.7(*b*) shows the error ellipses of the same network that were observed using triangulation and a baseline from stations Red to Bug. This survey includes 19 observed angles, and thus also has 5 degrees of freedom.

With respect to these two figures, and keeping in mind that the smaller the ellipse, the higher the precision, the following general observations can be made

1. In both figures, stations Sand and Birch have the highest precisions. This, of course, is expected due to their proximity to control station Bug, and because of the density of observations made to these stations, which included direct measurements from both control stations.

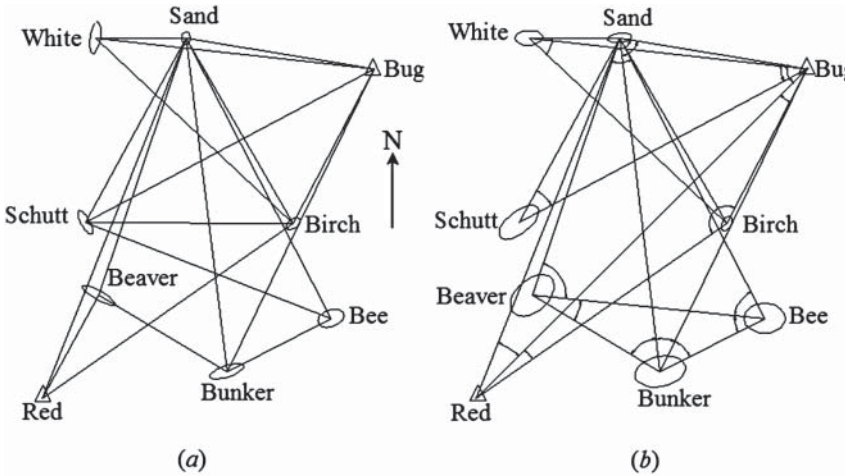


FIGURE 19.7 Network analysis using error ellipses: (*a*) trilateration for 19 distances; (*b*) triangulation for 19 angles.

2. The large size of error ellipses at stations Beaver, Schutt, Bunker, and Bee of Figure 19.7(*b*) shows that they have lower precision. This, too, is expected because there were fewer observations made to those stations. Also, neither Beaver nor Bee was connected directly by observations to either of the control stations.
3. Stations White and Schutt of Figure 19.7(*a*) have relatively high East–West precisions and relatively low North–South precisions. Examination of the network geometry reveals that this could be expected. Distance measurements to those two points from station Red, plus an observed distance between White and Schutt would have greatly improved the North–South precision.
4. Stations Beaver and Bunker of Figure 19.7(*a*) have relatively low precisions East–West and relatively high precisions North–South. Again, this is expected when examining the network geometry.
5. The smaller error ellipses of Figure 19.7(*a*) suggest that the trilateration survey will yield superior precision to the triangulation survey of Figure 19.7(*b*). This is expected since the distance measurements had a stated uncertainty of ±(5 mm + 5 ppm). In a 5000 ft distance, this yields an uncertainty of ±0.030 ft. To achieve the same precision, the comparable angle uncertainty would need to be

$$\theta'' = \frac{S}{R}\rho = \pm\frac{\pm 0.030}{5000}206,264.8''/\text{rad} = \pm 1.2''$$

The proposed instrument and field procedures for the project that yielded the error ellipses of Figure 19.7(*b*) had an expected uncertainty of only ±6″. The ultimate design would include both observed distances and angles.

These examples serve to illustrate the value of computing station error ellipses in an *a priori* analysis. The observations were easily and quickly made by a comparison of the ellipses in the two figures. Similar information would have been difficult, if not impossible, to determine from standard deviations. By varying the survey, it is possible ultimately to find a design that provides optimal results in terms of meeting a uniformly acceptable precision and survey economy.

19.7 OTHER MEASURES OF STATION UNCERTAINTY

Other measures of accuracies are sometimes called for in specifications and standards. As discussed in Section 19.5, the standard error ellipse has a *c*-multiplier of 1.00 and a probability between 35% and 39%. Other common errors and probabilities are given in Table 19.3.

The orthogonalization of the submatrices demonstrated in Section 19.3.1 can be used to compute the error ellipsoids for three-dimensional coordinates from a GNSS adjustment or the three-dimensional geodetic network adjustment discussed in Chapter 23. That is, the uncertainties along the three orthogonal axes of the error

TABLE 19.3 Other Measures of Two-Dimensional Positional Uncertainties

Probability (%)	*C*	Name
35–39	1.00	Standard error ellipse
50.0	1.18	Circular error probable (CEP)
63.2	1.41	Distance RMS (DRMS)
86.5	2.00	2-sigma ellipse
95.0	2.45	95% confidence level
98.2	2.83	2DRMS
98.9	3.00	3-sigma ellipse

TABLE 19.4 Measures of Three-Dimensional Positional Uncertainties

Probability (%)	*C*	Name
19.9	1.00	Standard ellipsoid
50.0	1.53	Spherical error probable (SEP)
61.0	1.73	Mean radical spherical error (MRSE)
73.8	2.00	2-sigma ellipsoid
95.0	2.80	95% confidence level
97.1	3.00	3-sigma ellipsoid

ellipsoid can be computed using the eigenvalues of the 3×3 block diagonal submatrix appropriate for each station. The common measures for ellipsoids are listed in Table 19.4.

The required accuracy for a control survey depends primarily on its purpose. Some major factors that affect accuracy are the type and condition of equipment used, field procedures adopted, and the experience and capabilities of personnel employed. In 1998, the Federal Geographic Data Committee (FGDC) published detailed standards of accuracy and specifications for data disseminated by the federal government.[1] The rationale for the standards was to provide a uniform set of standards specifying accuracies of several data capture methods for control surveys along with other analog and digital representations of survey data such as maps.

Table 19.5 lists the 1998 FGDC accuracy standards for control surveys. These standards are independent of the method of survey and based on a 95% confidence level. To meet these standards, control points in the survey must be consistent with all other points in the geodetic control network and not merely those within the particular survey. This is known as network accuracy and requires a comparison of survey results against network values or values captured using more accurate

[1]The 1998 FGDC standards titled Geospatial Positioning Accuracy Standards can be downloaded at the following website: http://www.fgdc.gov/standards/standards_publications.

TABLE 19.5 1998 FGDC Accuracy Standards: Horizontal, Ellipsoid Height, and Orthometric Height

Accuracy Classifications	95% Confidence Less Than or Equal to (m)
1 mm	0.001
2 mm	0.002
5 mm	0.005
1 cm	0.010
2 cm	0.020
5 cm	0.050
1 dm	0.100
2 dm	0.200
5 dm	0.500
1 m	1.000
2 m	2.000
5 m	5.000
10 m	10.000

survey procedures. For example, the quality of a map can be determined by comparing at least 20 identifiable-point coordinate values against the locations of these same points as determined by an optical or GNSS survey.

For horizontal surveys, the accuracy standard specifies the 95% radial error of the survey within which the true or theoretical location of the survey point falls 95% of the time. As shown in Figure 19.8, the 95% radial error is always shorter than the semimajor axes of the 95% error ellipse. To compute the probability of the radial error for a point, a constant C is determined using the semimajor (S_u) and semiminor (S_v) axes of the standard error ellipse from the least-squares adjustment as

$$C = S_v / S_u \tag{19.24}$$

Using a polynomial from Leenhouts (1985), the radial error of a point is approximated as

$$r = S_u \left(q_0 + q_1 C + q_2 C^2 + q_3 C^3\right) \tag{19.25}$$

where r is the radial error for a station; S_u the semimajor axis of the standard error ellipse; C as defined in Equation (19.24); and coefficients q_0 through q_3 are as listed in Table 19.6 for various probability levels. Thus, the 95% radial error for a station is computed as

$$r_{95\%} = S_u(1.960790 + 0.004071C + 0.114276C^2 + 0.371625C^3) \tag{19.26}$$

The vertical accuracy standard specifies a linear value (plus or minus) within which the true or theoretical location of the point falls 95% of the time. It is

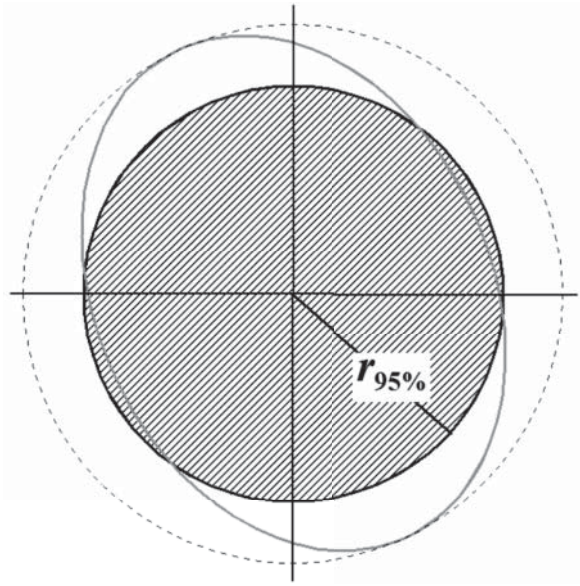


FIGURE 19.8 A 95% circular error overlaying the 95% error ellipse for Station *S*.

TABLE 19.6 Coefficients for a Third-Order Polynomial Approximations of Radial Errors

% Probability	q_0	q_1	q_2	q_3
50	0.671956	0.011407	1.013704	−0.523457
75	1.153896	−0.121102	0.925793	−2.198069
90	1.647743	−0.62698	0.437189	0.127667
95	1.960790	0.004071	0.114276	0.371625
99	2.572111	0.127744	−0.394634	0.727603
99.5	2.801799	0.165577	−0.538393	0.821817
99.9	3.282670	0.228114	−0.767617	0.964153

computed using the normal distribution multiplier of 1.9600 times the standard deviation for the elevation of the benchmark.

Procedures leading to classification of a survey to these standards involve four steps:

1. The survey observations, field records, sketches, and other documentation are examined to ensure their compliance with specifications for the intended accuracy of the survey.

2. A minimally constrained least-squares adjustment of the survey observations is analyzed to guarantee that the observations are free from blunders and have been correctly weighted.
3. The accuracy of control points in the local existing network to which the survey is tied is computed by random error propagation and weighted accordingly in the least-squares adjustment of the survey network.
4. The survey accuracy is checked at the 95% confidence level by comparing minimally constrained adjustment results against established control. The comparison considers the network accuracy of the existing control as well as systematic effects such as crustal motion or datum distortion.

Following the adjustment, a local accuracy for the survey can be determined using an average 95% radial error for the station coordinates or benchmarks in the survey. The local accuracy represents the correctness of the coordinates of the stations relative to the coordinates of other stations in the survey. The network accuracy represents the correctness of the coordinates in the survey with respect to the geodetic datum. Thus, it is possible that a survey may have a very good local accuracy but not have a good network accuracy. To understand this, consider a survey for an engineering project that is based on a few local control stations in the project area. To qualify for determination of network accuracy, the survey should be connected to 20 or more control stations in the geodetic datum. Since this goes beyond the scope of the survey, which may be supporting a construction project, these standards also have guidelines for architectural, engineering, construction, and facilities management projects.

As stated previously, these standards can also be used to classify maps, scanning projects, and other data capture or representation processes. The importance of classifying the accuracy of maps with statistical values is that all forms of maps, whether hardcopy or digital, can have their accuracy stated so that users understand the product and its potential uses. Newer standards such as the *ASPRS Positional Accuracy Standards for Digital Geospatial Data Edition 2 Version 2* (ASPRS 2024), USACE New Orleans District Minimum Survey Standards (USACE 2022), and the Transportation Research Board Mobile Lidar Guidelines (Olsen et al. 2013) to report the accuracy of geospatial data products. In addition to providing requirements for quality control procedures and metrics, these standards contain terminology and definitions as well as provide the mathematical and statistical theorical background for implementation of the standard. The ASPRS standards also provide examples of application for different types of geospatial data products such as UAS photogrammetry and airborne lidar. The root-mean-square error (RMSE) is a common way to communicate dataset accuracy within both the scientific and practitioner community. It is the metric used for reporting in the new ASPRS standards. It is defined as the square root of the average of squared discrepancies for points tested as discussed in Section 2.10. Discrepancies are the differences between coordinates and elevations of points taken from the map, and

TABLE 19.7 Map Coordinates versus Surveyed Checkpoint Coordinates

	Map Coordinates			Surveyed Checkpoint Coordinates			Discrepancies/ Residuals		
Point	(1) x (m)	(2) y (m)	(3) z (m)	(4) X (m)	(5) Y (m)	(6) H (m)	(7) Δx	(8) Δy	(9) Δz
1	326,064.94	27,695.00	604.22	326,069.477	27,700.068	594.056	−4.537	−5.068	10.164
2	332,436.19	27,596.91	578.46	332,438.494	27,605.343	587.351	−2.304	−8.433	−8.891
3	329,244.19	25,219.12	585.16	329,235.172	25,217.616	583.693	9.018	1.504	1.467
4	326,054.07	22,891.32	558.35	326,059.779	22,890.734	563.272	−5.709	0.586	−4.922
5	332,431.37	22,795.94	571.03	332,437.285	22,796.579	575.159	−5.915	−0.639	−4.129
				Number of Checkpoints			5	5	5
				RMSE (m)			5.91	7.76	6.72
				$RMSE_r$			7.41		
				95% RMSE			12.83		13.14

their values as determined by check surveys. Mathematically, it is denoted as

$$\text{RMSE} = \sqrt{\sum_{i=1}^{n} \frac{[f(x_i) - x_i]^2}{n}} \tag{19.27}$$

where n is the number of tested samples from the map; $f(x_i)$ is the position obtained from the map or geodetic datum; and x_i is the position obtained from the check survey. For example, consider the data shown in Table 19.7. The map-derived values for coordinates of the points are shown in columns (1)–(3). The surveyed checkpoint coordinate values are shown in columns (4)–(6). The discrepancies shown in columns (7)–(9) are computed as the difference between the map and checkpoint coordinate values. Since the checkpoints must be surveyed at a higher accuracy than the map, these discrepancies can be thought of as residuals Thus, the residual for the x coordinate of point 1 is computed as $326{,}064.94 - 32{,}069.477 = -4.537$m. Similarly, all other coordinate differences are determined. Using Equation (19.27), the RMSE in x ($RMSE_x$), y ($RMSE_y$), and z ($RMSE_z$) are computed from their corresponding residuals as

$$\text{RMSE}_x = \sqrt{\frac{(-4.537)^2 + (-2.304)^2 + 9.018^2 + (-5.709)^2 + (-5.915)^2}{5}}$$
$$= \pm 5.91 \text{ m}$$

$$\text{RMSE}_y = \sqrt{\frac{(-5.068)^2 + (-8.433)^2 + 1.504^2 + 0.586^2 + (-0.639)^2}{5}}$$
$$= \pm 4.47 \text{ m}$$

$$\text{RMSE}_z = \sqrt{\frac{(10.164)^2 + (-8.891)^2 + 1.467^2 + (-4.922)^2 + (-4.129)^2}{5}}$$
$$= \pm 6.72 \text{ m}$$

From the RMSE_x and RMSE_y, a radial value for the horizontal positional accuracy can be determined as

$$\text{RMSE}_r = \sqrt{\text{RMSE}_x^2 + \text{RMSE}_y^2} = \sqrt{5.91^2 + 4.47^2} = \pm 7.41 \text{ m}$$

If it is necessary to scale accuracies to 95% confidence to comply with an earlier standard than the ASPRS (2024) standards, the values should be scaled. Note that the scale factor values (C) in Tables 19.3 and 19.4 assume the standard deviation is equal in all dimensions and the standard deviation is a 1-dimensional value. When SLOPOV is used to combine different standard deviations or RMSE values across dimensions into a single value, circular (2-dimensional) and spherical error (3-dimeinsional) error probabilities are used to scale from RMSE to 95% confidence using the values presented in Table 19.8, derived based on the methodologies outlined in the Aeronautical Chart and Information Center Technical Report No. 96 (Greenwalt and Schult 1962) and the Transportation Research Board Mobile Lidar Guidelines (Olsen et al. 2013). Thus, to obtain the 95% positional horizontal accuracy for the map, the RMSE_r is multiplied by 1.7308 (2-dimensional), which in this case results in a 95% positional horizontal accuracy of ±12.83 m for this map. To compute the 95% positional vertical accuracy for the map (RMSE_z) is multiplied by 1.9600 (1-dimensional), which in this case yields ±13.17 m. Should a 3-dimensional value be desired, one would compute the RMSE3D as

$$\text{RMSE}_{3D} = \sqrt{\text{RMSE}_x^2 + \text{RMSE}_y^2 + \text{RMSE}_z^2}$$
$$= \sqrt{5.91^2 + 4.47^2 + 6.72^2} = \pm 10.00 \text{ m}$$

and then multiply by 1.6140 (3-dimensional, Table 19.8) to obtain the 95% positional 3D accuracy, which is ±16.14 m. Table 19.8 provides the scale factors to obtain the 95% confidence level for each dimension, and Table 19.9 provides the probable error multiplier for each dimension and for various confidence levels. Users can use Table 19.9 to derive a scale factor for a desired confidence level.

TABLE 19.8 RMSE to 95% Confidence Scale Factors

Dimensions	Error Theory	C	Scale Factor = $C/\sqrt{\text{Dimensions}}$
1	Linear	1.9600	1.9600
2	Circular	2.4477	1.7308
3	Spherical	2.7955	1.6140

TABLE 19.9 *C* factor (probable error multiplier) for each dimension and for various confidence levels (Greenwalt and Schultz 1962)

Confidence Level	Linear (Dimension = 1)	Circular (Dimension = 2)	Spherical (Dimension = 3)
50%	0.6745	1.1774	1.5382
90%	1.6449	2.1460	2.5003
95%	1.9600	2.4477	2.7955
99%	2.5758	3.0349	3.3683
99.9%	3.2905	3.7169	4.0345

19.8 CHECK POINTS

Validation against known 'ground truth' measurements called check points is important to survey and mapping data quality control (Olsen et al., 2013; Nolan et al. 2015; Nolan et al. 2017; ASPRS 2024) by comparing the data product results with high precision direct measurements taken on the ground called check points. Note that a statistical assessment of residuals of control points used in the adjustment should not be utilized in place of check points as the check points should be independent of the network, especially in high precision applications. Both are important to understanding accuracy. The ASPRS (2024) standards require obtaining a minimum of 30 and maximum of 120 check points, depending on project size compared with the minimum 20 points required in most previous standards based on the FGDC 1998. Accuracy is reported as the Root Mean Square (RMS) differences between the check points and geospatial product (e.g., DEM). The ASPRS standard also provides additional quality control measures such as lidar relative swath to swath precision, minimum nominal pulse density, and spatial distribution of check points. These products are helpful to understand the overall data quality compared with a single reported accuracy given that the quality can vary substantially across a dataset.

For RMSEs to be credible, the check point survey should be performed at a higher level of accuracy than the map product. However, this has become challenging as sensor capabilities have improved to the level that it can be difficult to obtain check points at a higher accuracy with another technology. Hence, the ASPRS 2024 standards now require reporting of the check point accuracy, which should be at least two times more accurate than the product being tested. Check points should be obtained on clearly identifiable features in the geospatial data product. For example, assume a laser scanner is used to develop a map of a structure. In this instance, a conventional survey with a reflectorless total station can provide checkpoints at a higher level of accuracy and could be used to establish coordinates on clearly defined points (e.g., areas where there will be a strong intensity contrast on a smooth surface rather than edges of an object), which are also visible and measured in the scan of the site. If a GNSS survey is

to be used, special procedures as outlined in the NGS' *Guidelines for Establishing GPS-Derived Ellipsoid Heights* (Zilkoski, 1997) should be used to establish geodetic heights on clearly defined points. Additionally, the most recent geoid model must be applied to geodetic heights to derive orthometric heights. However, if this method is used, realize that the geoid model for the area of interest may contain more errors than that is allowable. Alternately, a GNSS survey could be used to establish horizontal positioning while differential leveling or trigonometric leveling could be used to establish orthometric heights on points. It should also be recognized here that the network accuracy of a geodetic survey can be determined using this same method, with check coordinates coming from network stations.

PROBLEMS

Note: For problems below requiring least squares adjustment, if a computer program is not distinctly specified for use in the problem, it is expected that the least-squares algorithm will be solved using the program MATRIX, which is included with on the book's companion website. Partial answers are provided for problems marked with an asterisk in Appendix G.

***19.1** Calculate the semiminor and semimajor axes of the standard error ellipse for the adjusted position of station *U* in trilateration Example 14.2. Plot the figure using a scale of 1:10,000 and the error ellipse using an appropriate scale.

***19.2** Calculate the semiminor and semimajor axes of the 95% confidence error ellipse for Problem 19.1. Plot this ellipse superimposed over the ellipse of Problem 19.1.

19.3 Repeat Problem 19.1 using the resection problem in Example 15.2.

19.4 Repeat Problem 19.1 using the data for Problem 19.3.

19.5 Same as Problem 19.1 except for the adjusted position of stations *B* and *C* of Example 15.3. Use a scale of 1:24,000 for the figure and plot the error ellipses using an appropriate multiplication factor.

Calculate the error ellipse data for the unknown stations in Problems 19.6 to 19.21.

***19.6** Problem 14.1.

19.7 Problem 14.4.

19.8 Problem 14.8.

19.9 Problem 15.2.

19.10 Problem 15.5.

19.11 Problem 15.9.

19.12 Problem 16.1.

19.13 Problem 16.2.

19.14 Problem 16.3.

19.15 Problem 16.4.

19.16 Problem 16.5.

19.17 What is the 95% radial error at Station *U* in Problem 19.16?

19.18 Problem 16.7.

19.19 What is the 95% radial error at station *C* in Problem 19.18?

19.20 What is the 95% radial error at station *B* in Problem 19.12?

19.21 What is the 95% radial error at station *U* in Problem 19.15?

Using a level of significance of 0.05, compute the 95% probable error ellipse for the station in each problem.

***19.22** Problem 19.12.

19.23 Problem 19.15.

19.24 Problem 19.16.

19.25 Using the program STATS, determine the percent probability of the standard error ellipse for a horizontal survey with

***(a)** 6 degrees of freedom

(b) 12 degrees of freedom

(c) 18 degrees of freedom

(d) 64 degrees of freedom

(e) 120 degrees of freedom

PROGRAMMING PROBLEMS

19.26 Develop a computational program that takes the Q_{xx} matrix and S_0^2 from a horizontal adjustment and computes error ellipse data for the unknown stations.

19.27 Develop a computational program that solves Problems 19.15 and 19.16.

CHAPTER 20

CONSTRAINT EQUATIONS

20.1 INTRODUCTION

When doing an adjustment, it is sometimes necessary to fix an observation to a specific value. For instance, in Chapter 14, it was shown that the coordinates of a control station can be fixed by setting its *dx* and *dy* corrections to zero, and thus the corrections and their corresponding coefficients in the *J* matrix were removed from the solution. This is called a *constrained adjustment*. Another constrained adjustment occurs when the direction or length of a line is held to a specific value, or when an elevation difference between two stations is fixed in differential leveling. In this chapter, methods available for developing observational constraints are discussed. However, before discussing constraints, the procedure for including control station coordinates in an adjustment is described.

20.2 ADJUSTMENT OF CONTROL STATION COORDINATES

 In examples in preceding chapters, when the coordinates of a control station were excluded from the adjustments, hence their values held fixed, constrained

adjustments were being performed. That is, the observations were being forced to fit the control coordinates. However, control is not perfect, and not all control is of equal reliability. This fact is evidenced by the fact that different orders of accuracy are used to classify control.

When more than minimal control is held fixed in an adjustment, the observations are forced to fit this control. For example, if the coordinates of two control stations are held fixed, but their actual positions are not in agreement with the values given by their held coordinates, the observations will be adjusted to match the erroneous coordinates. Simply stated, precise observations may be forced to fit less precise control. This was not a major problem in the days of transits and tapes, but it does happen with modern instrumentation. This topic is discussed in more detail in Chapter 21.

To clarify the problem further, suppose that a new survey is tied to two existing control stations set from two previous surveys. Assume that the precision of the existing control stations is only 1:10,000. A new survey uses equipment and field procedures designed to produce a survey of 1:50,000. Thus, it will have a higher accuracy than either of the control stations to which it must fit. If both existing control stations are fixed in the adjustment, the new observations must distort to fit the errors of the control stations. After the adjustment, their residuals will show a lower order fit that matches the control. In this case, it would be better to allow the control coordinates to adjust according to their assigned quality so that the observations are not distorted. However, it should be stated that the precision of the new coordinates relative to stations not in the adjustment can only be as good as the initial control.

The observation equations for control station coordinates are

$$\begin{aligned} x' &= x + v_x \\ y' &= y + v_y \end{aligned} \tag{20.1}$$

In Equation (20.1), x' and y' are the observed coordinate values of the control station; x and y are the published coordinate values for the station; and v_x and v_y are the residuals for the respective published coordinate values.

To allow the control to adjust, Equations (20.1) must be included in the adjustment for each control station. To *fix* a control station in this scheme, high weights are assigned to the station's coordinates. Conversely, low weights will allow a control station's coordinates to adjust. In this manner, all control stations are allowed to adjust in accordance with their expected levels of accuracy. In Chapter 21, it will be shown that when the control is included as observations, *poor* observations and control stations can be isolated in the adjustment by using weights.

Example 20.1 A trilateration survey was completed for the network shown in Figure 20.1, and the following observations were collected.

Control Stations

Station	X (ft)	Y (ft)
A	10,000.00	10,000.00
C	12,487.08	10,528.65

Distance Observations

From	To	Distance (ft)	σ (ft)	From	To	Distance (ft)	σ (ft)
A	B	1400.91	0.023	B	E	1644.29	0.023
A	E	1090.55	0.022	B	F	1217.54	0.022
B	C	1723.45	0.023	D	F	842.75	0.022
C	F	976.26	0.022	D	E	1044.99	0.022
C	D	1244.40	0.023	E	F	930.93	0.022

Perform a least-squares adjustment of this survey, holding the control coordinates of stations A and C by appropriate weights (Assume that these control stations have a precision of 1:10,000).

SOLUTION The first iteration linearized observation equations for the control coordinates of Equation (20.1) are

$$dx_A = 10,000.00 - x'_{A_0}$$
$$dy_A = 10,000.00 - y'_{A_0}$$
$$dx_C = 12,487.08 - x'_{C_0}$$
$$dy_C = 10,528.65 - y'_{C_0}$$

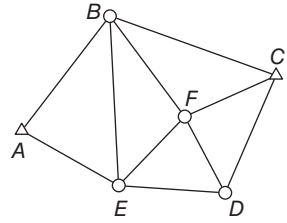


FIGURE 20.1 Trilateration network.

The J, X, and K matrices formed in this adjustment are:

$$J = \begin{bmatrix}
\frac{\partial D_{AB}}{\partial x_A} & \frac{\partial D_{AB}}{\partial y_A} & \frac{\partial D_{AB}}{\partial x_B} & \frac{\partial D_{AB}}{\partial y_B} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{\partial D_{AE}}{\partial x_A} & \frac{\partial D_{AE}}{\partial y_A} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial D_{AE}}{\partial x_E} & \frac{\partial D_{AE}}{\partial y_E} & 0 & 0 \\
0 & 0 & \frac{\partial D_{BE}}{\partial x_B} & \frac{\partial D_{BE}}{\partial y_B} & 0 & 0 & 0 & 0 & \frac{\partial D_{BE}}{\partial x_E} & \frac{\partial D_{BE}}{\partial y_E} & 0 & 0 \\
0 & 0 & \frac{\partial D_{BF}}{\partial x_B} & \frac{\partial D_{BF}}{\partial y_B} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial D_{BF}}{\partial x_F} & \frac{\partial D_{BF}}{\partial y_F} \\
0 & 0 & \frac{\partial D_{BC}}{\partial x_B} & \frac{\partial D_{BC}}{\partial y_B} & \frac{\partial D_{BC}}{\partial x_C} & \frac{\partial D_{BC}}{\partial y_C} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\partial D_{CF}}{\partial x_C} & \frac{\partial D_{CF}}{\partial y_C} & 0 & 0 & 0 & 0 & \frac{\partial D_{CF}}{\partial x_F} & \frac{\partial D_{CF}}{\partial y_F} \\
0 & 0 & 0 & 0 & \frac{\partial D_{CD}}{\partial x_C} & \frac{\partial D_{CD}}{\partial y_C} & \frac{\partial D_{CD}}{\partial x_D} & \frac{\partial D_{CD}}{\partial y_D} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial D_{DF}}{\partial x_D} & \frac{\partial D_{DF}}{\partial y_D} & 0 & 0 & \frac{\partial D_{DF}}{\partial x_F} & \frac{\partial D_{DF}}{\partial y_F} \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial D_{DE}}{\partial x_D} & \frac{\partial D_{DE}}{\partial y_D} & \frac{\partial D_{DE}}{\partial x_E} & \frac{\partial D_{DE}}{\partial y_E} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\partial D_{EF}}{\partial x_E} & \frac{\partial D_{EF}}{\partial y_E} & \frac{\partial D_{EF}}{\partial x_F} & \frac{\partial D_{EF}}{\partial y_F} \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

$$X = \begin{bmatrix} dx_A \\ dy_A \\ dx_B \\ dy_B \\ dx_C \\ dy_C \\ dx_D \\ dy_D \\ dx_E \\ dy_E \\ dx_F \\ dy_F \end{bmatrix} \qquad K = \begin{bmatrix} L_{AB} - AB_0 \\ L_{AE} - AE_0 \\ L_{BE} - BE_0 \\ L_{BF} - BF_0 \\ L_{BC} - BC_0 \\ L_{CF} - CF_0 \\ L_{CD} - CD_0 \\ L_{DF} - DF_0 \\ L_{DE} - DE_0 \\ L_{EF} - EF_0 \\ x_A - x_{A_0} \\ y_A - y_{A_0} \\ x_C - x_{C_0} \\ y_C - y_{C_0} \end{bmatrix}$$

Notice that the last four rows of the J matrix correspond to linearized observation Equation (20.1) for the coordinates of control stations A and C. Each

coordinate has a row with *one* in the column corresponding to its correction. Obviously, by including the control station coordinates, four unknowns have been added to the adjustment: dx_A, dy_A, dx_C, and dy_C. However, four observations have also been added. Therefore, the number of redundancies is unaffected by adding the coordinate observation equations. That is, the adjustment has the same number of redundancies with or without the control equations.

It is possible to weight a control station according to the precision of its coordinates. Unfortunately, control stations are sometimes published with distance precision rather than the covariance matrix elements that are required for proper weighting. However, estimates of the standard deviations of the coordinates can be computed from the published distance precisions. That is, if the distance precision between stations A and C is 1:10,000 or better, their coordinates should have estimated errors that yield a distance precision of 1:10,000 between the stations. To find the estimated errors in the coordinates that yield the appropriate distance precision between the stations, Equation (6.16) can be applied to the distance formula, resulting in

$$\sigma_{D_{ij}}^2 = \left(\frac{\partial D_{ij}}{\partial x_i}\right)^2 \sigma_{x_i}^2 + \left(\frac{\partial D_{ij}}{\partial y_i}\right)^2 \sigma_{y_i}^2 + \left(\frac{\partial D_{ij}}{\partial x_j}\right)^2 \sigma_{x_j}^2 + \left(\frac{\partial D_{ij}}{\partial y_j}\right)^2 \sigma_{y_j}^2 \tag{20.2}$$

In Equation (20.2), $\sigma_{D_{ij}}^2$ is the variance in distance D_{ij} and $\sigma_{x_i}^2$, $\sigma_{y_i}^2$, $\sigma_{x_j}^2$, and $\sigma_{y_j}^2$ the variances in the coordinates of the end points of the line. If the estimated errors for the coordinates of Equation (20.2) are equal, then substituting in the appropriate partial derivatives yields

$$\begin{aligned}\sigma_{D_{ij}}^2 &= 2\left(\frac{\partial D_{ij}}{\partial x}\right)^2 \sigma_x^2 + 2\left(\frac{\partial D_{ij}}{\partial y}\right)^2 \sigma_y^2 \\ &= 2\left(\frac{\Delta x}{IJ}\sigma_c\right)^2 + 2\left(\frac{\Delta y}{IJ}\sigma_c\right)^2\end{aligned} \tag{20.3}$$

In Equation (20.3), σ_c is the standard deviation in the x and y coordinates. [Note that the partial derivatives appearing in Equation (20.3) were described in Section 14.2]. Factoring $2\sigma_c^2$ from Equation (20.3) yields

$$\sigma_{D_{ij}}^2 = 2\sigma_c^2 \left(\frac{\Delta x^2 + \Delta y^2}{IJ^2}\right) = 2\sigma_c^2 \tag{20.4}$$

where $\sigma_c^2 = \sigma_x^2 = \sigma_y^2$.

From the coordinates of A and C, the distance AC is 2542.65 ft. To get a distance precision of 1:10,000, a maximum distance error of ±0.25 ft may exist. Assuming equal coordinate errors, then

$$\pm 0.25 = \sigma_x\sqrt{2} = \sigma_y\sqrt{2}$$

**** Adjusted Distance Observations ****

No.	From	To	Distance	Residual
1	A	B	1,400.910	-0.000
2	A	E	1,090.550	-0.000
3	B	E	1,644.288	-0.002
4	B	F	1,217.544	0.004
5	B	C	1,723.447	-0.003
6	C	F	976.263	0.003
7	C	D	1,244.397	-0.003
8	D	E	1,044.988	-0.002
9	E	F	930.933	0.003
10	D	F	842.753	0.003

****** Adjusted Control Stations ******

No.	Sta.	Northing	Easting	N Res	E Res
1	C	10,528.650	12,487.080	0.000	0.002
2	A	10,000.000	10,000.000	-0.000	-0.002

Reference Standard Deviation = 0.25
Degrees of Freedom = 2

******* Adjusted Unknowns ******

Station	Northing	Easting	? North	? East	t ang°	A axis	B axis
A	10,000.000	9,999.998	0.033	0.045	168.000	0.046	0.033
B	11,103.933	10,862.483	0.039	0.034	65.522	0.040	0.033
C	10,528.650	12,487.082	0.033	0.045	168.000	0.046	0.033
D	9,387.462	11,990.882	0.040	0.038	49.910	0.044	0.033
E	9,461.900	10,948.549	0.039	0.034	110.409	0.039	0.033
F	10,131.563	11,595.223	0.033	0.034	17.967	0.034	0.033

FIGURE 20.2 The results of the adjustment of data for Figure 20.1.

Thus, $\sigma_x = \sigma_y = \pm 0.18$ ft. The computed standard deviations are used to weight the control in the adjustment. The weight matrix for this adjustment is

$$W = \begin{bmatrix}
\frac{1}{0.023^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{0.022^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{0.023^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{0.022^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{0.023^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{0.022^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{0.022^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{0.022^2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{0.022^2} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{0.022^2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{0.18^2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{0.18^2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{0.18^2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{0.18^2}
\end{bmatrix}$$

The adjustment results, obtained using the program ADJUST, are shown in Figure 20.2. Notice that the control stations were adjusted slightly, as evidenced by their residuals. Also note the error ellipse data computed for each control station.

20.3 HOLDING CONTROL STATION COORDINATES AND DIRECTIONS OF LINES FIXED



As demonstrated in Example 14.2, the coordinates of a control station are easily fixed during an adjustment. This is accomplished by assigning values of zero to the coefficients of the *dx* and *dy* correction terms. This method removes their corrections from the equations. In that particular example, each observation equation had only two unknowns, since one end of each observed distance was a control station that was held fixed during the adjustment. This was a special case of a method known as the *solution by elimination of constraints*.

This method can be shown in matrix notation as

$$A_1X_1 + A_2X_2 = L_1 + V \tag{20.5}$$

$$C_1X_1 + C_2X_2 = L_2 \tag{20.6}$$

In Equations (20.2) and (20.6), A_1, A_2, X_1, X_2, L_1, and L_2 are the A, X, and L matrices partitioned by the constraint equations, as shown in Figure 20.3; C_1 and C_2 are the partitions of the matrix C consisting of the coefficients of the constraint equations; and V is the residual matrix. In this method, matrices A, C, and X are partitioned into two matrix equations that separate the constrained and unconstrained observations. Careful consideration should be given to the partition of C_1 since this matrix cannot be singular. If singularity exists, a new set of constraint equations that are mathematically independent must be determined. Also, since each constraint equation will remove one parameter from the adjustment, the number of constraints must not be so large that the remaining A_1 and X_1 have no independent equations or are themselves singular.

From Equation (20.6), solve for X_1 in terms of C_1, C_2, X_2, and L_2 as

$$X_1 = C_1^{-1}(L_2 - C_2X_2) \tag{20.7}$$

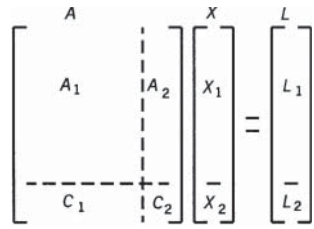


FIGURE 20.3 A, X, and L matrices partitioned.

Substituting Equation (20.7) into Equation (20.5) yields

$$A_1\left[C_1^{-1}(L_2 - C_2X_2)\right] + A_2X_2 = L_1 + V \tag{20.8}$$

Rearranging Equation (20.8), regrouping, and dropping V for the time being gives

$$\left(-A_1C_1^{-1}C_2 + A_2\right)X_2 = L_1 - A_1C_1^{-1}L_2 \tag{20.9}$$

Letting $A' = -A_1C_1^{-1}C_2 + A_2$, Equation (20.9) can be rewritten as

$$A'X_2 = L_1 - A_1C_1^{-1}L_2 \tag{20.10}$$

Now Equation (20.10) can be solved for X_2, which, in turn, is substituted into Equation (20.7) to solve for X_1.

In the solution by elimination of constraint, the constraint equations are used to eliminate unknown parameters from the adjustment, thereby fixing certain geometric conditions during the adjustment. This method was used when the coordinates of the control stations were *removed* from the adjustments in previous chapters. In the following subsection, this method is used to hold the azimuth of a line during an adjustment.

20.3.1 Holding the Direction of a Line Fixed by Elimination of Constraints

Using this method, constraint equations are written and then substituted functionally into the observation equations to eliminate unknown parameters. To illustrate this, suppose that the direction of line IJ shown in Figure 20.4 is fixed. Thus, the position of J is constrained to move linearly along IJ during the adjustment. If J moves to J' after adjustment, then the relationship between the direction of IJ and dx_j and dy_j is

$$dx_j = dy_i \tan\alpha \tag{20.11}$$

For example, suppose that the direction of line AB in Figure 20.5 is to be held fixed during a trilateration adjustment. Noting that station A is to be held fixed

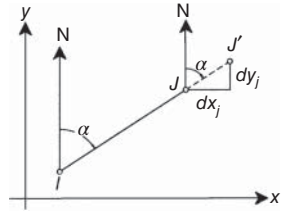


FIGURE 20.4 Holding direction IJ fixed.

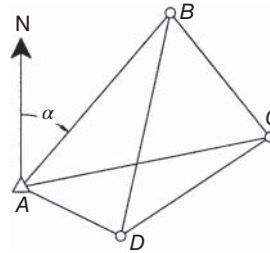


FIGURE 20.5 Holding direction *AB* fixed in a trilateration adjustment.

and using prototype Equation (14.9), the following linearized observation equation results for observed distance *AB*.

$$k_{l_{ab}} + v_{l_b} = \frac{x_{b_0} - x_a}{AB_0} dx_b + \frac{y_{b_0} - y_a}{AB_0} dy_b \tag{20.12}$$

Now, based on Equation (20.11), the following relationship is written for line *AB*:

$$dx_b = dy_b \tan \alpha \tag{20.13}$$

Substituting Equation (20.13) into Equation (20.12) yields

$$k_{l_{ab}} + v_{l_{ab}} = \frac{x_{b_0} - x_a}{AB_0} \tan \alpha \, dy_b + \frac{y_{b_0} - y_a}{AB_0} dy_b \tag{20.14}$$

Factoring dy_b in Equation (20.14), the constraint observation equation is

$$k_{l_{ab}} + v_{l_{ab}} = \frac{(x_{b_0} - x_a) \tan \alpha + (y_{b_0} - y_a)}{AB_0} dy_b \tag{20.15}$$

Using this same method, the coefficients of dy_b for lines *BC* and *BD* are also determined, resulting in the *J* matrix shown in Table 20.1.

TABLE 20.1 The *J* Matrix of Figure 20.3

	Unknowns				
Distance	dy_b	dx_c	dy_c	dx_d	dy_d
AB	$[(x_b - x_a)\tan \alpha + (y_b - y_a)]/AB$	0	0	0	0
AC	0	$(x_c - x_a)/AC$	$(y_c - y_a)/AC$	0	0
AD	0	0	0	$(x_d - x_a)/AD$	$(y_d - y_a)/AD$
BC	$[(x_b - x_c)\tan \alpha + (y_b - y_c)]/BC$	$(x_c - x_b)/BC$	$(y_c - y_b)/BC$	0	0
BD	$[(x_b - x_d)\tan \alpha + (y_b - y_d)]/BD$	0	0	$(x_d - x_b)/BD$	$(y_d - y_b)/BD$
CD	0	$(x_c - x_d)/CD$	$(y_c - y_d)/CD$	$(x_d - x_c)/CD$	$(y_d - y_c)/CD$

For this example, the K, X, and V matrices are

$$K = \begin{bmatrix} AB - AB_0 \\ AC - AC_0 \\ AD - AD_0 \\ BC - BC_0 \\ BD - BD_0 \\ CD - CD_0 \end{bmatrix} \quad X = \begin{bmatrix} dy_b \\ dy_c \\ dx_c \\ dy_d \\ dx_d \end{bmatrix} \quad V = \begin{bmatrix} v_{ab} \\ v_{ac} \\ v_{ad} \\ v_{bc} \\ v_{bd} \\ v_{cd} \end{bmatrix}$$

20.4 HELMERT'S METHOD



Another method of introducing constraints was originally presented by Friedrich R. Helmert in 1872. In this procedure, the constraint equation(s) border the reduced normal equations as

$$\begin{bmatrix} A^{\mathrm{T}}WA & \vdots & C^{\mathrm{T}} \\ \cdots & \cdots & \cdots \\ C & \vdots & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ \cdots \\ X_2 \end{bmatrix} = \begin{bmatrix} A^{\mathrm{T}}WL_1 \\ \cdots \\ L_2 \end{bmatrix} \tag{20.16}$$

To establish this matrix, the normal matrix and its matching constants matrix are formed as has been done in Chapters 13 through 19. Following this, the observation equations for the constraints are formed. These observation equations are then included in the normal matrix as additional rows [C] and columns in Equation (20.16), and their constants are added to the constants' matrix as additional rows [L_2] in Equation (20.16). The inverse of this bordered normal matrix is computed. The matrix solution of the Equation (20.16) is

$$\begin{bmatrix} X_1 \\ \cdots \\ X_2 \end{bmatrix} = \begin{bmatrix} A^{\mathrm{T}}WA & \vdots & C^{\mathrm{T}} \\ \cdots & \cdots & \cdots \\ C & \vdots & 0 \end{bmatrix}^{-1} \begin{bmatrix} A^{\mathrm{T}}WL_1 \\ \cdots \\ L_2 \end{bmatrix} \tag{20.17}$$

In Equation (20.17), X_2 is not used in the subsequent solution for the unknowns. This procedure is illustrated in the following examples. Both Example 20.2 and 20.3 are solved in the spreadsheet *Chapter 18.xls*, which is available on the book's companion website (http://www.wiley.com/college/ghilani). In this spreadsheet, the examples are first solved using Helmert's method. Following this, they are solved using the elimination of constraints method. No matter the method used, the same results are always obtained.

Example 20.2 A constrained differential leveling adjustment is shown in Figure 20.6. Differential elevations were observed for a network where the elevation difference between stations B and E is to be held at -17.60 ft. The elevation of A is 1300.62 ft, and the observed elevation differences for each line are shown below.

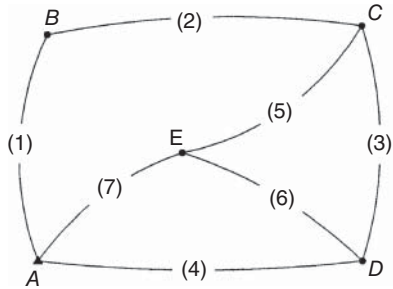


FIGURE 20.6 Differential leveling network.

Line	From	To	Elevation (ft)	S (ft)
1	A	B	25.15	0.07
2	B	C	−10.57	0.05
3	C	D	−1.76	0.03
4	D	A	−12.65	0.08
5	C	E	−7.06	0.03
6	E	D	5.37	0.05
7	E	A	−7.47	0.05

Perform a least-squares adjustment of this level net, constraining the required elevation difference.

SOLUTION The A, X, and L matrices are

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad X = \begin{bmatrix} B \\ C \\ D \\ E \end{bmatrix} \quad L = \begin{bmatrix} 1325.77 \\ -10.57 \\ -1.76 \\ -1313.27 \\ -7.06 \\ 5.37 \\ -1308.09 \end{bmatrix}$$

The weight matrix (W) is

$$W = \begin{bmatrix} 204.08 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 400 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1111.11 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 156.25 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1111.11 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 400 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 400 \end{bmatrix}$$

The reduced normal equations are

$$\begin{bmatrix} 604.08 & -400.00 & 0.00 & 0.00 \\ -400.00 & 2622.22 & -1111.11 & -1111.11 \\ 0.00 & -1111.11 & 1667.36 & -400.00 \\ 0.00 & -1111.11 & -400.00 & 1911.11 \end{bmatrix} \begin{bmatrix} B \\ C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 274,793.30 \\ 5572.00 \\ 205,390.90 \\ 513,243.60 \end{bmatrix} \qquad (a)$$

The reduced normal matrix is now bordered by the constraint equation

$$E - B = -17.60$$

which has a matrix form of

$$[-1 \;\; 0 \;\; 0 \;\; 1] \begin{bmatrix} B \\ C \\ D \\ E \end{bmatrix} = [-17.60] \qquad (b)$$

The left side of Equation (*b*) is now included as an additional row and column to the reduced normal matrix in Equation (*a*). The lower-right corner diagonal element of the newly bordered normal matrix is assigned a value of 0. Similarly, the right-hand side of Equation (*b*) is added as an additional row in the right-hand side of Equation (*a*). Thus, the bordered-normal equations are

$$\begin{bmatrix} 604.80 & -400.00 & 0.00 & 0.00 & -1 \\ -400.00 & 2622.22 & -1111.11 & -1111.11 & 0 \\ 0.00 & -1111.11 & 1667.36 & -400.00 & 0 \\ 0.00 & -1111.11 & -400.00 & 1911.11 & 1 \\ -1 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} B \\ C \\ D \\ E \\ X_2 \end{bmatrix} = \begin{bmatrix} 274,793.30 \\ 5572.00 \\ 205,390.90 \\ 513,243.60 \\ -17.60 \end{bmatrix} \qquad (c)$$

Notice in Equation (*c*) that an additional unknown, X_2, is added at the bottom of the X to make the X matrix dimensionally consistent with the bordered-normal matrix of Equation (*c*). Similarly, the right-hand side of the constraint Equation (*b*) is added to the bottom of the matrix. Using Equation (20.17), the resulting solution is

$$X = \begin{bmatrix} 1325.686 \\ 1315.143 \\ 1313.390 \\ 1308.086 \\ -28.003 \end{bmatrix} \qquad (d)$$

From the X matrix in Equation (*d*), the elevation of station B is 1325.686, and that for station E is 1308.086. Thus, the elevation difference between stations B and E is exactly -17.60, which was required by the constraint condition.

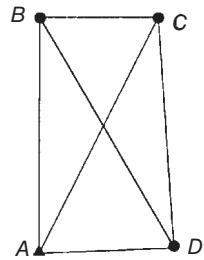


FIGURE 20.7 Network for Example 20.3

Example 20.3 Helmert's method can also be used to constrain the direction of a line. In Figure 20.7, the bearing of line *AB* is to remain at its record value of N 0°04′E. The data for this trilaterated network are:

Control Stations

Station	*X* (m)	*Y* (m)
A	1000.000	1000.000

Station Initial Approximations

Station	*X* (m)	*Y* (m)
B	1003.07	3640.00
C	2323.07	3638.46
D	2496.08	1061.74

Distance Observations

From	To	Distance (m)	σ (m)	From	To	Distance (m)	σ (m)
A	*C*	2951.604	0.025	*C*	*D*	2582.534	0.024
A	*B*	2640.017	0.024	*D*	*A*	1497.360	0.021
B	*C*	1320.016	0.021	*B*	*D*	2979.325	0.025

Adjust this figure by the method of least-squares holding the direction of the line *AB* using Helmert's method.

SOLUTION Using procedures discussed in Chapter 14, the reduced normal equations for the trilaterated system are

$$\begin{bmatrix} 2690.728 & -706.157 & -2284.890 & 0.000 & -405.837 & 706.157 \\ -706.157 & 2988.234 & 0.000 & 0.000 & 706.157 & -1228.714 \\ -2284.890 & 0.000 & 2624.565 & 529.707 & -8.737 & 124.814 \\ 0.000 & 0.000 & 529.707 & 3077.557 & 124.814 & -1783.060 \\ 405.837 & 706.157 & -8.737 & 124.814 & 2636.054 & -742.112 \\ 706.157 & -1228.714 & 124.814 & -1783.060 & -742.112 & 3015.328 \end{bmatrix} \times \begin{bmatrix} dx_b \\ dy_b \\ dx_c \\ dy_c \\ dx_d \\ dy_d \end{bmatrix} = \begin{bmatrix} -6.615 \\ 6.944 \\ 21.601 \\ 17.831 \\ 7.229 \\ 11.304 \end{bmatrix} \qquad (e)$$

Following prototype Equation (15.9), the linearized equation for the azimuth of line AB is

$$\begin{bmatrix} 78.13 & -0.09 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} dx_b \\ dy_b \\ dx_c \\ dy_c \\ dx_d \\ dy_d \end{bmatrix} = [0.139] \qquad (f)$$

The observation equation for the constrained direction [Equation (f)] is then added to the border of the matrix of reduced normal Equations (e), which yields

$$\begin{bmatrix} 2690.728 & -706.157 & -2284.890 & 0.000 & -405.837 & 706.157 & 78.13 \\ -706.157 & 2988.234 & 0.000 & 0.000 & 706.157 & -1228.714 & -0.09 \\ -2284.890 & 0.000 & 2624.565 & 529.707 & -8.737 & 124.814 & 0.000 \\ 0.000 & 0.000 & 529.707 & 3077.557 & 124.814 & -1783.060 & 0.000 \\ 405.837 & 706.157 & -8.737 & 124.814 & 2636.054 & -742.112 & 0.000 \\ 706.157 & -1228.714 & 124.814 & -1783.060 & -742.112 & 3015.328 & 0.000 \\ 78.13 & -0.09 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \end{bmatrix}$$

$$\times \begin{bmatrix} dx_b \\ dy_b \\ dx_c \\ dy_c \\ dx_d \\ dy_d \\ dx_2 \end{bmatrix} = \begin{bmatrix} -6.615 \\ 6.944 \\ 21.601 \\ 17.831 \\ 7.229 \\ 11.304 \\ 0.139 \end{bmatrix} \qquad (g)$$

This is a nonlinear problem, and thus the solution must be iterated until convergence. The first two iterations yielded the X matrices listed as X_1 and X_2 below. The third iteration resulted in negligible corrections to the unknowns. The total of these corrections is shown as X_T below.

$$X_1 = \begin{bmatrix} 0.00179 \\ 0.00799 \\ 0.00636 \\ 0.01343 \\ 0.00460 \\ 0.01540 \\ -0.00337 \end{bmatrix} \quad X_2 = \begin{bmatrix} -0.00001 \\ -0.00553 \\ 0.00477 \\ -0.00508 \\ -0.00342 \\ -0.00719 \\ -0.00359 \end{bmatrix} \quad X_T = \begin{bmatrix} 0.00178 \\ 0.00247 \\ 0.01113 \\ 0.00835 \\ 0.00117 \\ 0.00821 \\ -0.00696 \end{bmatrix}$$

Adding the coordinate corrections of X_T to the initial approximations results in the final coordinates for stations B, C, and D of

B: (1003.072, 3640.003) C: (2323.081, 3638.468) D: (2496.081, 1061.748)

Checking the solution: Using Equation (15.1), check to see that the direction of line *AB* was held to the value of the constraint.

$$\text{Az}_{AB} = \tan^{-1}\left(\frac{3.072}{2640.003}\right) = 0°04'00'' \text{ (Check!)}$$

20.5 REDUNDANCIES IN A CONSTRAINED ADJUSTMENT



The number of redundancies in an adjustment increases by one for each parameter that is removed by a constraint equation. An expression for determining the number of redundancies is

$$r = m - n + c \tag{20.11}$$

where r is the number of redundancies in the system; m is the number of observations in the system; n is the number of unknown parameters in the system; and c is the number of mathematically independent constraints applied to the system. In Example 20.2, there were seven observations in a differential leveling network that had four stations with unknown elevations. One constraint was added to the system of equations that fixed the elevation difference between *B* and *E* as −17.60. In this way, the elevation of *B* and *E* became mathematically dependent. By applying Equation (20.18), the number of redundancies in the system is $r = 7 - 4 + 1 = 4$. Without the constraint, this adjustment would have only $7 - 4 = 3$ redundancies. Thus, the constraint added one redundant observation to the adjustment while making the elevations of *B* and *E* mathematically dependent.

Care must be used when adding constraints to an adjustment. It would be possible to add as many mathematically independent constraint equations as there are unknown parameters. If that were done, all unknowns would be constrained or fixed, and it would be impossible to perform an adjustment. Care should also be exercised when adding observations between constraints. For example, it is impossible to adjust the distance between two control stations whose coordinates are fixed since it is impossible to change the distance. Furthermore, it is also possible to add constraints that are mathematically dependent equations. Under these circumstances, even if the system of equations has a solution, two mathematically dependent constraints would remove only one unknown parameter, and thus the redundancies in the system would increase by only one.

20.6 ENFORCING CONSTRAINTS THROUGH WEIGHTING

The methods described previously for handling constraint equations can often be avoided simply by overweighting the observations to be constrained in a weighted least-squares adjustment. This was done in Example 16.2 to fix the direction of

a line. As a further demonstration of the procedure of enforcing constraints by overweighting, Example 20.3 will be adjusted by writing observation equations for azimuth AB and the control station coordinates X_A and Y_A. These observations will be fixed by assigning a $0.001''$ standard deviation to the azimuth of line AB and standard deviations of 0.001 ft to the coordinates of Station A.

The J, K, and W matrices for the first iteration of this problem are listed below. Note that the numbers have been rounded to three decimal places for display purposes only.

$$J = \begin{bmatrix} -0.448 & -0.894 & 0.000 & 0.000 & 0.448 & 0.894 & 0.000 & 0.000 \\ -0.001 & -1.000 & 0.001 & 1.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 0.000 & -1.000 & 0.001 & 1.000 & -0.001 & 0.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 0.000 & -0.067 & 0.998 & 0.067 & -0.997 \\ -0.999 & -0.041 & 0.000 & 0.000 & 0.000 & 0.000 & 0.999 & 0.041 \\ 0.000 & 0.000 & -0.050 & 0.865 & 0.000 & 0.000 & 0.050 & -0.865 \\ -78.130 & 0.091 & 78.130 & -0.091 & 0.000 & 0.000 & 0.000 & 0.000 \\ 1.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \\ 0.000 & 1.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 & 0.000 \end{bmatrix}$$

$$K = \begin{bmatrix} -0.003 \\ 0.015 \\ 0.015 \\ 0.012 \\ 0.007 \\ -0.021 \\ 0.139 \\ 0.000 \\ 0.000 \end{bmatrix}$$

$$W = \begin{bmatrix} \frac{1}{0.025^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{0.024^2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{0.021^2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{0.024^2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{0.021^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{0.025^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{\rho^2}{0.001^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{0.001^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{0.001^2} \end{bmatrix}$$

The results of the adjustment (from the program ADJUST) are shown in Figure 20.8. Notice in the previous adjustment that the control station coordinates remained fixed, and the residual of the azimuth of line AB is zero. Thus, the azimuth of line AB was held fixed without the inclusion of any constraint equation. It was simply constrained by overweighting the observation. Also note that the final adjusted coordinates of stations B, C, and D match the solution in Example 20.3.

```
*****************
Adjusted stations
*****************
Standard error ellipses computed
Station          X              Y          Sx         Sy         Su         Sv       t
=============================================================================================
     A      1,000.000      1,000.000     0.0010     0.0010     0.0010     0.0010 135.00°
     B      1,003.072      3,640.003     0.0010     0.0217     0.0217     0.0010   0.07°
     C      2,323.081      3,638.468     0.0205     0.0248     0.0263     0.0186 152.10°
     D      2,496.081      1,061.748     0.0204     0.0275     0.0281     0.0196  16.37°

*****************************
Adjusted Distance Observations
*****************************
   Station    Station
  Occupied   Sighted        Distance          V          S
=============================================================
         A          C      2,951.620     0.0157     0.0215
         A          B      2,640.004    -0.0127     0.0217
         B          C      1,320.010    -0.0056     0.0205
         C          D      2,582.521    -0.0130     0.0215
         D          A      1,497.355    -0.0050     0.0206
         B          D      2,979.341     0.0159     0.0214

*****************************
Adjusted Azimuth Observations
*****************************
   Station    Station
  Occupied   Sighted           Azimuth       V        S"
=========================================================
         A          B       0° 04' 00"     0.0"     0.0"

                  ****************************************
                          Adjustment Statistics
                  ****************************************
                             Iterations = 2
                           Redundancies = 1
                      Reference Variance = 1.499
                           Reference So = ±1.2

Passed X² test at 95.0% significance level!
X² lower value = 0.00   X² upper value = 5.02
A priori value of 1 used for reference variance in computations of statistics.
Convergence!
```

FIGURE 20.8 Results of Example 20.3's adjustment by overweighting.

PROBLEMS

Note: For problems requiring least-squares adjustment, if a computer program is not distinctly specified for use in the problem, it is expected that the least-squares algorithm will be solved using the MATRIX program, which is available on the book's companion website. Partial answers to problems that are marked with an asterisk are provided in Appendix G.

***20.1** Given the following observed lengths in a trilateration survey, adjust the survey by least-squares using the elimination of constraints method to hold the coordinates of A at $x_a = 30{,}000.00$ ft and $y_a = 30{,}000.00$ ft, and

the azimuth of line AB to $2°31'56'' \pm 0.001''$ from North. Find the adjusted coordinates of B, C, and D.

Distance Observations

Course	Distance (ft)	S (ft)	Course	Distance (ft)	S (ft)
AB	9,813.20	0.033	BC	9,629.69	0.033
AC	13,764.18	0.044	BD	15,006.82	0.047
AD	11,395.99	0.037	CD	9,943.63	0.033

Initial Coordinates

Station	X (ft)	Y (ft)
B	30,433.56	39,803.63
C	40,054.84	39,399.65
D	41,386.93	29,545.64

20.2 Do Problem 20.1 using Helmert's method.

20.3 Do Problem 20.1 using the method of weighting for the azimuth.

20.4 Repeat Problem 20.1 using the following data.

Control Station

Station	X (ft)	Y (ft)
A	1,569,233.09	23,782.75

Approximate Coordinates

Station	X (ft)	Y (ft)
B	1,569,892.48	23,813.96
C	1,570,009.13	24,290.38
D	1,569,470.19	24,353.92

Distance Observations

Occupied	Sighted	Distance (ft)	S (ft)
A	B	660.12	0.015
B	C	490.48	0.015
C	D	542.67	0.015
D	A	618.40	0.015
A	C	927.34	0.015

Azimuth Observation

Course	Azimuth	S
AB	$87°17'22''$	$4.1''$

20.5 Do Problem 20.4 using Helmert's method.

20.6 Do Problem 20.4 using the method of weighting for the azimuth.

***20.7** Adjust the following differential leveling data using the elimination of constraints method. Hold the elevation of A to 1314.568 m and the elevation difference $\Delta Elev_{BD}$ to −7.804 m.

From	To	ΔElev (m)	S (mm)
A	B	−19.858	0.56
B	C	9.877	0.66
D	E	−4.092	0.50
E	A	16.147	0.71
C	D	−2.072	0.56

20.8 Repeat Problem 20.7 using Helmert's method.

20.9 Repeat Problem 20.7 using the overweighting technique.

20.10 Do Problem 14.9, holding distance AB to 3243.014 m and distance EF to 3569.299 m using Helmert's method.

20.11 Repeat Problem 20.10 using the overweighting technique.

***20.12** If Stations A and F had relative precisions of 1:20,000, what would be the appropriate standard deviations for the stations' coordinates if they were adjusted in Problems 20.10 and 20.11?

20.13 Repeat Problem 20.11 but adjust the control coordinates for stations A and F with relative precisions of 1:50,000.

20.14 In Problem 15.8, station A has standard deviations in X and Y of ±0.003 ft and ±0.002 ft, respectively. Station D has standard deviations in X and Y of ±0.015 ft and ±0.012 ft, respectively. Adjust the data using the overweighting method.

20.15 Using ADJUST, adjust Problem 15.11, assuming the control stations A and B have uncertainties in X and Y of ±0.0007 m and ±0.0009 m, respectively. Stations G and H have uncertainties of ±0.006 m and ±0.008 m, respectively. Use the overweighting method to determine the coordinates of the stations, their uncertainties, and error ellipses.

PRACTICAL PROBLEMS

20.16 Develop a computational program that computes the coefficients for the J matrix in a trilateration adjustment with a constrained azimuth. Use the program to solve Problem 20.9.

20.17 Develop a computational program that computes a constrained least-squares adjustment of a trilateration network using Helmert's method. Use this program to solve Problem 20.13.

CHAPTER 21

BLUNDER DETECTION IN HORIZONTAL NETWORKS

21.1 INTRODUCTION

Up to this point, data sets were assumed to be free of blunders. However, when adjusting real observations, the data sets are seldom blunder-free. Not all blunders are large, but no matter their sizes, it is desirable to remove them from the data set. In this chapter, methods used to detect blunders before and after an adjustment are discussed.

Many examples can be cited that illustrate mishaps that have resulted from undetected blunders in survey data. However, a few could have been more costly and embarrassing than a blunder of about 1 mile that occurred in an early nineteenth-century survey on the border between the United States and Canada near the north end of Lake Champlain. Following the survey, construction of a US military fort was begun. The project was abandoned two years later when the blunder was detected, and a resurvey showed that the fort was actually located on Canadian soil. The abandoned facility was subsequently named "Fort Blunder!"

As discussed in previous chapters, observations are normally distributed. This means that occasionally large random errors will occur. However, in accordance with theory, this seldom happens. Thus, large errors in data sets are more likely to be blunders than random errors. Common blunders in data sets include number

transposition, entry and recording errors, station misidentifications, and others. When blunders are present in a data set, a least-squares adjustment may not be possible or will, at a minimum, produce poor or invalid results. To be safe, the results of an adjustment should never be accepted without an analysis of the post-adjustment statistics.

21.2 A PRIORI METHODS FOR DETECTING BLUNDERS IN OBSERVATIONS

 In performing adjustments, it should always be assumed that there are possible observational blunders in the data. Thus, appropriate methods should be used to isolate and remove them. It is especially important to eliminate large blunders prior to the adjustment of nonlinear equations, since blunders can cause the solution to diverge. In this section, several methods are discussed that can be used to isolate blunders in a horizontal adjustment.

21.2.1 Use of the *K* Matrix

In horizontal surveys, the easiest method available for detecting blunders is to use redundant observations. When the approximate station coordinates are computed using standard surveying methods, they should be *close* to their final adjusted values. Thus, the differences between observations computed from the approximate coordinates and their observed values (K matrix) are expected to be small. If an observational blunder is present, there are two possible situations that can occur about K-matrix values. If the observation containing a blunder is not used to compute initial coordinates, its corresponding K-matrix value will be relatively large. However, if an observation with a blunder is used in the computation of the initial station coordinates, the remaining redundant observations to that station will have relatively large values.

Figure 21.1 shows the two possible situations. In Figure 21.1(a), a distance blunder is present in line BP and is shown by the length PP'. However, this distance was not used in computing the coordinates of station P, and thus the K-matrix value for $BP' - BP_0$ will suggest the presence of a blunder by its relatively large size. In Figure 21.1(b), the distance blunder in BP was used to compute the initial coordinates of station P'. In this case, the redundant angle and distance observations connecting P with A, C, and D will show large discrepancies in the K matrix. In the latter case, it is possible that some redundant observations may agree reasonably with their computed values, since a shift in a station's position can occur along a sight line for an angle, or along a radius for a distance. Still, most redundant observations will have large K-matrix values, and thus raise suspicions that a blunder exists in one of the observations used to compute the coordinates of station P.

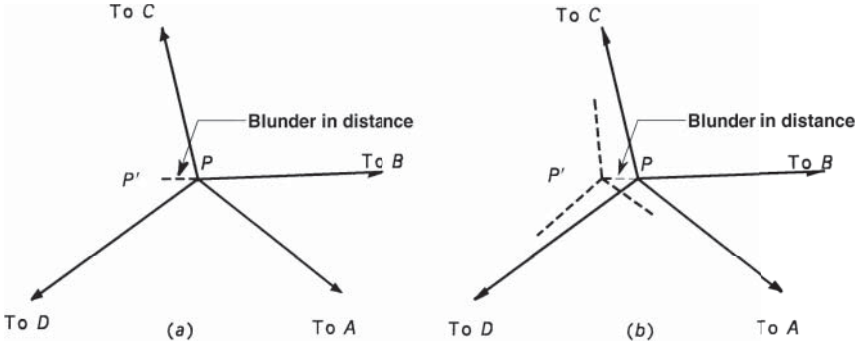


FIGURE 21.1 Presence of a distance blunder in computations; (a) the distance blunder was not used in the computations of coordinates for station *P* and (b) the distance blunder was used in the computations of coordinates *P*.

21.2.2 Traverse Closure Checks

As mentioned in Chapter 8, errors can be propagated throughout a traverse to determine the anticipated closure error. Large complex networks can be broken into smaller link and loop traverses to check estimated closure errors against their actual values. When a loop fails to meet its estimated closure, the observations included in the computations should be checked for blunders.

Figure 21.2(*a*) and (*b*) shows a graphical technique to isolate a traverse distance blunder and an angular blunder, respectively. In Figure 21.2(*a*), a blunder in distance *CD* is shown. Notice that the remaining courses, *DE* and *EA*, are translated by the blunder in the direction of course *CD*. Thus, the length of the closure line ($A'A$) will be nearly equal to the length of the blunder in *CD,* with a direction that is consistent with the azimuth of *CD*. Since other observations contain small random

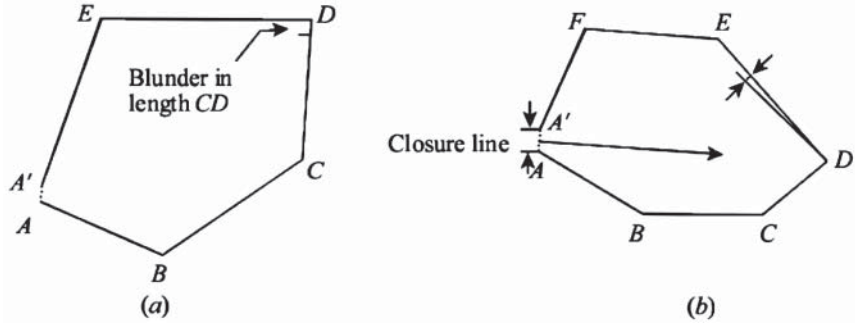


FIGURE 21.2 Effects of a single blunder on the traverse closure error; (a) a distance blunder and (b) an angular blunder.

errors, the length and direction of the closure line, $A'A$, will not match the blunder exactly. However, when one blunder is present in a traverse, the misclosure and the blunder will be close in both length and direction.

In the traverse of Figure 21.2(*b*), the effect of an angular blunder at traverse station *D* is illustrated. As shown, the courses *DE*, *EF*, and FA' will be rotated about station *D*. Thus the perpendicular bisector of the closure line AA' will point to station *D*. Again, due to random errors in other observations, the perpendicular bisector may not intersect the observation with the blunder precisely, but it should be close enough to identify the angle with the blunder. Since the angle at the initial station is not used in traverse computations, it is possible to isolate a single angular blunder by beginning traverse computations at the station with the suspected blunder. In this case, when the blunder is not used in the computations, estimated misclosure errors will be met, and the blunder can be isolated to the single unused angle. Thus, in Figure 21.2(*b*), if the traverse computations were started at station *D* and used an assumed azimuth for the course of *CD*, the traverse misclosure when returning to *D* would be within estimated tolerance since the angle at *D* is not used in the traverse computations. Of course, this means that the angles should not be adjusted prior to the traverse computations.

21.3 A POSTERIORI BLUNDER DETECTION

When doing a least-squares adjustment involving more than the minimum amount of control, both a *minimally constrained* and an *over-constrained* adjustment should be performed. In a *minimally constrained adjustment*, the data needs to satisfy the appropriate geometric closures and are not influenced by any control errors. After the adjustment, a χ^2 test[1] can be used to check the *a priori* value of the reference variance against its *a posteriori* estimate. However, this test is not a good indicator of the presence of a blunder since it is sensitive to poor relative weighting. Thus, the a posteriori residuals should also be checked for the presence of large discrepancies. If no large discrepancies are present, the observational weights should be altered, and the adjustment rerun. Since the χ^2 test is sensitive to weights, the procedures described in Chapters 7 through 10 should be used for building the stochastic model of the adjustment.

Besides the sizes of the residuals, the *signs of the residuals* may also indicate a problem in the data. From normal probability theory, residuals are expected to be small and randomly distributed. In Figure 21.3, a small section of a larger network is shown. Notice that the distance residuals between stations *A* and *B* are all positive. This is not expected from normally distributed data. Thus, it is possible that either a blunder or systematic error is present in some or all of the survey. If both *A* and *B* are control stations, part of the problem could stem from control coordinate discrepancies. This possibility can be isolated by doing a minimally constrained adjustment.

[1] Statistical testing was discussed in Chapter 5.

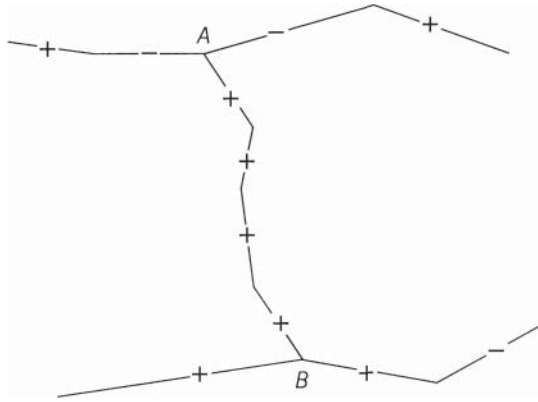


FIGURE 21.3 Distribution of residuals by sign.

Although residual sizes can suggest observational errors, they do not necessarily identify the observations that contain blunders. This is because least-squares solution generally spreads a large observational error or blunder out from its source radially. However, this condition is not unique to least-squares adjustments, since any arbitrary adjustment method, including the compass rule for traverse adjustment, will also spread a single observational error throughout the entire observational set.

While an abnormally large residual may suggest the presence of a blunder in an observation, this is not always the case. One reason for this could be poor relative weighting in the observations. For example, suppose that angle *GAH* in Figure 21.4 has a small blunder but has been given a relatively high weight. In this case, the

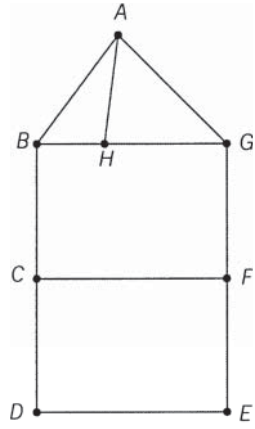


FIGURE 21.4 Survey network.

largest residual may well appear in a length between stations G and H, B and H, C and F, and most noticeably between D and E due to their distances from station A. This is because the angular blunder will cause the network to spread or compress. When this happens, the sign of the distance residuals between G and H, B and H, C and F, and D and E may all be the same and thus, may indicate the problem. Again, this situation can be minimized by using proper methods to determine observational variances so they are truly reflective of the estimated errors in the observations.

21.4 DEVELOPMENT OF THE COVARIANCE MATRIX FOR THE RESIDUALS

 In Chapter 5, it was shown how a sample data set could be tested at any confidence level to isolate observational residuals that were too large. The concept of statistical blunder detection in surveying was introduced in the mid-1960s and utilizes the cofactor matrix for the residuals. To develop this matrix, the adjustment of a linear problem can be expressed in matrix form as

$$L + V = AX + C \tag{21.1}$$

where C is a vector of constants; A is the coefficient matrix; X is the estimated parameter matrix; L is the observation matrix; and V is the residual vector. Equation (21.1) can be rewritten in terms of V as

$$V = AX - T \tag{21.2}$$

where $T = L - C$ that has a covariance matrix of $W^{-1} = S^2 Q_{ll}$. The solution of Equation (21.2) results in the expression

$$X = (A^T WA)^{-1} A^T WT \tag{21.3}$$

Letting ε represent a vector of true errors for the observations, Equation (21.1) can be written as

$$L - \varepsilon = A\overline{X} + C \tag{21.4}$$

where $\overline{X}$ is the true value for the unknown parameter X and thus

$$T = L - C = A\overline{X} + \varepsilon \tag{21.5}$$

Substituting Equations (21.3) and (21.5) into Equation (21.2) yields

$$V = A(A^T WA)^{-1} A^T W(AX + \varepsilon) - (A\overline{X} + \varepsilon) \tag{21.6}$$

Expanding Equation (21.6) results in

$$V = A(A^T WA)^{-1}A^T W\varepsilon - \varepsilon + A(A^T WA)^{-1}A^T WA\overline{X} - A\overline{X} \tag{21.7}$$

Since $(A^{\mathrm{T}}WA)^{-1} = A^{-1}W^{-1}A^{-\mathrm{T}}$, Equation (21.7) can be simplified to

$$V = A(A^T WA)^{-1}A^T W\varepsilon - \varepsilon \tag{21.8}$$

Factoring W_ε from Equation (21.8) yields

$$V = -(W^{-1} - A(A^T WA)^{-1}A^T)W\varepsilon \tag{21.9}$$

Recognizing $(A^{\mathrm{T}}WA)^{-1} = Q_{xx}$ and defining $Q_{vv} = W^{-1} - AQ_{xx}A^{\mathrm{T}}$, Equation (21.9) can be rewritten as

$$V = -Q_{vv}W\varepsilon \tag{21.10}$$

where $Q_{vv} = W^{-1} - AQ_{xx}A^{\mathrm{T}} = W^{-1} - Q_{ll}$.

If we let R be the product of W and Q_{vv}, the R matrix is both singular and *idempotent*. Being singular, it has no inverse. When a matrix is idempotent, the following properties exist for the matrix: (1) The square of the matrix is equal to the original matrix (i.e., $RR = R$), (2) every diagonal element is between zero and 1, and (3) the sum of the diagonal elements, known as the *trace of the matrix*, equals the degrees of freedom in the adjustment. This latter property is expressed mathematically as

$$r_{11} + r_{22} + \cdots + r_{mm} = \text{degrees of freedom} \tag{21.11}$$

(4) The sum of the square of the elements in any single row or column equals the diagonal element. That is,

$$r_{ii} = r_{i1}^2 + r_{i2}^2 + \cdots + r_{im}^2 = r_{1i}^2 + r_{2i}^2 + \cdots + r_{mi}^2 \tag{21.12}$$

Now consider the case when all observations have zero errors except for a particular observation l_i that contains a blunder of size Δl_i. A vector of the true errors is expressed as

$$\Delta\varepsilon = \Delta l_i \varepsilon_i = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \Delta l_i \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \Delta l_i \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{21.13}$$

If the original observations are uncorrelated, the specific correction for Δv_i can be expressed as

$$\Delta v_i = -q_{ii} w_{ii} \Delta l_i = -r_i \Delta l_i \tag{21.14}$$

where q_{ii} is the ith diagonal of the Q_{vv} matrix; w_{ii} is the ith diagonal term of the weight matrix, W; and $r_i = q_i w_i$ is the observation's *redundancy number*.

When the system has a unique solution, r_i will equal zero. If the observation is over-constrained, r_i would equal one. The redundancy numbers provide insight into the geometric strength of the adjustment. An adjustment that, in general, has low redundancy numbers will have observations that lack sufficient checks to isolate blunders, and thus the chance for undetected blunders to exist in the observations is high. Conversely, a high overall redundancy number enables a high level of internal checking of the observations, and thus there is a lower chance of accepting observations that contain blunders. The quotient of r/m is called the *relative redundancy* of the adjustment, where r is the total number of redundant observations in the system and m is the number of observations. As discussed in Section 21.9.1, a well-designed network will have individual observational redundancy numbers that are all close to the relative redundancy number.

21.5 DETECTION OF OUTLIERS IN OBSERVATIONS: DATA SNOOPING

 Equation (21.10) defines the covariance matrix for the vector of residuals, v_i. From this, the *standardized residual* is computed using the appropriate diagonal element of the Q_{vv} matrix as

$$\overline{v}_i = \frac{v_i}{\sqrt{q_{ii}}} \tag{21.15}$$

where $\overline{v}_i$ is the standardized residual; v_i is the computed residual; and q_{ii} is the diagonal element of the Q_{vv} matrix. Using the Q_{vv} matrix, the standard deviation in the residual is $S_0\sqrt{q_{ii}}$. Thus, as suggested by Willem Baarda (1968), if the denominator of Equation (21.15) is multiplied by S_0, a *t statistic* is defined. If the residual is significantly different from zero, the observation used to derive the statistic might be a blunder. The *test statistic* for this hypothesis test is

$$t_i = \frac{v_i}{S_0\sqrt{q_{ii}}} = \frac{v_i}{S_v} = \frac{\overline{v}_i}{S_0} \tag{21.16}$$

Baarda computed rejection criteria for various significance levels (see Table 21.1), determining the α and β levels for the Type I and Type II errors. The interpretation of these criteria is shown in Figure 21.5. When a blunder is present in the data set, the t distribution is shifted, and a statistical test for this shift may be performed. As with any other statistical test, two types of errors can occur.

TABLE 21.1 Rejection Criteria with Corresponding Significance Levels

α	$1-\alpha$	β	$1-\beta$	Rejection Criteria
0.05	0.95	0.80	0.20	2.8
0.001	0.999	0.80	0.20	4.1
0.001	0.999	0.999	0.001	6.6

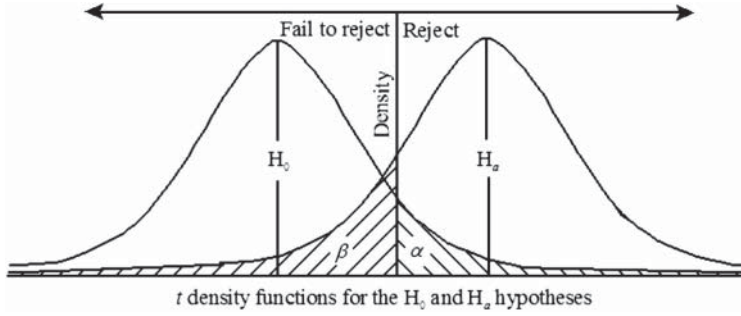


FIGURE 21.5 Effects of a blunder on the t distribution.

A Type I error occurs when data are rejected that do not contain blunders, and a Type II error occurs when a blunder is not detected in a data set where one is actually present. The rejection criteria are represented by the vertical line in Figure 21.5, and their corresponding significance levels are shown in Table 21.1. In practice, authors[2] have reported that a value of 3.29 also works as a criterion for rejection of blunders.

Thus, the approach is to use a rejection level given by a t distribution with $r-1$ degrees of freedom. The observation with the largest absolute value of t_i, as given by Equation (21.17), is rejected when it is greater than the rejection level. That is, the observation is rejected when

$$\frac{|v_i|}{S_0\sqrt{q_{ii}}} > \text{Rejection level} \tag{21.17}$$

Since the existence of any blunder in the data set will affect the remaining observations, and since Equation (21.17) is dependent on S_0 whose value was computed from data containing blunders, all observations that are detected as blunders should not be removed in a single pass. Instead, only the largest or the largest independent group of observations should be deleted. Furthermore, since Equation (21.17) is dependent on S_0, it is possible to rewrite the equation so that it can be computed

[2]References relating to the use of 3.29 as the rejection criteria are made in Amer (1979) and Harvey (1994).

during the final iteration of a nonlinear adjustment. In this case, the appropriate equation is

$$\overline{v}_i = \frac{|v_i|}{\sqrt{q_{ii}}} > S_0 \times \text{Rejection level} \tag{21.18}$$

A summary of procedures for this manner of blunder detection is as follows:

Step 1: Locate all standardized residuals that meet the rejection criteria of Equation (21.17) or (21.18).

Step 2: Remove the largest detected blunder or unrelated blunder groups.

Step 3: Rerun the adjustment.

Step 4: Continue steps 1 through 3 until all detectable blunders are removed.

Step 5: If more than one observation is removed in steps 1 through 4, reenter the observations in the adjustment in a *one-at-a-time* fashion. Check the observation after each adjustment to see if it is again detected as a blunder. If it is, remove it from the adjustment or have that observation reobserved.

Again, it should be noted that this form of blunder detection is sensitive to improper relative weighting in observations. Thus, it is important to use weights that are reflective of the observational errors. Again, proper methods of computing estimated errors in observations and weighting were discussed in Chapters 7 through 10.

21.6 DETECTION OF OUTLIERS IN OBSERVATIONS: THE TAU CRITERION



Data snooping is based on Equation (21.16) being a t statistic. However, when a blunder is present in a data set, both S_0 and S_v are affected by the blunder. Thus, Alan J. Pope (1976) stated that Equation (21.16) is instead a τ (tau) statistic, where the critical τ value is computed as

$$\tau_{\alpha/2} = \frac{t_{\alpha/2,r-1}\sqrt{r}}{\sqrt{r - 1 + t^2_{\alpha/2,r-1}}} \tag{21.19}$$

In Equation (21.19), the probability of the computed τ from Equation (21.16) being greater than $\tau_{\alpha/2}$ is $1 - (1 - \alpha)^{\frac{1}{n}}$, which is referred to as the transformation for control of the Type I error, where n is conventionally taken to be the number of *nonspur* observations in the adjustment. A nonspur observation is any observation, where σ_v and v are nonzero. Recalling Equation (21.16) and using the τ criterion,

observations are considered for rejection when

$$\frac{|\overline{v}_i|}{S_0} > \tau_{\alpha/2} \tag{21.20}$$

The procedures for eliminating and removing blunders using the τ statistic are the same as those used in data snooping. That is, the observation with the largest standardized residuals that is detected as an outlier or blunder using Equation (21.20) is removed from the data set, and the adjustment is rerun. This procedure is continued, removing a single observation at a time until no more observations are detected as outliers or blunders. Again, the rejected observations should be reinserted into the adjustment in a one-at-a-time fashion to determine if they are still detected as blunders. Any observations that are detected as a blunder a second time are discarded as blunders or must be reobserved.

Even though data snooping and the τ criterion are theoretically different, they have been shown in practice to yield similar results. Thus, the matter of which method of statistical blunder detection to use is a matter of personal preference. As shown in Figure 21.6, the software ADJUST allows the user to select the blunder detection method of their choice.

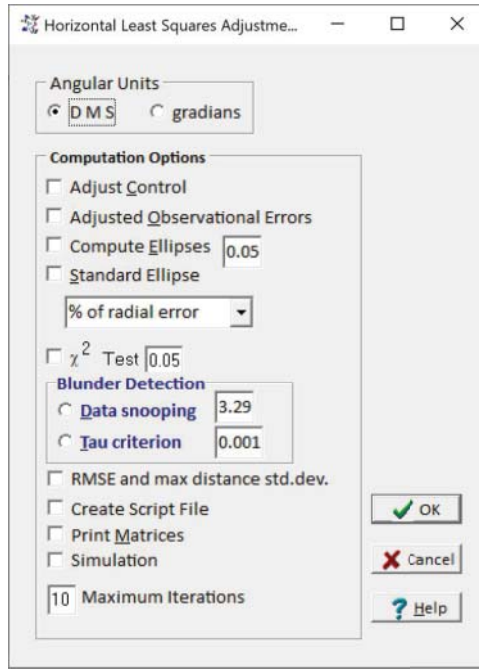


FIGURE 21.6 Options for horizontal adjustment in ADJUST.

21.7 TECHNIQUES USED IN ADJUSTING CONTROL

As discussed in Chapter 20, some control is necessary in each adjustment. However, since control itself is not perfect, this raises the question of how control should be managed. If control stations that contain errors are overweighted, the adjustment will associate the control errors with the observations. This effect can be removed by using only the minimum amount of control required to fix the project. Table 21.2 lists the type of survey versus the minimum amount of control. Thus, in a horizontal adjustment, if the coordinates of only one station and the direction of only one line are held fixed, the observations will not be constricted by the control. That is, the observations will only need to satisfy the internal geometric constraints of the network. If more than minimum control is used, these additional constraints will be factored into the adjustment.

Statistical blunder detection can help identify weak control or the presence of systematic errors in observations. Using a minimally constrained adjustment, the data set is screened for blunders. After becoming confident that the blunders are removed from the data set, an over-constrained adjustment is performed. Following the over-constrained adjustment, an F test is used to compare the ratio of the reference variances from the minimally constrained and over-constrained adjustments. The ratio of the two reference variances should be statistically equal to 1.[3]

If the two reference variances are found to be statistically different, two possible causes might exist. The first is that there are errors in the control that must be isolated and removed. The second is that the observations contain systematic errors. Since systematic errors are not compensating in nature, they will appear as blunders in the over-constrained adjustment. If systematic errors are suspected, they should be identified and removed from the original data set, and the entire adjustment

TABLE 21.2 Requirements for a Minimally Constrained Adjustment

Survey Type	Minimum Amount of Control
Differential leveling	1 benchmark
Horizontal survey	1 point with known *xy* coordinates
	1 course with known azimuth
GNSS survey	1 point with known geodetic coordinates

[3]The ratio of the reference variances from the minimally constrained and constrained adjustments should be one since both reference variances should be statistically equal. That is, $\sigma^2_{minimally\ constrained} = \sigma^2_{over\ constrained}$.

procedure performed again. If no systematic errors are identified,[4] different combinations of control stations should be used in the constrained adjustments until the problem is isolated. By following this type of systematic approach, a single control station that has inconsistent coordinates can be isolated.

With this stated, it should be realized that the ideal amount of control in each survey type is greater than the minimum. In fact, for all three survey types, a minimum of three fixed stations (control) is always preferable. For example, in a differential leveling survey with only two benchmarks, it would be impossible to isolate the problem by simply removing one benchmark from the adjustment. However, if three benchmarks are used, a separate adjustment containing only two of the benchmarks can be run until the offending benchmark is isolated. Obviously, if only one benchmark is available for an adjustment and it contains a blunder, then the results of the leveling adjustment will be inaccurate, no matter how well the network was executed.

Extreme caution should always be used when dealing with control stations. Although it is possible that a control station was disturbed or that the original published coordinates contained errors, this is not always the case. A prudent surveyor should check for physical evidence of disturbance and talk with other surveyors before deciding to discard control. If the station was set by a local, state, or federal agency, the surveyor should contact the proper authorities and report any suspected problems. People in the agency familiar with the control may help explain any apparent problem. For example, it is possible that the control used in the survey was established by two previously nonconnecting surveys. In this case, the relative accuracy of the stations was never checked when they were established. Another common problem with control is the connection of two control points from different realizations or epochs in the same reference frame. As an example, suppose two surveys both use the North American Terrestrial Reference Frame of 2022. However, a conventional survey has coordinates from epoch 2025.0, and a GNSS survey has coordinates from epoch 2035.0. The coordinates from the control in these two surveys will most likely not agree with each other due to plate tectonic motions.

21.8 A DATA SET WITH BLUNDERS

Example 21.1 The network shown in Figure 21.7 was established to provide control for mapping in the area of stations 1 through 6. It began from two National Geodetic Survey, Second Order, Class II (1:20,000 precision) control stations, 2000 and 2001. The data for the job were gathered by five different field crews in a class environment. The procedures discussed in Chapter 7 were used to estimate the observational errors. The problem is to check for blunders in the data set using the data snooping method with a rejection level of $3.29S_0$.

[4]When adjusting data that cover a large region (e.g., spherical excess, reduction to these ellipsoid, etc.), it is essential that geodetic corrections to the data be considered and applied where necessary. These corrections are systematic in nature and can cause errors when fitting to more than minimal control.

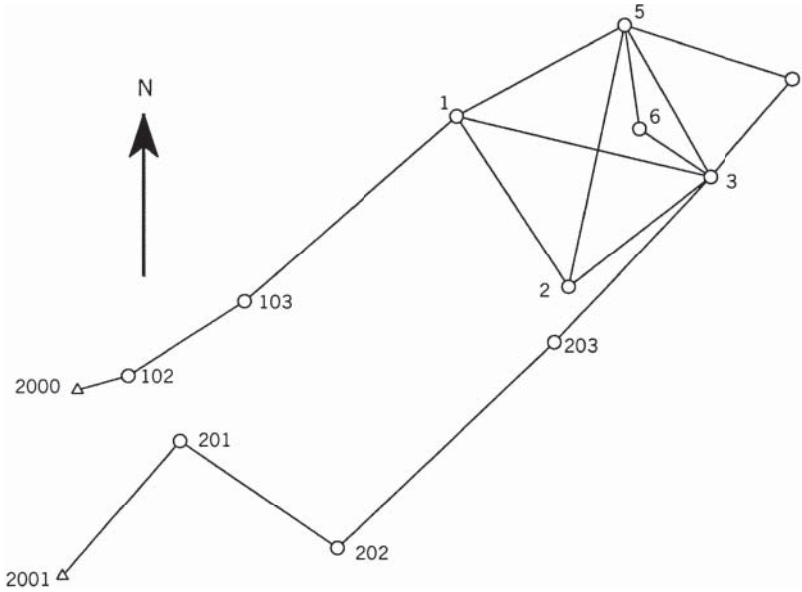


FIGURE 21.7 Standard error ellipse data for Example 21.1.

Control Stations

Station	Northing (ft)	Easting (ft)
2000	419,710.09	2,476,334.60
2001	419,266.82	2,476,297.98

Angle Observations

Backsight	Occupied	Foresight	Angle	S (″)
102	2000	2001	109°10′54.0″	25.5
2000	102	103	162°58′16.0″	28.9
102	103	1	172°01′43.0″	11.8
2000	2001	201	36°04′26.2″	7.4
2001	201	202	263°54′18.7″	9.7
201	202	203	101°49′55.0″	8.1
202	203	3	176°49′10.0″	8.4
203	3	2	8°59′56.0″	6.5
2	1	3	316°48′00.5″	6.3
3	5	4	324°17′44.0″	8.1
6	5	3	338°36′38.5″	10.7
1	5	3	268°49′32.5″	9.8
2	5	3	318°20′54.5″	7.0
2	3	1	51°07′11.0″	7.2
2	3	5	98°09′36.5″	10.3
2	3	6	71°42′51.5″	15.1
2	3	4	167°32′28.0″	14.5

Distance Observations

From	To	Distance (ft)	*S* (ft)
2001	201	425.90	0.022
201	202	453.10	0.022
202	203	709.78	0.022
203	3	537.18	0.022
5	3	410.46	0.022
5	4	397.89	0.022
5	6	246.61	0.022
5	1	450.67	0.022
5	2	629.58	0.022
3	2	422.70	0.022
3	1	615.74	0.022
3	5	410.44	0.022
3	6	201.98	0.022
3	4	298.10	0.022
1	2	480.71	0.022
1	3	615.74	0.022
2000	102	125.24	0.022
102	103	327.37	0.022
103	1	665.79	0.022

The approximate coordinates were computed for the stations as follows:

Station	Northing (ft)	Easting (ft)
1	420,353.62	2,477,233.88
2	419,951.98	2,477,497.99
3	420,210.17	2,477,832.67
4	420,438.88	2,478,023.86
5	420,567.44	2,477,630.64
6	420,323.31	2,477,665.36
102	419,743.39	2,476,454.17
103	419,919.69	2,476,728.88
201	419,589.24	2,476,576.25
202	419,331.29	2,476,948.76
203	419,819.56	2,477,463.90

SOLUTION Do the *a priori* check of the computed observations versus their *K*-matrix values. In this check, a multiplier of 1000*S*, where *S* is the standard deviation of the observation, was used as a tolerance. An ADJUST data file, *Example 21-1.dat*, is available on the companion website for this book. This file has the same format as the data file used in a horizontal least-squares adjustment. The option to perform an a priori check of data files is found under the *estimated errors* submenu of the *programs* menu in ADJUST.

In this check, the distance of 3–4 had a discrepancy of 50 ft from its computed value. Additionally, the angle 102–103–1 differed from its computed value by more than 2°. These differences indicate that problems exist in the observations.

However, these differences should not create a problem with convergence during the adjustment, since the observations remained in the data set when the adjustment was attempted. The results of the first trial adjustment are shown below. The software used the rejection criteria procedure based on Equation (21.18) for its blunder detection. The column headed Std. Res. represents the standardized residual of the observation as defined in Equation (21.15) and the column headed by *Red. Num*. represents the redundancy number of the observation as defined in Equation (21.14).

**** Adjusted Distance Observations ****

No.	From	To	Distance	Residual	Std. Res.	Red. Num.
1	1	3	616.234	0.494	26.148	0.7458
2	1	2	480.943	0.233	12.926	0.6871
3	1	3	616.234	0.494	26.148	0.7458
4	**3**	**4**	**267.044**	**-31.056**	**-1821.579**	**0.6169**
5	3	6	203.746	1.766	107.428	0.5748
6	3	5	413.726	3.286	171.934	0.7719
7	3	2	422.765	0.065	3.500	0.7312
8	5	2	630.949	1.369	75.909	0.6791
9	5	1	449.398	-1.272	-79.651	0.5377
10	5	6	247.822	1.212	75.418	0.5488
11	5	4	407.125	9.235	631.032	0.4529
12	5	3	413.726	3.266	170.888	0.7719
13	102	103	327.250	-0.120	-17.338	0.1018
14	103	1	665.702	-0.088	-12.395	0.1050
15	201	202	453.362	0.262	91.903	0.0172
16	202	203	709.856	0.076	10.737	0.1048
17	203	3	537.241	0.061	8.775	0.1026
18	2000	102	125.056	-0.184	-28.821	0.0868
19	2001	201	425.949	0.049	7.074	0.1008

**** Adjusted Angle Observations ****

No.	From	Occ	To	Angle	Residual	Std. Res.	Red Num
1	2	1	3	316°49′55.1″	114.6″	28.041	0.4164
2	2	3	4	167°36′00.2″	212.2″	25.577	0.3260
3	2	3	6	71°43′01.5″	10.0″	1.054	0.3990
4	2	3	5	97°55′09.3″	-867.2″	-101.159	0.6876
5	2	3	1	51°06′14.6″	-56.4″	-11.156	0.4985
6	203	3	2	8°59′36.3″	-19.7″	-13.003	0.0550
7	2	5	3	318°25′14.4″	259.9″	44.471	0.6949
8	1	5	3	268°58′49.8″	557.3″	78.590	0.5288
9	6	5	3	338°42′53.4″	374.9″	63.507	0.3058
10	**3**	**5**	**4**	**322°02′24.7″**	**-8119.3″**	**-1781.060**	**0.3197**
11	2000	102	103	162°23′50.9″	-2065.1″	-110.371	0.4194
12	102	103	1	171°57′46.9″	-236.1″	-112.246	0.0317
13	2001	201	202	263°58′31.6″	252.9″	104.430	0.0619
14	201	202	203	101°52′56.4″	181.4″	57.971	0.1493
15	202	203	3	176°50′15.9″	65.9″	23.278	0.1138
16	102	2000	2001	109°40′18.6″	1764.6″	106.331	0.4234
17	2000	2001	201	36°07′56.4″	210.2″	104.450	0.0731

****** Adjustment Statistics ******
Adjustment's Reference Standard Deviation = 487.79
Rejection Level = 1604.82

The proper procedure for removing blunders is to remove the single observation that is greater in magnitude than the rejection level selected for the adjustment and is greater in magnitude than the value of any other standardized residual in the

adjustment. This procedure prevents removing observations that are connected to blunders, and thus are inherently affected by their presence. By comparing the values of the standardized residuals against the rejection level of the adjustment, it can be seen that both a single distance (3–4) and angle (3–5–4) are possible blunders since their standardized residuals are greater than the rejection level chosen. However, upon inspection of Figure 21.6, it can be seen that a blunder in distance 3–4 will directly affect angle 3–5–4, and distance 3–4 has the standardized residual that is greatest in magnitude. This explains the previous a priori rejection of this distance observation. Since distance 3–4 affects the size of angle 3–5–4 in the adjustment directly, this angle was also flagged as a possible blunder. However, its standardized residual is less than that of distance 3–4, and thus only this distance should be removed from the observations. After removing this distance from the observations, the adjustment was rerun with the results shown below.

**** Adjusted Distance Observations ****

No.	From	To	Distance	Residual	Std. Res.	Red. Num.
1	1	3	615.693	-0.047	-2.495	0.7457
2	1	2	480.644	-0.066	-3.647	0.6868
3	1	3	615.693	-0.047	-2.495	0.7457
4	2001	201	425.902	0.002	0.265	0.1009
5	3	6	201.963	-0.017	-1.032	0.5765
6	3	5	410.439	-0.001	-0.032	0.7661
7	3	2	422.684	-0.016	-0.858	0.7314
8	5	2	629.557	-0.023	-1.280	0.6784
9	5	1	450.656	-0.014	-0.858	0.5389
10	5	6	246.590	-0.020	-1.241	0.5519
11	5	4	397.885	-0.005	-0.380	0.4313
12	5	3	410.439	-0.021	-1.082	0.7661
13	102	103	327.298	-0.072	-10.380	0.1018
14	103	1	665.751	-0.039	-5.506	0.1049
15	201	202	453.346	0.246	86.073	0.0172
16	202	203	709.807	0.027	3.857	0.1049
17	203	3	537.193	0.013	1.922	0.1027
18	2000	102	125.101	-0.139	-21.759	0.0868

**** Adjusted Angle Observations ****

No.	From	Occ	To	Angle	Residual	Std. Res.	Red Num
1	2	1	3	316°47′54.2″	-6.3″	-1.551	0.4160
2	2	3	4	167°32′31.0″	3.0″	0.380	0.2988
3	2	3	6	71°42′46.0″	-5.5″	-0.576	0.3953
4	2	3	5	98°09′18.6″	-17.9″	-2.088	0.6839
5	2	3	1	51°07′04.1″	-6.9″	-1.360	0.4978
6	203	3	2	8°59′26.7″	-29.3″	-19.340	0.0550
7	2	5	3	318°20′51.4″	-3.1″	-0.532	0.6933
8	1	5	3	268°50′03.4″	30.9″	4.353	0.5282
9	6	5	3	338°36′37.1″	-1.4″	-0.238	0.3049
10	3	5	4	324°17′43.6″	-0.4″	-0.381	0.0160
11	**2000**	**102**	**103**	**162°24′10.2″**	**-2045.8″**	**-109.353**	**0.4193**
12	**102**	**103**	**1**	**171°57′51.2″**	**-231.8″**	**-110.360**	**0.0316**
13	2001	201	202	263°58′20.3″	241.6″	99.714	0.0619
14	201	202	203	101°52′34.7″	159.7″	51.023	0.1494
15	202	203	3	176°49′56.1″	46.1″	16.273	0.1138
16	**102**	**2000**	**2001**	**109°40′17.7″**	**1763.7″**	**106.280**	**0.4233**
17	2000	2001	201	36°07′46.9″	200.7″	99.688	0.0732

****** Adjustment Statistics ******

Adjustment's Reference Standard Deviation = 30.62

Rejection Level = 100.73

After this adjustment, analysis of standardized residuals indicates that the most likely angles to yet contain blunders are observations 11, 12, and 16. Of these, observation 12 containing the previously identified angle 102–103–11 in line 12 of the angle observations displays the highest standardized residual. Looking at Figure 21.6, it is seen that this angle attaches the northern traverse leg to control station 2000. This is a crucial observation in the network, if any hopes of redundancy in the orientation of the network are to be maintained. Since this is a flat angle (i.e., nearly 180°), it is possible that the backsight and foresight stations were reported incorrectly, which could be checked by reversing stations 102 and 1. However, without further field checking, it cannot be guaranteed that this occurred. A decision must be made ultimately about whether this angle should be reobserved. For now, though, this observation will be discarded and another adjustment performed.

In this stepwise blunder detection process, it is always wise to remove as few observations as possible. In no case should observations that are blunder free be deleted. This can and does happen, however, in stepwise blunder detection process. But through persistent and careful processing, only those observations that contain blunders are identified and eliminated. The results of the adjustment after removing angle 12 are shown below.

```
*****************
Adjusted stations
*****************
Standard error ellipses computed
  Station       X            Y          Sx      Sy      Su       Sv      t
=============================================================================
        1   2,477,233.72  420,353.59   0.071   0.069   0.092   0.036  133.47°
        2   2,477,497.89  419,951.98   0.050   0.083   0.090   0.037  156.01°
        3   2,477,832.55  420,210.21   0.062   0.107   0.119   0.034  152.80°
        4   2,477,991.64  420,400.58   0.077   0.121   0.138   0.039  149.71°
        5   2,477,630.43  420,567.45   0.088   0.093   0.123   0.036  136.74°
        6   2,477,665.22  420,323.32   0.071   0.096   0.114   0.036  145.44°
      102   2,476,455.89  419,741.38   0.024   0.018   0.024   0.017   80.86°
      103   2,476,735.05  419,912.42   0.051   0.070   0.081   0.031  147.25°
      201   2,476,576.23  419,589.23   0.020   0.022   0.024   0.017   37.73°
      202   2,476,948.74  419,331.29   0.029   0.041   0.042   0.029   14.24°
      203   2,477,463.84  419,819.58   0.040   0.077   0.081   0.032  160.84°

******************************
Adjusted Distance Observations
******************************
  Station  Station
 Occupied  Sighted   Distance      V Std.Res.      Red.#
=================================================
   2001       201     425.88    -0.023    -3.25    0.102
    201       202     453.09    -0.005    -3.25    0.006
```

202	203	709.76	-0.023	-3.25	0.104
203	3	537.16	-0.023	-3.25	0.103
5	3	410.45	-0.011	-0.60	0.767
5	4	397.89	-0.003	-0.19	0.436
5	6	246.60	-0.014	-0.83	0.556
5	1	450.68	0.013	0.80	0.542
5	2	629.58	0.003	0.15	0.678
3	2	422.70	0.003	0.16	0.736
3	1	615.75	0.008	0.40	0.745
3	5	410.45	0.009	0.44	0.767
3	6	201.97	-0.013	-0.78	0.580
1	2	480.71	-0.003	-0.19	0.688
1	3	615.75	0.008	0.40	0.745
2000	102	125.26	0.020	3.25	0.082
102	103	327.39	0.023	3.25	0.101
103	1	665.81	0.023	3.25	0.104

****
Adjusted Angle Observations
****

Station Backsighted	Station Occupied	Station Foresighted	Angle	V	Std.Res.	Red.#
102	2000	2001	109°11′11.1″	17.06″	3.25	0.042
2000	102	103	162°58′05.1″	-10.95″	-3.25	0.014
2000	2001	201	36°04′23.8″	-2.45″	-3.25	0.010
2001	201	202	263°54′15.7″	-2.97″	-3.25	0.009
201	202	203	101°49′46.3″	-8.72″	-3.25	0.110
202	203	3	176°49′01.0″	-8.98″	-3.25	0.109
203	3	2	8°59′51.1″	-4.91″	-3.25	0.054
2	1	3	316°48′02.8″	2.29″	0.57	0.410
3	5	4	324°17′43.8″	-0.19″	-0.19	0.016
6	5	3	338°36′37.0″	-1.51″	-0.26	0.302
1	5	3	268°49′43.7″	11.20″	1.57	0.528
2	5	3	318°20′51.1″	-3.44″	-0.59	0.691
2	3	1	51°07′14.4″	3.45″	0.68	0.497
2	3	5	98°09′22.0″	-14.55″	-1.71	0.680
2	3	6	71°42′48.5″	-2.97″	-0.31	0.392
2	3	4	167°32′29.5″	1.48″	0.19	0.294

****
Adjustment Statistics
****

Iterations = 4
Redundancies = 12
Reference Variance = 1.316
Reference So = ±1.1
Possible blunder in observations with Std.Res. > 4.33
Convergence!

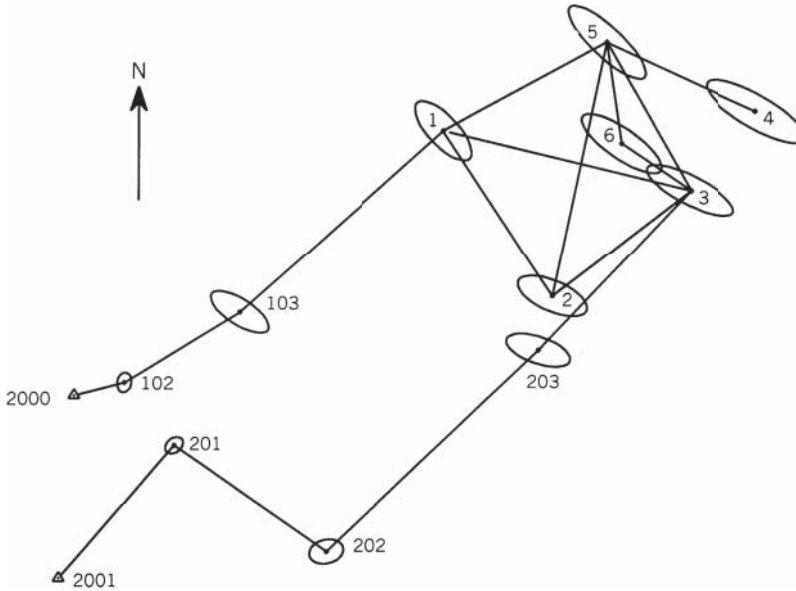


FIGURE 21.8 Horizontal least-squares option screen shows blunder detection options in ADJUST.

From the analysis of the results above, all observations containing blunders appear to have been removed. However, it should also be noted that several remaining distance and angle observations have very low redundancy numbers. This identifies them as unchecked observations, which is an undesirable situation also. Thus, good judgment dictates reobservation of the observations deleted. This weakness can also be seen in the size of the standard error ellipses for the stations shown in Figure 21.8.

Note especially the rotation of the error ellipses. That is, the uncertainty is primarily in a direction perpendicular to the line between stations 1 and 102. This condition was predictable due to the network geometry and since the angle 102–103–1 had been removed from the data set. Furthermore, the crew on the northern leg never observed an angle at station 1, which would tie into station 103, and thus the position of station 103 was found by the intersection of two distances that nearly form a straight line. This results in a larger error in the direction perpendicular to the lines at this station.

This example demonstrates the process used to statistically detect and remove observational blunders. Whether the observations should be reobserved depends on the intended use of the survey. Obviously, additional observations can strengthen the network and probably reduce the size of the error ellipses. However, the geometry of this particular survey lends itself to large rotational errors.

Observations between stations 102 and 201 also contribute to the overall strength in the network. However, because a building obstructs that line, these

observations could not be obtained. Similarly, the network strength could be improved if a connection to control east of the survey could be made. However, in this case, no reasonable connection to additional control could be made. This is a common problem in network design. That is, it is sometimes physically impossible to gather observations that would contribute to the total network strength. Thus, a compromise must be made between the *ideal* network and what is physically obtainable. Balancing these aspects requires careful planning before the observations are collected. Of course today, poor location of control and line obstructions that occur due to terrain, vegetation, or buildings, can be overcome using GNSS.

21.9 SOME FURTHER CONSIDERATIONS

 Equation (21.14) shows the relationship between blunders and their effects on residuals as $\Delta v_i = -r_i \Delta l_i$. From this relationship, note that the effect of the blunder, Δl_i, on the residual, Δv_i, is directly proportional to the redundancy number, r_i. Therefore,

1. If r_i is large ($\approx$1), the blunder greatly affects the residual and should be easy to find.
2. If r_i is small ($\approx$0), the blunder has little effect on the residual and will be hard to find.
3. If $r_i = 0$ as in the case of spur observations, then the blunder is undetectable, and the solved parameters will be incorrect since the error cannot been detected.

Since redundancy numbers can range from 0 to 1, it is possible to compute the minimum detectable error for a single blunder. For example, suppose that a value of 4.0 is used to isolate observational blunders. Then, if the reference variance of the adjustment is 6, all observations that have standardized residuals greater than 24.0 (4.0×6) are possible blunders. However, from Equation (21.14), for an observation with a redundancy number of 0.2 ($r_i = 0.2$) and a standardized residual of $\Delta v_i = 24.0$, the minimum detectable error is 24.0/0.2, or 120! Thus, a blunder, Δl_i, in this observation as large as five times the desired level can go undetected due to its low redundancy number. This situation can be extended to observations that have no observational checks (spur observations), that is, r_i is 0. In this case, Equation (21.14) shows that it is impossible to detect any blunder, Δl_i, in the observation since $\Delta v_i / \Delta r_i$ is indeterminate.

With this taken into consideration, it has been shown (Amer, 1979) that a marginally detectable blunder in an individual observation is

$$\Delta l_i = S \sqrt{\frac{\lambda_0}{q_{ii} w_{ii}^2}} \tag{21.21}$$

where λ_0 is the mean of the noncentral normal distribution shown in Figure 21.5 known as the *noncentrality parameter*. This parameter is the translation of the normal distribution that leads to rejection of the null hypothesis, and whose values can be taken from nomograms developed by Baarda (1968). The sizes of the values obtained from Equation (21.21) provide a clear insight into the weak areas of the network.

21.9.1 Internal Reliability

Internal reliability is found by examining how well observations check themselves geometrically. As mentioned previously, if a station is uniquely determined, q_{ii} will be zero in Equation (21.21) and the computed value of Δl_i is infinity. This indicates a lack of self-checking observations. Since Equation (21.21) is independent of the actual observations, it provides a method of detecting weak areas in networks. To minimize the sizes of the undetected blunders in a network, the redundancy numbers of the individual observations should approach their maximum value of 1. Furthermore, for uniform network strength, the individual redundancy numbers, r_i, should be close to the global relative redundancy of r/m, where r is the number of redundant observations and m is the number of observations in the network. Weak areas in the network are located by finding regions where the redundancy numbers become small in comparison to relative redundancy.

21.9.2 External Reliability

An undetected blunder of size Δl_i has a direct effect on the adjusted parameters. External reliability is the effect of the undetected blunders on these parameters. As Δl_i (a blunder) increases, so will its effect on ΔX. The value of ΔX is given by

$$\Delta X = (A^T W A)^{-1} A^T W \Delta \varepsilon \tag{21.22}$$

Again, this equation is datum independent. From Equation (21.22), to minimize the value of ΔX_i, the size of redundancy numbers must be increased. Baarda suggested using average coordinate values in determining the effect of an undetected blunder with the following equation:

$$\lambda = \Delta X^T Q_{xx} \Delta X \tag{21.23}$$

where λ represents the noncentrality parameter.

The noncentrality parameter should remain as small as possible to minimize the effects of undetected blunders on the coordinates. Note that as the redundancy numbers on the observations become small, the effects of undetected blunders become large. Thus, the effect on the coordinates of a station from an undetected blunder is greater when the redundancy number is low. In fact, an observation with a high redundancy number is likely to be detected as a blunder.

A traverse sideshot (spur observation) can be used to explain this phenomenon. Since the angle and distance to the station are unchecked in a sideshot, the

coordinates of the station will change according to the size of the observational blunders. The observations creating the sideshot will have redundancy numbers of zero since they are geometrically unchecked. This situation is neither good nor acceptable in a well-designed observational system. In network design, one should always check the redundancy numbers of the anticipated observations and strive to achieve uniformly high values for all observations. Redundancy numbers above 0.5 are generally sufficient to provide well-checked observations.

21.10 SURVEY DESIGN

In Chapters 8 and 20, the topic of observational system design was discussed. Redundancy numbers can now be added to this discussion. A well-designed network will provide sufficient geometric checks to allow observational blunders to be detected and removed. In Section 21.9.1, it was stated that if blunder removal is to occur uniformly throughout the system, the redundancy numbers should be close to the system's global relative redundancy. Furthermore, in Section 21.9.2, it was noted that redundancy numbers should be about 0.5 or greater. By combining these two additional concepts with the error ellipse sizes and shapes, and stochastic model planning, an overall methodology for designing observational systems can be obtained.

To begin the design process, the approximate positions for stations to be included in the survey must be determined. These locations can be determined from topographic maps, photographic measurements, code-based GNSS, online maps, or previous survey data. The approximate locations of the control stations should be dictated by their desired locations, the surrounding terrain, vegetation, soils, sight line obstructions, and so on. For conventional surveys, field reconnaissance at this phase of the design process is generally worthwhile to verify sightlines and accessibility of stations. It is also worthwhile in GNSS surveys to ensure adequate sky views are available from the prospective station locations. At this stage, moving a station only a small distance from the original design location may greatly enhance visibility, but not change the geometry of the network significantly. In conventional surveys, using topographic maps in this process allows one to check sight line ground clearances by constructing profiles between stations. Additionally, online software such as Google Earth provides easy access to recent photography to check overhead visibility in GNSS surveys.

When approximate station coordinates are determined, a stochastic model for the observational system can be designed following the procedures discussed in Chapters 7 and 11. In this design process, considerations should be given to the abilities of the field personnel, quality of the equipment, and observational procedures. After the design is completed, specifications for field crews can be written based on these design parameters. These specifications should include the type of instrument used, number of turnings for angle observations, accuracy of instrument leveling and centering, horizon closure requirements, as well as many other items.

These items should be written in a form that field crews will understand. For example, it is not sufficiently clear to state that the setup error of an instrument must be under ±0.005 ft. It is better to state that level vials and optical plummets for all instruments used in the survey must be calibrated, and special care must be taken in setting and leveling instruments over a station. If one expects good fieldwork, then the instructions to the field crews must be understandable and verifiable.

Once the stochastic model is designed, simulated observations can be computed from the station coordinates, and a least-squares adjustment of the simulated observations performed. Since actual observations have not been made, their values are computed from the station coordinates. The adjustment will converge in a single iteration with all residuals equaling zero. Thus, the reference variance must be assigned the a priori value of 1 to compute the error ellipse axes and the standard deviations of the station coordinates. Having completed the adjustment, the network can be checked for geometrically weak areas, unacceptable error ellipse size or shape, and so on. This inspection may dictate the need for any or all of the following: (1) more observations, (2) different observational procedures, (3) different equipment, (4) more stations, (5) different network geometry, and so on. In any event, a clear picture of results obtainable from the observational system will be provided by the simulated adjustment and additional observations, or a different network geometry can be used.

In GNSS surveys, it should be realized that the design of the network provides checks on the observational procedures. However, the accuracy of the derived positions is more dependent on the geometry of the satellites, the length of the sessions, and the type of processing than it is on the actual geometry of the survey. Thus, when designing a GNSS survey, strong considerations should be given to station accessibility, satellite visibility, lack of multipath conditions caused by reflective surfaces, positional dilution of precision (PDOP), and length of sessions.

It should be noted that what is expected from the design may not actually occur due to numerous and varied reasons. Thus, systems are generally overdesigned. However, this tendency to overdesign should be tempered with the knowledge that it will raise the costs of the survey. Thus, a balance should be found between the design and costs. Experienced surveyors know what level of accuracy is necessary for each job and design observational systems to achieve the desired accuracy. It would be cost-prohibitive and foolish to always design an observational system for maximum accuracy, regardless of the survey's intended use. As an example, the final adjustment of the survey in Section 21.7 had sufficient accuracy to be used in a mapping project with a final scale of 1:1200 since the largest error ellipse semimajor axis (0.138 ft) would only plot as 0.0014 in. and is thus less than the width of a line on the final map product.

For convenience, the steps involved in network design are summarized as follows:

Step 1: Using a topographic map, aerial photos, or website, lay out possible positions for stations.

Step 2: Use the topographic map together with air photos to check sight lines for possible obstructions and ground clearance.

Step 3: Do field reconnaissance, checking sight lines for obstructions not shown on the map or photos, and adjust positions of stations as necessary.

Step 4: Determine approximate coordinates for the stations from the map or photos.

Step 5: Compute values of observations using the coordinates from step 4.

Step 6: Using methods discussed in Chapter 6 or 11, compute the standard deviation of each observation based on available equipment and field measuring procedures.

Step 7: Perform a least-squares adjustment to compute observational redundancy numbers, standard deviations of station coordinates, and error ellipses at a specified percent probability.

Step 8: Inspect the solution for weak areas based on redundancy numbers and ellipse shapes. Add or remove observations as necessary, or reevaluate observational procedures and equipment.

Step 9: Evaluate the costs of the survey and determine if some other method of measurement (GNSS, for example) may be more cost-effective.

Step 10: Write specifications for field crews.

21.11 SOFTWARE

Data snooping and the tau criterion have been implemented in ADJUST for differential leveling, horizontal, and GNSS network adjustments. As shown in Figure 21.8, the user can implement either option by selecting it. Once selected, a text box will appear to the right of the option, which defaults to an appropriate rejection criterion. For example, when data snooping is selected, the default rejection level is 3.29. The default level of significance for the tau criterion is 0.001. The user may modify these defaults to suit their situation. As discussed in Section 21.8, the software will compute the standardized residuals and the redundancy numbers for each observation. Additionally, this option should be selected when performing a simulated adjustment so that the redundancy numbers can be scanned for observations that lack sufficient checks.

On the book's companion website is the Mathcad® file *C21.xmcd* that demonstrates the statistical blunder detection concepts covered in this chapter. Figure 21.9 shows the section of the Mathcad code that is used to compute the standardized residuals and redundancy numbers. For those wishing to program blunder detection using a higher-level language, the Mathcad file *C21.xmcd* demonstrates the use of blunder detection in analyzing a horizontal network. Furthermore, the computation of the critical tau value has been added to the program STATS. Both of these programs can be found on the book's companion website.

Compute Q_{vv} for blunder detection

$$Q_{vv} := W^{-1} - Q_{11}$$

Compute standardized residuals and redundancy numbers

$$i := 0 \,..\, nobs - 1$$

$$SV_i := \frac{|V_i|}{S_0 \cdot \sqrt{Q_{vv_{i,i}}}} \qquad \text{Standardized residuals}$$

$$R_i := Q_{vv_{i,i}} \cdot W_{i,i} \qquad \text{Redundancy numbers}$$

FIGURE 21.9 Mathcad code to compute standardized residuals and redundancy numbers.

PROBLEMS

Note: ADJUST, which is available on the book's companion website, should be used to solve the problems requiring a least-squares adjustment. Partial answers are provided for problems marked with an asterisk in Appendix G.

***21.1** Discuss the effects of a distance blunder on a traverse closure and explain how it can be identified.

21.2 Discuss the effects of an angle blunder on a traverse closure and explain how it can be identified.

21.3 Explain why it is not wise to have observational redundancy numbers near zero in a well-designed network.

21.4 Create a list of items that should be included in field specifications for a crew in a designed network and then create a sample set of specifications for a horizontal survey.

21.5 Summarize the general procedures used in isolating observational blunders.

21.6 How are control problems isolated in an adjustment?

21.7 Discuss possible causes for control problems in an adjustment.

21.8 Why is it recommended that there be at least three control stations in a least-squares adjustment?

21.9 Outline the procedures used in survey network design.

***21.10** Using data snooping, analyze the data in Problem 15.10. Hold the control fixed during the adjustment.

21.11 Repeat Problem 21.10 using the tau criterion.

21.12 Analyze the data in Problem 16.8.

(a) data snooping.

(b) tau criterion.

21.13 Analyze the data in Problem 15.14 using

(a) data snooping.

(b) tau criterion.

21.14 Using data snooping, analyze the data in Problem 16.14 for blunders. Hold the control fixed during the adjustment.

21.15 Analyze the data in Problem 16.9 while holding the control station coordinates fixed using

(a) data snooping.

(b) tau criterion.

***21.16** Two control stations that are 12,203.02 ft apart have a published precision of 1:50,000. What are the appropriate estimated errors for the coordinates of these points?

21.17 Repeat Problem 21.16 for two stations that are 40,503.645 m apart with a published precision of 1:50,000.

21.18 Analyze the data in Problem 16.10, assuming the control points have uncertainties as follows:

Station	S_x	S_y
A	0.255	0.248
D	0.213	0.157
H	0.315	0.189
K	0.159	0.230

(a) data snooping.

(b) tau criterion.

21.19 The standard deviations on the *xy* coordinates for stations *A*, *D*, *H*, and *K* in Problem 16.10 are listed in the following table. Adjust and analyze the data. Be sure to handle the control correctly. Compare this solution with that in Problem 21.18.

Station	S_x	S_y
A	0.016	0.020
D	0.028	0.022
H	0.021	0.015
K	0.033	0.038

21.20 The following data was collected. Apply the procedures discussed in this chapter to isolate and remove any apparent blunders in the data using either data snooping or the tau criterion.

(a) Discussion on apparent blunders.

(b) Analyze statistically the constrained and minimally constrained adjustments. Did the two adjustments pass the F test at a level of significance of 0.05?

(c) At a level of significance of 0.05, does the constrained adjustment pass the χ^2 test?

Control Stations

Station	Easting (m)	Northing (m)
A	1105.687 ± 0.005	1000.204 ± 0.008
D	4609.645 ± 0.011	256.789 ± 0.015

Approximate Station Coordinates

Station	Easting (m)	Northing (m)
B	3205.687	1602.593
C	1597.600	200.000
E	1597.600	200.000
F	2814.698	842.358

Distance Observations

From	To	Distance (m)	S (m)	From	To	Distance (m)	S (m)
A	*B*	2184.688	0.008	*E*	*F*	1376.207	0.006
A	*F*	1716.281	0.007	*E*	*A*	939.301	0.005
B	*C*	1527.384	0.006	*D*	*F*	1888.046	0.007
B	*F*	854.876	0.005	*D*	*E*	3012.591	0.010
C	*D*	741.819	0.005	*C*	*F*	1800.671	0.007

Angle Observations

Backsight	Occupied	Foresight	Angle	S ($''$)
F	*A*	*E*	53°08′33.6″	2.3
B	*A*	*F*	21°16′55.2″	2.2
F	*B*	*A*	46°46′38.5″	2.3
C	*B*	*F*	93°55′23.9″	2.3
F	*C*	*B*	28°16′35.8″	2.2
D	*C*	*F*	85°06′21.2″	2.3
F	*D*	*C*	71°50′57.6″	2.3
E	*D*	*F*	19°08′51.8″	2.2
A	*E*	*F*	93°45′22.8″	2.3
F	*E*	*D*	26°44′36.3″	2.2
A	*F*	*B*	111°56′25.0″	2.3
B	*F*	*C*	57°48′19.5″	2.3
C	*F*	*D*	23°02′45.8″	2.2
D	*F*	*E*	134°06′27.3″	2.2
E	*F*	*A*	33°06′04.7″	2.2

21.21 Apply to procedures discussed in this chapter to isolate any blunders in the following data. Assume that station *A* has a relative precision of 1:100,000 and *H* has a relative precision of 1:50,000.

(a) Analyze the data using data snooping using a rejection criterion of 3.29.

(b) Analyze the data using the tau criterion at a level of significance of 0.001.

(c) Analyze statistically the constrained and minimally constrained adjustments. Did the two adjustments pass the *F* test at a level of significance of 0.05?

(d) At a level of significance of 0.05, does the constrained adjustment pass the χ^2 test?

Control Stations

Station	*X* (ft)	*Y* (ft)
A	45,806.98 ± 0.241	80,465.38 ± 0.147
H	147,608.99 ± 0.087	202,654.36 ± 0.098

Approximate Coordinates

Station	*X* (ft)	*Y* (ft)
B	36,489.48	151,309.33
C	49,263.50	202,648.34
D	98,056.35	153,648.04
E	99,121.34	76,098.35
F	144,512.87	74,215.66
G	150,608.77	142,516.35

Distance Observations

Course	Distance (ft)	*S* (ft)	Course	Distance (ft)	*S* (ft)
AB	71,454.13	0.215	*BC*	52,904.31	0.159
CD	69,150.62	0.208	*DE*	77,557.09	0.233
EF	45,430.65	0.137	*FG*	68,571.96	0.206
GH	60,212.72	0.181	*CH*	98,345.84	0.295
AE	53,492.61	0.161	*AD*	89,920.33	0.270
DH	69,692.94	0.210	*BD*	61,611.51	0.185
DG	53,718.64	0.162	*DF*	92,022.35	0.276

Angle Observations

Backsight	Occupied	Foresight	Angle	*S* (″)
B	*A*	*D*	43°01′02.7″	2.1
D	*B*	*A*	84°40′58.5″	2.1
C	*B*	*D*	73°51′06.5″	2.1
D	*C*	*B*	58°51′02.5″	2.1
H	*C*	*D*	45°07′31.2″	2.1
D	*H*	*C*	44°40′44.3″	2.1

Backsight	Occupied	Foresight	Angle	S (″)
G	*H*	*D*	48°10′24.0″	2.1
D	*G*	*H*	75°11′06.6″	2.1
F	*G*	*D*	96°51′34.3″	2.1
D	*F*	*G*	35°25′18.1″	2.1
E	*F*	*D*	57°18′12.4″	2.1
A	*E*	*D*	84°31′49.4″	2.1
D	*E*	*F*	93°09′44.6″	2.3
D	*A*	*E*	59°09′26.9″	2.1
A	*D*	*B*	52°18′00.4″	2.1
B	*D*	*C*	47°17′50.0″	2.1
C	*D*	*H*	90°11′45.9″	2.1
H	*D*	*G*	56°38′30.7″	2.1
G	*D*	*F*	47°43′04.5″	2.1
F	*D*	*E*	29°32′07.2″	2.1
E	*D*	*A*	36°18′39.8″	2.1

21.22 Using the data set *Blunder.dat*, which is part of the installation of the software ADJUST that accompanies this book, isolate any blunders that are detectable using data snooping and a rejection criterion of 3.29.

21.23 Repeat Problem 21.22 using the tau criterion and a level of significance of 0.001.

21.24 The following approximate station coordinates were determined from a map of an area where a second-order, class I survey (1:50,000) is to be performed. All sight lines to neighboring stations have been checked and are available for conventional observations of distances, angles, and azimuths. The control stations are visible to their nearest neighbors. Design a control network that will meet the specified accuracy at a 95% confidence level and have sufficient checks to ensure the reliability of the results. Stations J218, J219, and ROCK are first-order control stations (1:100,000).

Control Stations

Station	X (m)	Y (m)
J218	283,224.223	116,202.946
J219	154,995.165	330,773.314
ROCK	521,287.251	330,276.310

Approximate Station Locations

Station	X (m)	Y(m)
101	280,278	194,109
102	276,350	278,887
103	360,147	121,768
104	356,219	195,090
105	352,291	274,304
106	455,728	132,570
107	446,563	198,036
108	440,671	270,700

PRACTICAL PROBLEMS

21.26 Design a 6-mile by 6-mile control network having a minimum of eight control stations using a topographic map of your local area. Design a traditional measurement network made up of angles, azimuths, and distances so that the largest ellipse axis at a 95% confidence level is less than 0.20 ft and so that all observations have redundancy numbers greater than 0.5. In the design, specify the shortest permissible sight distance, the largest permissible errors in pointing, reading, and instrument and target setup errors, the number of repetitions necessary for each angle measurement, and the necessary quality of angle and distance measuring instruments. Use realistic values for the instruments. Plot profiles of sight lines for each observation.

21.27 Design a 6-mile by 6-mile GNSS control network to be established by differential GNSS that has a minimum of eight control stations using a topographic map of your local area to select station locations. Design the survey so that all baseline observations included in the network have redundancy numbers greater than 0.5. In the design, use a unit matrix for the covariance matrix of the baselines.

CHAPTER 22

THE GENERAL LEAST-SQUARES METHOD AND ITS APPLICATION TO CURVE FITTING AND COORDINATE TRANSFORMATIONS

22.1 INTRODUCTION TO GENERAL LEAST-SQUARES



When fitting points to a straight line recognize that both the x and y coordinates contain errors. Yet in the mathematical model presented in Section 11.11.1, the residuals, as illustrated in Figure 11.2, are only applied to the y coordinate. Because both coordinates contain errors, this mathematical model fails to account for the x coordinate being an observation. In this chapter, the *general least-squares method* is presented, and its use in performing adjustments, where the observation equations involve more than a single observation, is demonstrated.

22.2 GENERAL LEAST-SQUARES EQUATIONS FOR FITTING A STRAIGHT LINE

Consider the data illustrated in Figure 11.2. To account properly for both the x and y coordinates being observations, the observation equation must contain residuals for all observations. That is, Equation (11.40) must be rewritten as

$$F(x, y) = (y + v_y) - m(x + v_x) - b = 0 \tag{22.1}$$

In Equation (22.1), x and y are a point's coordinate pair with residuals v_x and v_y, respectively; m is the slope of the line; and b is the y intercept. Equation (22.1) contains v_x, v_y, m, and b as unknowns and is nonlinear. Thus, its solution is obtained

by using the methods outlined in Section 12.10. The resulting linearized form of Equation (22.1) is

$$\frac{\partial F}{\partial x}v_x + \frac{\partial F}{\partial y}v_y + \frac{\partial F}{\partial m}dm + \frac{\partial F}{\partial b}db = -(m_0x + b_0 - y) \tag{22.2}$$

where the partial derivatives are

$$\frac{\partial F}{\partial x} = -m \quad \frac{\partial F}{\partial y} = 1 \quad \frac{\partial F}{\partial m} = -x \quad \frac{\partial F}{\partial b} = -1 \tag{22.3}$$

For the four data points A, B, C, and D, substituting Equation (22.3) into (22.2), the following four equations can be written:

$$\begin{aligned}
-m_0v_{x_A} + v_{y_A} - x_A dm - db &= -(m_0x_A + b_0 - y_A)\\
-m_0v_{x_B} + v_{y_B} - x_B dm - db &= -(m_0x_B + b_0 - y_B)\\
-m_0v_{x_C} + v_{y_C} - x_C dm - db &= -(m_0x_C + b_0 - y_C)\\
-m_0v_{x_D} + v_{y_D} - x_D dm - db &= -(m_0x_D + b_0 - y_D)
\end{aligned} \tag{22.4}$$

In matrix form, Equations (22.4) can be written as

$$BV + JX = K \tag{22.5}$$

where

$$B = \begin{bmatrix} -m_0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -m_0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -m_0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & -m_0 & 0 \end{bmatrix} \quad J = \begin{bmatrix} -x_A & -1\\ -x_B & -1\\ -x_C & -1\\ -x_D & -1 \end{bmatrix}$$

$$V = \begin{bmatrix} v_{x_A}\\ v_{y_A}\\ v_{x_B}\\ v_{y_B}\\ v_{x_C}\\ v_{y_C}\\ v_{x_D}\\ v_{y_D} \end{bmatrix} \quad X = \begin{bmatrix} dm\\ db \end{bmatrix} \quad K = \begin{bmatrix} -(m_0x_A + b_0 - y_A)\\ -(m_0x_B + b_0 - y_B)\\ -(m_0x_C + b_0 - y_C)\\ -(m_0x_D + b_0 - y_D) \end{bmatrix} \tag{22.6}$$

Now, since both x and y are observed coordinates, they can have individual estimated standard errors. Assuming that the coordinates are from independent

observations, the four points will have eight observed coordinates and a cofactor matrix of

$$Q = \begin{bmatrix} \sigma_{x_A}^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_{y_A}^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_{x_B}^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_{y_B}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_{x_C}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_{y_C}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{x_D}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{y_D}^2 \end{bmatrix}$$

22.3 GENERAL LEAST-SQUARES SOLUTION

In solving the general least-squares problem, an *equivalent solution* is achieved. For this solution, the following *equivalent weight matrix* is created for the system

$$W_e = (BQB^{\mathrm{T}})^{-1} \tag{22.7}$$

where B is as defined in Equation (22.6). Using the equivalent weight matrix in Equation (22.7), the *equivalent matrix system* is

$$J^{\mathrm{T}} W_e J X = J^{\mathrm{T}} W_e K \tag{22.8}$$

Equation (22.8) has the solution

$$X = (J^{\mathrm{T}} W_e J)^{-1} J^{\mathrm{T}} W_e K \tag{22.9}$$

Since this is a nonlinear equation system, the corrections in matrix X are applied to the initial approximations, and the method is repeated until the system converges. The *equivalent residuals* vector V_e is found following the usual procedure of

$$V_e = JX - K \tag{22.10}$$

Using Equation (22.10), the observational residuals are

$$V = QB^{\mathrm{T}} W_e V_e \tag{22.11}$$

Also, since this is a nonlinear problem and the observations are being adjusted, the observations should also be updated according to their residuals. Thus, the updated observations for the second iteration are

$$l_i' = l_i + v_i \tag{22.12}$$

where l_i' are the second iteration's observations; l_i are the previous iteration's observations; and v_i are the observations' corresponding residuals. Generally, in practice, the original observations are "close" to their final adjusted values, and thus Equation (22.12) is not actually used since the second iteration is only a check for convergence.

Finally, the reference variance for the adjustment can be computed using the equivalent residuals and weight matrix employing the equation

$$S_0^2 = \frac{V_e^{\mathrm{T}} W_e V_e}{r} \tag{22.13}$$

where r is the number of redundancies in the system.

It should be noted that the same results can be obtained using the observational residuals and the following expression:

$$S_0^2 = \frac{V^{\mathrm{T}} W V}{r} \tag{22.14}$$



Example 22.1 **Numerical Solution of the Straight-Line-Fit Problem**

Recall the least-squares fit of points to a line in Section 11.11.1. In that example, the observed coordinate pairs were

A: (3.00, 4.50) B: (4.25, 4.35) C: (5.50, 5.50) D: (8.00, 5.50)

The solution for the slope of the line, m, and y intercept, b, were

$$\begin{aligned} m &= 0.246 \\ b &= 3.663 \end{aligned} \tag{a}$$

Additionally, the residuals were

$$V = AX - L = \begin{bmatrix} -0.10 \\ 0.46 \\ -0.48 \\ 0.13 \end{bmatrix}$$

Now, this problem will be solved using the general least-squares method. Assume that the following covariance matrix is given:

$$Q = \begin{bmatrix} 0.020^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.015^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.023^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.036^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.033^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.028^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.016^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.019^2 \end{bmatrix}$$

SOLUTION The step-by-step procedure for solving this problem using general least-squares follows.

Step 1: Compute the initial approximations. Initial approximations for both m and b are found by using two points and solving the unique system. For this example, the values from the solution of Section 11.11.1, as given above, will be used. It is often the case where the standard least-squares methods are used to determine the initial approximations for the general least-squares adjustment.

Step 2: Develop *the appropriate matrices*. In accordance with Equation (22.6), the B matrix is

$$B = \begin{bmatrix} -0.246 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.246 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.246 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.246 & 1 \end{bmatrix}$$

Using Equation (22.7), the equivalent weight matrix is

$$W_e = (BQB^{\mathrm{T}})^{-1} = \left[\frac{1}{10,000} \begin{bmatrix} 2.5 & 0 & 0 & 0 \\ 0 & 13.3 & 0 & 0 \\ 0 & 0 & 8.5 & 0 \\ 0 & 0 & 0 & 3.8 \end{bmatrix} \right]^{-1}$$

$$= \begin{bmatrix} 4012.7 & 0 & 0 & 0 \\ 0 & 753.0 & 0 & 0 \\ 0 & 0 & 1176.6 & 0 \\ 0 & 0 & 0 & 2656.1 \end{bmatrix}$$

Step 3: Solve the system. The first iteration corrections are found using Equation (22.9) as

$$X = (J^{\mathrm{T}} W_e J)^{-1} J^{\mathrm{T}} W_e K = \begin{bmatrix} -0.0318 \\ 0.1907 \end{bmatrix}$$

where the J, K, and $J^{\mathrm{T}} W_e J$ matrices for the first iteration were

$$J = \begin{bmatrix} -3.00 & -1 \\ -4.25 & -1 \\ -5.50 & -1 \\ -8.00 & -1 \end{bmatrix} \quad K = \begin{bmatrix} 0.246(3.00) + 3.663 - 4.50 \\ 0.246(4.25) + 3.663 - 4.25 \\ 0.246(5.50) + 3.663 - 5.50 \\ 0.246(8.00) + 3.663 - 5.50 \end{bmatrix} = \begin{bmatrix} -0.099 \\ 0.458 \\ -0.484 \\ 0.131 \end{bmatrix}$$

$$J^{\mathrm{T}} W_e J = \begin{bmatrix} 255,297.91 & 42,958.45 \\ 42,958.45 & 8598.40 \end{bmatrix}$$

$$J^{\mathrm{T}} W_e K = \begin{bmatrix} 72.9737 \\ 273.5321 \end{bmatrix}$$

Step 4: Apply the corrections to m_0 and b_0 to update the initial approximations for the second iteration

$$m = 0.246 - 0.0318 = 0.2142$$

$$b = 3.663 + 0.1907 = 3.8537$$

Second Iteration. During the second iteration, only the unknown parameters are updated, and thus, only B, W_e, and K matrices differ from their first iteration counterparts. Their second iteration values are

$$B = \begin{bmatrix} -0.2142 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.2142 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.2142 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.2142 & 1 \end{bmatrix}$$

$$W_e = \begin{bmatrix} 4109.3 & 0 & 0 & 0 \\ 0 & 757.4 & 0 & 0 \\ 0 & 0 & 199.1 & 0 \\ 0 & 0 & 0 & 2682.8 \end{bmatrix} \quad K = \begin{bmatrix} -0.00370 \\ 0.51405 \\ -0.46820 \\ 0.06730 \end{bmatrix}$$

The corrections after this iteration are

$$X = \begin{bmatrix} 0.00002 \\ 0.00068 \end{bmatrix}$$

Thus, m and b are 0.214 and 3.854 to three decimal places. Using Equation (22.10), the equivalent residual vector is

$$V_e = \begin{bmatrix} -0.0030 \\ 0.5148 \\ -0.4674 \\ 0.0681 \end{bmatrix}$$

Using Equation (22.11), the observational residuals are

$$V = QB^{\mathrm{T}}W_eV_e = \begin{bmatrix} 0.0010 \\ -0.0027 \\ -0.0442 \\ 0.5053 \\ 0.1307 \\ -0.4394 \\ -0.0100 \\ 0.0660 \end{bmatrix}$$

A graphical interpretation of the residuals is shown in Figure 22.1. Notice how the equivalent residuals are aligned with the y-axis and that the observational

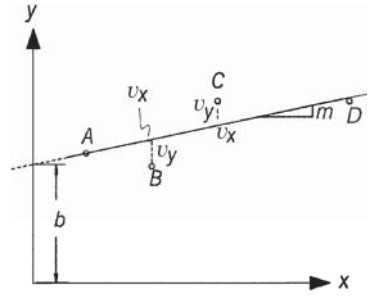


FIGURE 22.1 Residuals for point *C*.

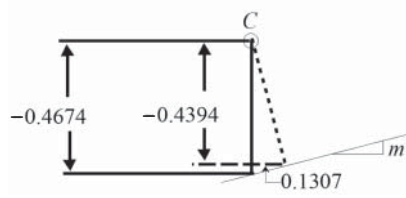


FIGURE 22.2 General least-squares fits of points to a line.

residuals exist in the cardinal *x*- and *y*-axes directions. These residuals are shown more clearly in Figure 22.1, which is an enlarged view of the portion of Figure 22.2 that surrounds point *C*. The equivalent residual of *C* is −0.4674 from the line, and the observational residuals $v_{x_c} = 0.1307$ and $v_{y_c} = -0.4394$ are parallel to the *x*- and *y*-axes, respectively. This general solution is more appropriate for adjusting the actual coordinate observations. Note that this solution is slightly different from that determined previously in Example 11.3.

Using Equation (22.13), the reference standard deviation is

$$S_0 = \sqrt{\frac{475.2}{4-2}} = \pm 15.4$$

22.4 TWO-DIMENSIONAL COORDINATE TRANSFORMATION BY GENERAL LEAST-SQUARES

As presented in Chapter 18, two-dimensional coordinate transformations are commonly used to convert points from one two-dimensional coordinate system to another. Again, the general least-squares method is a more appropriate method for these transformations since the coordinates in both systems are observations that contain errors.

22.4.1 Two-Dimensional Conformal Coordinate Transformation

The two-dimensional conformal coordinate transformation, presented in Chapter 18, has four unknowns, consisting of a scale factor, a rotation angle, and two translations. Equations (18.5) express this transformation, and they are repeated here for convenience. The transformation equations are

$$\begin{aligned} X &= ax - by + c \\ Y &= bx + ay + d \end{aligned} \tag{22.15}$$

Equations (22.15) can be rearranged as

$$\begin{aligned} F\colon\ & ax - by + c - X = 0 \\ G\colon\ & bx + ay + d - Y = 0 \end{aligned} \tag{22.16}$$

Since the coordinates from both the xy and XY systems contain errors, Equations (22.16) are rewritten as

$$\begin{aligned} F(x, y, X, Y) &= a(x + v_x) - b(y + v_y) + c - (X + v_X) = 0 \\ G(x, y, X, Y) &= b(x + v_x) + a(y + v_y) + d - (Y + v_Y) = 0 \end{aligned} \tag{22.17}$$

These equations are nonlinear in terms of their observations and residuals. They are solved by linearizing the equations and iterating to a solution. The partial derivatives with respect to each unknown are

$$\begin{aligned} &\frac{\partial F}{\partial x} = a \quad \frac{\partial F}{\partial y} = -b \quad \frac{\partial F}{\partial X} = -1 \quad \frac{\partial G}{\partial x} = b \quad \frac{\partial G}{\partial y} = a \quad \frac{\partial G}{\partial Y} = -1 \\ &\frac{\partial F}{\partial a} = x \quad \frac{\partial F}{\partial b} = -y \quad \frac{\partial F}{\partial c} = 1 \quad \frac{\partial G}{\partial a} = y \quad \frac{\partial G}{\partial b} = x \quad \frac{\partial G}{\partial d} = 1 \end{aligned} \tag{22.18}$$

Using the partial derivatives in Equation (22.18), a matrix for each point can be built as

$$\begin{bmatrix} a_0 & -b_0 & -1 & 0 \\ b_0 & a_0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_X \\ v_Y \end{bmatrix} + \begin{bmatrix} x & -y & 1 & 0 \\ y & x & 0 & 1 \end{bmatrix} \begin{bmatrix} da \\ db \\ dc \\ dd \end{bmatrix} = \begin{bmatrix} X - (a_0 x + b_0 y + T_x) \\ Y - (b_0 x + a_0 y + T_y) \end{bmatrix} \tag{22.19}$$

For a redundant system, the matrices are solved following the matrix procedures outlined in Section 22.3.

Example 22.2 Four fiducial points are digitized from an aerial photo, and their observed (x, y) and control (X, Y) values are listed in Table 22.1. The standard deviations of these observations are also listed. What are the most probable values for the transformation parameters and the resulting residuals?

TABLE 22.1 Data for a Two-Dimensional Conformal Coordinate Transformation

Point	$X \pm S_X$	$Y \pm S_Y$	$x \pm s_x$	$y \pm s_y$
1	-113.000 ± 0.002	0.003 ± 0.002	0.7637 ± 0.026	5.9603 ± 0.028
3	0.001 ± 0.002	112.993 ± 0.002	5.0620 ± 0.024	10.5407 ± 0.030
5	112.998 ± 0.002	0.003 ± 0.002	9.6627 ± 0.028	6.2430 ± 0.022
7	0.001 ± 0.002	-112.999 ± 0.002	5.3500 ± 0.024	1.6540 ± 0.026

SOLUTION In this problem, initial approximations for a, b, c, and d must first be computed. These values can be found using a standard least-squares adjustment or by solving the system with only two points. The former procedure was demonstrated in Example 18.1. Using standard least-squares, initial approximations for the parameters are determined to be

$$a_0 = 25.386458$$
$$b_0 = -0.8158708$$
$$c_0 = -137.216$$
$$d_0 = -150.600$$

Now the B, Q, J, and K matrices described in Section 22.3 can be formed. They are listed below (note that the numbers are rounded to three decimal places for publication purposes only):

$$B = \begin{bmatrix}
25.386 & 0.816 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-0.816 & 25.386 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 25.386 & 0.816 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -0.816 & 25.386 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 25.386 & 0.816 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.816 & 25.386 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 25.386 & 0.816 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.816 & 25.386 & 0 & -1
\end{bmatrix}$$

$$J = \begin{bmatrix}
0.764 & -5.960 & 1 & 0 \\
5.960 & 0.764 & 0 & 1 \\
5.062 & -10.541 & 1 & 0 \\
10.541 & 5.062 & 0 & 1 \\
9.663 & -6.243 & 1 & 0 \\
6.243 & 9.663 & 0 & 1 \\
5.350 & -1.654 & 1 & 0 \\
1.654 & 5.350 & 0 & 1
\end{bmatrix} \qquad K = \begin{bmatrix}
-0.03447270 \\
-0.08482509 \\
0.11090026 \\
0.13190015 \\
-0.18120912 \\
-0.00114251 \\
0.04999940 \\
-0.02329275
\end{bmatrix}$$

Also, the Q matrix is

$$Q = \begin{bmatrix} 0.026^2 & & & & & & & & & & & & & & \\ & 0.028^2 & & & & & & & & & & & & & \\ & & 0.002^2 & & & & & & & & & & & & \\ & & & 0.002^2 & & & & & & & & & & & \\ & & & & 0.024^2 & & & & & & & & & & \\ & & & & & 0.030^2 & & & & & & & & & \\ & & & & & & 0.002^2 & & & & & & & & \\ & & & & & & & 0.002^2 & & & & & & & \\ & & & & & & & & 0.028^2 & & & & & & \\ & & & & & & & & & 0.022^2 & & & & & \\ & & & & & & & & & & 0.002^2 & & & & \\ & & & & & & & & & & & 0.002^2 & & & \\ & & & & & & & & & & & & 0.024^2 & & \\ & & & & & & & & & & & & & 0.026^2 & \\ & & & & & & & & & & & & & & 0.002^2 \\ & & & & & & & & & & & & & & 0.002^2 \end{bmatrix}$$

The solution for the first iteration is

$$X = \begin{bmatrix} -0.000124503 \\ -0.000026212 \\ -0.000325016 \\ -0.000029546 \end{bmatrix}$$

Adding these corrections to the initial approximations gives

$$a = 25.38633347$$

$$b = -0.815897012$$

$$c = -137.2163$$

$$d = -150.6000$$

In the next iteration, only minor corrections occur, and thus the system has converged to a solution. The residuals and reference variance are computed as before. Although the solution has changed slightly from the standard least-squares method, it properly considers the fact that each observation equation contains four observations. Of course, once the transformation parameters have been determined, any points that exist only in the xy system can be transformed into the XY system by substitution into Equation (22.15) or (18.5). This part of the problem is not demonstrated in this example. In fact, for the remainder of the chapter, only the adjustment model will be developed.

22.4.2 Two-Dimensional Affine Coordinate Transformation



As discussed in Section 18.5, the main difference between conformal and affine transformations is that the latter allows for different scales along the x- and y-axes, and accounts for nonorthogonality in the axes. This results in

six parameters. Equations (18.9) express the affine transformation, and they are repeated here for convenience:

$$\begin{aligned} X &= ax + by + c \\ Y &= dx + ey + f \end{aligned} \tag{22.20}$$

Equation (22.20) can be rewritten as

$$\begin{aligned} F(x, y, X, Y) &= ax + by + c - X = 0 \\ G(x, y, X, Y) &= dx + ey + f - Y = 0 \end{aligned} \tag{22.21}$$

Again, Equation (22.21) consists of observations in both the x and y coordinates, and thus it is more appropriate to use the general least-squares method. Therefore, these equations can be rewritten as

$$\begin{aligned} F(x, y, X, Y) &= a(x + v_x) + b(y + v_y) + c - (X + v_X) = 0 \\ G(x, y, X, Y) &= d(x + v_x) + e(y + v_y) + f - (Y + v_Y) = 0 \end{aligned} \tag{22.22}$$

For each point, the linearized equations in matrix form are

$$\begin{bmatrix} a_0 & b_0 & -1 & 0 \\ d_0 & e_0 & 0 & -1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_X \\ v_Y \end{bmatrix} + \begin{bmatrix} x & y & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & x & y & 1 \end{bmatrix} \begin{bmatrix} da \\ db \\ dc \\ dd \\ de \\ df \end{bmatrix} = \begin{bmatrix} X - (a_0 x + b_0 y + c_0) \\ Y - (d_0 x + e_0 y + f_0) \end{bmatrix} \tag{22.23}$$

Two observation equations, like those of Equation (22.23), result for each control point. Since there are six unknown parameters, three control points are needed for a unique solution. With more than three, a redundant system exists, and the solution is obtained following the least-squares procedures outlined in Section 22.3.

22.4.3 Two-Dimensional Projective Transformation

The two-dimensional projective coordinate transformation converts a projection of one plane-coordinate system into another nonparallel plane system. This transformation was developed in Section 18.6. Equations (18.12) for this transformation are repeated here for convenience:

$$\begin{aligned} X &= \frac{a_1 x + b_1 y + c_1}{a_3 x + b_3 y + 1} \\ Y &= \frac{a_2 x + b_2 y + c_2}{a_3 x + b_3 y + 1} \end{aligned} \tag{22.24}$$

Unlike the conformal and affine types, this transformation is nonlinear in its standard form. In their general form, the projective equations become

$$F(x,y,X,Y) = \frac{a_1(x+v_x)+b_1(y+v_y)+c_1}{a_3(x+v_x)+b_3(y+v_y)+1} - (X+v_X) = 0$$
$$G(x,y,X,Y) = \frac{a_2(x+v_x)+b_2(y+v_y)+c_2}{a_3(x+v_x)+b_3(y+v_y)+1} - (Y+v_Y) = 0 \qquad (22.25)$$

Again, a linearized form for Equations (22.25) is needed. The partial derivatives for the unknown parameters were given in Section 18.6, and the remaining partial derivatives are given below:

$$\frac{\partial F}{\partial x} = \frac{a_1(b_3y+1)-a_3(b_1y+c_1)}{(a_3x+b_3y+1)^2} \quad \frac{\partial F}{\partial y} = \frac{b_1(a_3x+1)-b_3(a_1x+c_1)}{(a_3x+b_3y+1)^2}$$
$$\frac{\partial G}{\partial x} = \frac{a_2(b_3y+1)-a_3(b_2y+c_2)}{(a_3x+b_3y+1)^2} \quad \frac{\partial G}{\partial y} = \frac{b_2(a_3x+1)-b_3(a_2x+c_2)}{(a_3x+b_3y+1)^2}$$

In matrix form, the linearized equations for each point are

$$\begin{bmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & -1 & 0 \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} & 0 & -1 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_X \\ v_Y \end{bmatrix}$$
$$+ \begin{bmatrix} \frac{\partial F}{\partial a_1} & \frac{\partial F}{\partial b_1} & \frac{\partial F}{\partial c_1} & 0 & 0 & 0 & \frac{\partial F}{\partial a_3} & \frac{\partial F}{\partial b_3} \\ 0 & 0 & 0 & \frac{\partial G}{\partial a_2} & \frac{\partial G}{\partial b_2} & \frac{\partial G}{\partial c_2} & \frac{\partial G}{\partial a_3} & \frac{\partial G}{\partial b_3} \end{bmatrix} \begin{bmatrix} da_1 \\ db_1 \\ dc_1 \\ da_2 \\ db_2 \\ dc_2 \\ da_3 \\ db_3 \end{bmatrix} = K \qquad (22.26)$$

Equation (22.26) gives the observation equations for the two-dimensional projective transformation for one control point, and the K matrix is defined in Equation (18.13). Since there are eight unknown parameters, four control points are needed for a unique solution. More than four control points yield a redundant system that can be solved following the steps outlined in Section 22.3.

22.5 THREE-DIMENSIONAL CONFORMAL COORDINATE TRANSFORMATION BY GENERAL LEAST-SQUARES



As explained in Section 18.7, this coordinate transformation converts points from one three-dimensional coordinate system to another. Equations (18.15) express this transformation, and the matrix form of those equations is

$$X = SRx + T \tag{22.27}$$

where the individual matrices are as defined in Section 18.7.

Equation (18.15) gives detailed expressions for the three-dimensional coordinate transformation. Note that these equations involve six observations, *xyz* and *XYZ*. In the general least-squares method, these equations are rewritten as

$$\begin{aligned}
F(x,y,z,X,Y,Z) &= S[r_{11}(x+v_x) + r_{21}(y+v_y) + r_{31}(z+v_z)] + Tx - (X+v_X) = 0 \\
G(x,y,z,X,Y,Z) &= S[r_{12}(x+v_x) + r_{22}(y+v_y) + r_{32}(z+v_z)] + Ty - (Y+v_Y) = 0 \\
H(x,y,z,X,Y,Z) &= S[r_{13}(x+v_x) + r_{23}(y+v_y) + r_{33}(z+v_z)] + Tz - (Z+v_Z) = 0
\end{aligned} \tag{22.28}$$

Equations (22.28) are for a single point and again, they are nonlinear. They can be expressed in linearized matrix form as

$$\begin{bmatrix}
\frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} & \frac{\partial F}{\partial X} & 0 & 0 \\
\frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} & 0 & \frac{\partial G}{\partial Y} & 0 \\
\frac{\partial H}{\partial x} & \frac{\partial H}{\partial y} & \frac{\partial H}{\partial z} & 0 & 0 & \frac{\partial H}{\partial Z}
\end{bmatrix}
\begin{bmatrix} v_x \\ v_y \\ v_z \\ v_X \\ V_Y \\ v_Z \end{bmatrix}
+ [\text{Equation (18.6)}] \tag{22.29}$$

where

$$\begin{aligned}
\frac{\partial F}{\partial x} &= Sr_{11} & \frac{\partial F}{\partial y} &= Sr_{21} & \frac{\partial F}{\partial z} &= Sr_{31} & \frac{\partial F}{\partial X} &= -1 \\
\frac{\partial G}{\partial x} &= Sr_{12} & \frac{\partial G}{\partial y} &= Sr_{22} & \frac{\partial G}{\partial z} &= Sr_{32} & \frac{\partial G}{\partial Y} &= -1 \\
\frac{\partial H}{\partial x} &= Sr_{13} & \frac{\partial H}{\partial y} &= Sr_{23} & \frac{\partial H}{\partial z} &= Sr_{33} & \frac{\partial H}{\partial Z} &= -1
\end{aligned}$$

The remaining partial derivatives were given in Equation (18.6) of Section 18.7. This system is solved using the methods discussed in Section 22.3.

Example 22.3 Estimated errors were added to the control coordinates in Example 18.4. The control data are repeated in Table 22.2 and standard deviations of the control coordinates, needed to form the Q matrix, are also listed. Following Table 22.2, output from the program ADJUST is listed. The file *Example 22-3.dat* on the book's companion website uses the same format as was described in Example 18.4. However, in files using the general least-squares method, standard deviations can be given for the control coordinates also. Note that this solution shown in Figure 22.3 differs slightly from the one obtained by standard least-squares in Example 18.4.

3D Coordinate Transformation of File:
using generalized least squares method

Measured Points

NAME	x	y	z	Vx	Vy	Vz
1	1094.883	820.085	109.821	-0.001	-0.001	-0.000
2	503.891	1598.698	117.685	-0.002	0.000	0.000
3	2349.343	207.658	151.387	-0.001	0.000	0.000
4	1395.320	1348.853	215.261	0.001	0.000	-0.001

CONTROL POINTS

NAME	X	VX	Y	VY	Z	VZ
1	10037.810	-0.063	5262.090	-0.026	772.040	-0.001
2	10956.680	-0.019	5128.170	0.063	783.000	-0.027
3	8780.080	0.000	4840.290	0.038	782.620	-0.001
4	10185.800	0.032	4700.210	-0.085	851.320	0.007

Transformation Coefficients
Scale = 0.94996 +/- 0.00002
Omega = 2° 17' 00.0" +/- 0° 00' 26.7"
Phi = -0° 33' 05.6" +/- 0° 00' 06.1"
Kappa = 224° 32' 11.5" +/- 0° 00' 07.7"
Tx = 10233.855 +/- 0.066
Ty = 6549.964 +/- 0.055
Tz = 720.867 +/- 0.219

Reference Standard Deviation: 1.293
Degrees of Freedom: 5
Iterations: 3

Transformed Coordinates

NAME	X	Sx	Y	Sy	Z	Sz
1	10037.874	0.082	5262.116	0.063	772.041	0.275
2	10956.701	0.087	5128.106	0.070	783.027	0.286
3	8780.080	0.098	4840.251	0.087	782.622	0.314
4	10185.767	0.096	4700.296	0.072	851.313	0.324
5	10722.016	0.075	5691.210	0.062	766.067	0.246
6	10043.245	0.074	5675.887	0.060	816.857	0.246

FIGURE 22.3 Solution of Example 22.3 using ADJUST.

TABLE 22.2 Control Data for Three-Dimensional Conformal Coordinate Transformation

Point	X	Y	Z	S_X	S_Y	S_Z
1	10,037.81	5,262.09	772.04	0.05	0.06	0.05
2	10,956.68	5,128.17	783.00	0.04	0.06	0.09
3	8,780.08	4,840.29	782.62	0.02	0.04	0.02
4	10,185.80	4,700.21	851.32	0.03	0.05	0.03

PROBLEMS

Note: For problems below requiring least-squares adjustment, if a computer program is not distinctly specified for use in the problem, it is expected that the least-squares algorithm will be solved using the program MATRIX, which can be found on the book's companion website. Partial answers to problems marked with an asterisk can be found in Appendix G.

***22.1** Solve Problem 11.11 using the general least-squares method. Assign all coordinates a standard deviation of 1.

22.2 Solve Problem 11.16 using the general least-squares method. Assign the computed coordinates the following standard deviations.

Station	S_X	S_Y
A	0.001	0.001
C	0.020	0.013
E	0.020	0.027
G	0.020	0.035
I	0.020	0.046

Station	S_X	S_Y
B	0.020	0.012
D	0.020	0.021
F	0.020	0.028
H	0.020	0.040

22.3 Solve Problem 11.18 using the general least-squares method. Assign all coordinates a standard deviation of 0.1.

***22.4** Solve Problem 18.1 using the general least-squares method. Use the following standard deviations for the control coordinates.

Station	S_E	S_N
A	0.025	0.023
B	0.028	0.029
C	0.021	0.020
D	0.033	0.027

22.5 Solve Problem 18.2 using the general least-squares method given the following standard deviations for the control coordinates.

Point	S_X	S_Y
A	0.0010	0.0009
B	0.0010	0.0008
C	0.0009	0.0010
D	0.0011	0.0011

***22.6** Solve Problem 18.3 using the general least-squares method and the additional data in Problem 22.4.

22.7 Solve Problem 18.4 using the general least-squares method and the additional data in Problem 22.5.

***22.8** Solve Problem 18.12 using the general least-squares method and the following standard deviations for the control points.

Point	S_E	S_N
1	0.010	0.010
2	0.008	0.011
3	0.009	0.010
4	0.010	0.011

22.9 Solve Problem 18.13 using the general least-squares method and the following standard deviations for the control points.

Point	S_E	S_N
1	0.020	0.019
2	0.016	0.022
3	0.018	0.021
4	0.019	0.023

22.10 Solve Problem 18.14 using the general least-squares method and the following standard deviations for the control points.

Point	S_X	S_Y
1	0.0019	0.0019
2	0.0020	0.0016
3	0.0023	0.0019
4	0.0016	0.0019
5	0.0022	0.0021
6	0.0022	0.0018
7	0.0019	0.0021

22.11 Solve Problem 18.16 using the general least-squares method and the following standard deviations for the control points.

Point	S_E	S_N	S_H
A	0.005	0.004	0.006
B	0.005	0.005	0.007
C	0.005	0.005	0.005
D	0.005	0.005	0.005

22.12 Solve Problem 18.17 using the general least-squares method and the following standard deviations for the control points.

Point	S_E	S_N	S_h
A	0.010	0.009	0.012
B	0.010	0.010	0.011
C	0.010	0.010	0.009
D	0.011	0.010	0.008

22.13 Use the general least-squares method to transform the points from the measured system to the control system using a two-dimensional coordinate transformation. What method of adjustment is most appropriate for this data set.

(a) Conformal transformation.

(b) Affine transformation.

(c) Projective transformation.

	Measured Coordinates				Control Coordinates			
Point	x (mm)	y (mm)	S_x (mm)	S_y (mm)	X (mm)	Y (mm)	S_X (mm)	S_Y (mm)
6	103.555	−103.670	0.003	0.004	103.951	−103.969	0.009	0.009
7	0.390	−112.660	0.005	0.005	0.001	−112.999	0.009	0.009
3	0.275	111.780	0.004	0.007	0.001	112.993	0.009	0.009
4	103.450	102.815	0.003	0.004	103.956	103.960	0.009	0.009
5	112.490	−0.395	0.003	0.003	112.998	0.003	0.009	0.009
32	18.565	−87.580						
22	−5.790	2.035						
12	6.840	95.540						
13	86.840	102.195						
23	93.770	2.360						
33	92.655	−90.765						

22.14 Given the following data, transform the points from the measured system to the control system using a three-dimensional conformal coordinate transformation. Assume that all coordinates have estimated standard deviations of ±0.05 m. Use the general least- squares method.

	Measured Coordinates			Control Coordinates		
Point	x (m)	y (m)	z (m)	X (m)	Y (m)	Z (m)
1	607.54	501.63	469.09	390.13	499.74	469.32
2	589.98	632.36	82.81	371.46	630.95	81.14
3	643.65	421.28	83.50	425.43	419.18	82.38
4	628.58	440.51	82.27			
5	666.27	298.16	98.29			
6	632.59	710.62	103.01			

Solve each problem with the ADJUST program using the standard and general least-squares adjustment options. Compare and explain any noted differences in the solutions.

22.15 Problem 22.1.

22.16 Problem 22.2.

22.17 Problem 22.3.

22.18 Problem 22.4.

22.19 Problem 22.5.

22.20 Problem 22.6.

22.21 Problem 22.7.

22.22 Problem 22.8.

22.23 Problem 22.9.

22.24 Problem 22.10.

22.25 Problem 22.11.

22.26 Problem 22.12.

22.27 Problem 22.13.

22.28 Problem 22.14.

PROGRAMMING PROBLEMS

Develop a computational program that constructs the B, Q, J, and K matrices for each of the following transformations:

22.29 A two-dimensional conformal coordinate transformation.

22.30 A two-dimensional affine coordinate transformation.

22.31 A two-dimensional projective coordinate transformation.

22.32 A three-dimensional conformal coordinate transformation.

CHAPTER 23

THREE-DIMENSIONAL GEODETIC NETWORK ADJUSTMENT

23.1 INTRODUCTION



With the advent of total station instruments, survey data are collected in three dimensions. Thus, it is advantageous to develop an adjustment model that works in three dimensions. Rigorous triangulation adjustment models date back to Bruns (1878). The main observational data consists of horizontal angles, vertical angles, azimuths, and slant distances. It is also possible to include differential leveling observations in the model. Since all data are collected on the Earth's surface, the local geodetic coordinate system provides a natural system in which to perform the adjustment.

As shown in Figure 23.1, the local geodetic system is oriented such that the n-axis points along the meridian of the ellipse (local geodetic north), the u-axis is aligned along the normal of the ellipsoid, and the e-axis creates a right-handed coordinate system. The local geodetic coordinate system can be related to the geocentric coordinate system (see Section 17.5) through a series of three-dimensional rotations discussed in Section 18.7. To align the X-axis with the e-axis, the Z-axis is rotated by an amount of $\lambda - 180^\circ$. Then the Z-axis is aligned with the u-axis by a rotation of $\varphi - 90^\circ$ about the once rotated X-axis. The resultant expression is

$$\begin{bmatrix} \Delta n \\ -\Delta e \\ \Delta u \end{bmatrix} = R_2(\varphi - 90^\circ)\, R_3(\lambda - 180^\circ) \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix} \tag{23.1}$$

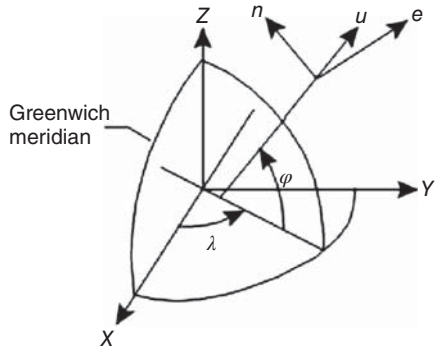


FIGURE 23.1 Relationship between the geocentric and local geodetic coordinate systems.

In Equation (23.1), changes in the local geodetic coordinate systems and geocentric coordinate system are represented by (Δn, Δe, Δu) and (ΔX, ΔY, ΔZ), respectively. Changing the algebraic sign of e and combining the rotation matrices given in Section 18.7 yields

$$\begin{bmatrix} \Delta n \\ -\Delta e \\ \Delta u \end{bmatrix} = \begin{bmatrix} -\sin\varphi\cos\lambda & -\sin\varphi\sin\lambda & \cos\varphi \\ -\sin\lambda & \cos\lambda & 0 \\ \cos\varphi\cos\lambda & \cos\varphi\sin\lambda & \sin\varphi \end{bmatrix} \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix} \tag{23.2}$$

The changes in the coordinates of the local geodetic system can be determined from the observation of azimuth Az, slant distance s, and altitude angle v. As shown in Figure 23.2, the changes in the local geodetic coordinate system can be computed as

$$\begin{aligned} \Delta n &= s\cos v\cos \mathrm{Az} \\ \Delta e &= s\cos v\sin \mathrm{Az} \\ \Delta u &= s\sin v \end{aligned} \tag{23.3}$$

Also, from Figure 23.2, the following inverse relationships can be developed:

$$\begin{aligned} s &= \sqrt{\Delta n^2 + \Delta e^2 + \Delta u^2} \\ \mathrm{Az} &= \tan^{-1}\left(\frac{\Delta e}{\Delta n}\right) \\ v &= \sin^{-1}\left(\frac{\Delta u}{s}\right) \end{aligned} \tag{23.4}$$

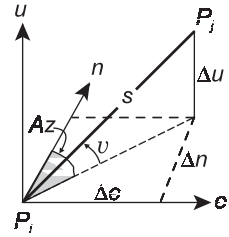


FIGURE 23.2 Reduction of observations in a local geodetic coordinate system.

By combining Equations (23.2) and (23.4), the reduced observations can be computed using changes in the geocentric coordinates. The resultant equations are

$$IJ = \sqrt{\Delta X^2 + \Delta Y^2 + \Delta Z^2} \tag{23.5}$$

$$\mathrm{Az}_{ij} = \tan^{-1}\left(\frac{-\Delta X \sin\lambda_i + \Delta Y \cos\lambda_i}{-\Delta X \sin\varphi_i \cos\lambda_i - \Delta Y \sin\varphi_i \sin\lambda_i + \Delta Z \cos\varphi_i}\right) \tag{23.6}$$

$$\upsilon_{ij} = \sin^{-1}\left(\frac{\Delta X \cos\varphi_i \cos\lambda_i + \Delta Y \cos\varphi_i \sin\lambda_i + \Delta Z \sin\varphi_i}{IJ}\right) \tag{23.7}$$

In Equations (23.5) to (23.7), φ_i and λ_i are the latitude and longitude of the observing station P_i; ΔX is $X_j - X_i$; ΔY is $Y_j - Y_i$; and ΔZ is $Z_j - Z_i$. For completeness, the equation for the zenith angle is

$$z_{ij} = \cos^{-1}\left(\frac{\Delta X \cos\varphi_i \cos\lambda_i + \Delta Y \cos\phi_i \sin\lambda_i + \Delta Z \sin\varphi_i}{IJ}\right) \tag{23.8}$$

Furthermore, since an angle is the difference between two azimuths, Equation (23.6) can be applied to horizontal angles.

23.2 LINEARIZATION OF EQUATIONS

Equations (23.5) through (23.8) can be linearized with respect to the local geodetic coordinates. The development of these equations is covered by Vincenty (1985) and Leick et al. (2015). The final prototype equations for slant distances, azimuths, horizontal angles, and altitude angles are listed in the following subsections. It is important to note that the residuals implicitly exist in each equation to make the equations consistent when using observations containing random errors.

23.2.1 Slant Distance Observations

In the three-dimensional model, slant distance (also known as *slope distance*) does not need to be reduced to either the station ground marks or ellipsoid in the functional model. However, if they are not reduced, then the computed values

from Equations (23.5) to (23.8) are based on the instrument and reflector locations and not the station's ground marks.[1] The prototype equation for the observed slant distances is

$$a_1 dn_i + a_2 de_i + a_3 du_i + a_4 dn_j + a_5 de_j + a_6 du_j = s_{ij} - IJ_0 \tag{23.9}$$

In Equation (23.9), the coefficients a_1 to a_6 are defined in Table 23.1, IJ_0 is the slant distance computed using Equation (23.5), and s_{ij} is the observed slant distance.

To demonstrate the derivation of the coefficients, a_1 is derived by taking the partial derivative of the slant distance formula in Equation (23.5) with respect to n_i, or

$$\frac{\partial s}{\partial n_i} = -\frac{\Delta n}{s} \tag{a}$$

TABLE 23.1 Coefficients for Linearized Equations in Equations (23.11) through (23.13)

$(a)\ a_1 = -(\cos v_{ij} \cos \mathrm{Az}_{ij})_0 \quad (b)\ a_2 = -(\cos v_{ij} \sin \mathrm{Az}_{ij})_0 \quad (c)\ a_3 = -(\sin v_{ij})_0$

$(d)\ a_4 = -(\cos v_{ji} \cos \mathrm{Az}_{ji})_0 \quad (e)\ a_5 = -(\cos v_{ji} \sin \mathrm{Az}_{ji})_0 \quad (f)\ a_6 = -(\sin v_{ji})_0$

$$(g)\ b_1 = \left(\frac{\sin \mathrm{Az}_{ij}}{IJ \cos v_{ij}}\right)_0 \quad (h)\ b_2 = -\left(\frac{\cos \mathrm{Az}_{ij}}{IJ \cos v_{ij}}\right)_0 \quad (i)\ b_3 = 0$$

$$(j)\ b_4 = -\left\{\left(\frac{\sin \mathrm{Az}_{ij}}{IJ \cos v_{ij}}\right)[\cos(\varphi_j - \varphi_i) + \sin \varphi_j \sin(\lambda_j - \lambda_i)/\tan \mathrm{Az}_{ij}]\right\}_0$$

$$(k)\ b_5 = \left\{\left(\frac{\cos \mathrm{Az}_{ij}}{IJ \cos v_{ij}}\right)[\cos(\lambda_j - \lambda_i) - \sin \varphi_i \sin(\lambda_j - \lambda_i) \tan \mathrm{Az}_{ij}]\right\}_0$$

$$(l)\ b_6 = \left\{\left(\frac{\cos \mathrm{Az}_{ij} \cos \varphi_j}{IJ \cos v_{ij}}\right)[\sin(\lambda_j - \lambda_i) + (\sin \varphi_i \cos(\lambda_j - \lambda_i) - \cos \varphi_i \tan \varphi_j) \tan \mathrm{Az}_{ij}]\right\}_0$$

$(m)\ c_1 = (\sin v_{ij} \cos \mathrm{Az}_{ij}/IJ)_0 \quad (n)\ c_2 = (\sin v_{ij} \sin \mathrm{Az}_{ij}/IJ)_0 \quad (o)\ c_3 = -(\cos v_{ij}/IJ)_0$

$$(p)\ c_4 = \left(\frac{-\cos \varphi_i \sin \varphi_j \cos(\lambda_j - \lambda_i) + \sin \varphi_i \cos \varphi_j + \sin v_{ij} \cos v_{ji} \cos \mathrm{Az}_{ji}}{IJ \cos v_{ij}}\right)_0$$

$$(q)\ c_5 = \left(\frac{-\cos \varphi_i \sin(\lambda_j - \lambda_i) + \sin v_{ij} \cos v_{ji} \sin \mathrm{Az}_{ji}}{IJ \cos v_{ij}}\right)_0$$

$$(r)\ c_6 = \left(\frac{\sin v_{ij} \sin v_{ji} + \sin \varphi_i \sin \varphi_j + \cos \varphi_i \cos \varphi_j \cos(\lambda_j - \lambda_i)}{IJ \cos v_{ij}}\right)_0$$

[1] This website can be found at https://hpiers.obspm.fr/eop-pc/index.php.

By substituting Δn from Equation (23.3), the resulting equation for a_1 is

$$\frac{\partial s}{\partial n_i} = -\cos v_{ij} \cos \mathrm{Az}_{ij} \qquad (b)$$

Following these procedures, the remaining coefficients for Equation (23.9) are derived.

23.2.2 Azimuth Observations

The prototype equation for the observed azimuths is

$$b_1 dn_i + b_2 de_i + b_3 du_i + b_4 dn_j + b_5 de_j + b_6 du_j = \alpha_{ij} - \alpha_0 \qquad (23.10)$$

In Equation (23.10), α_{ij} is the observed azimuth and α_0 its computed value based on Equation (23.6) using approximate values for station's I and J coordinates.

As an example, the coefficient for b_1 is computed by taking the partial derivative of the azimuth formula in Equation (23.4) with respect to n_i, or

$$\frac{\partial \mathrm{Az}_{ij}}{\partial n_i} = \frac{\Delta n^2}{\Delta n^2 + \Delta e^2} \frac{\Delta e}{\Delta n^2} = \frac{\Delta e}{\Delta n^2 + \Delta e^2} \qquad (c)$$

By substituting the appropriate formulas from Equation (23.3), the resulting equation for b_1 is

$$\begin{aligned} \frac{\partial \mathrm{Az}_{ij}}{\partial n_i} &= \frac{s \cos v_{ij} \sin \mathrm{Az}_{ij}}{s^2 \cos^2 v_{ij} \cos^2 \mathrm{Az}_{ij} + s^2 \cos^2 v_{ij} \sin^2 \mathrm{Az}_{ij}} \\ &= \frac{s \cos v_{ij} \sin \mathrm{Az}_{ij}}{s^2 \cos^2 v_{ij} (\cos^2 \mathrm{Az}_{ij} + \sin^2 \mathrm{Az}_{ij})} \\ &= \frac{\sin \mathrm{Az}_{ij}}{s \cos v_{ij}} \end{aligned} \qquad (d)$$

In a similar fashion, the remaining coefficients of Equation (23.10) can be derived. For the remaining linearized observation equations in this section, the derivation of the coefficients for each equation follows procedures like those presented in Equations (a) through (d). The derivations are left to the reader.

23.2.3 Altitude Angle Observations

Altitude angles are observed in the vertical plane and have a value of zero at the horizon. Of course, zenith angles, which are more commonly observed in practice, have their zero value in the direction of the zenith at the observing station.

As discussed in Section 23.7, all vertical angles can be subject to substantial systematic errors caused by deflection of the plumb line and refraction. Due to these errors, vertical angles should not be used in an adjustment on a regular basis. If these observations must be used in an adjustment, it is important to either correct the observations for the systematic errors caused by refraction or add the unknown parameters to the mathematical model to correct for the systematic errors. Since refraction is difficult to model along one's line of sight, it is always better to not include vertical angles in any adjustment. Also, as discussed in Section 23.7, adding correction parameters to a typical survey runs the risk of overparameterization. This occurs when there are more unknowns at a particular station than there are observations. In this case, the system is unsolvable. Thus, it is assumed that the corrections will be made before the adjustment. The appropriate prototype equation is

$$c_1 dn_i + c_2 de_i + c_3 du_i + c_4 dn_j + c_5 de_j + c_6 du_j = v_{ij} - v_0 \tag{23.11}$$

The coefficients for Equation (23.11) are listed in Table 23.1. Their values are evaluated using the approximate coordinate values, v_{ij} is the observed altitude angle, and v_0 is the altitude angle computed using Equation (23.5) and approximate station coordinates. However, given the problems with modeling refraction, this section is only provided for completeness since it is not recommended to include vertical angles in any adjustment.

23.2.4 Horizontal Angle Observations

As stated earlier, horizontal angles are the difference in two azimuths. That is, θ_{bif} is computed as $\mathrm{Az}_{if} - \mathrm{Az}_{ib}$ where b is the backsight station; i is the instrument station; and f is the foresight station. The prototype equation for a horizontal angle is

$$d_1 dn_b + d_2 de_b + d_3 du_b + d_4 dn_i + d_5 de_i + d_6 du_i + d_7 dn_f + d_8 de_f + d_9 du_f = \theta_{bif} - \theta_0 \tag{23.12}$$

The coefficients, d_1 through d_9, in Equation (23.12) are listed in Table 23.2; θ_{bif} is the observed horizontal angle; and θ_0 is the computed value for the angle based on difference between the foresight and backsight azimuths computed using Equation (23.6).

23.2.5 Differential Leveling Observations

Orthometric height differences as derived from differential leveling can also be included in the three-dimensional geodetic network adjustment model. However, as discussed in Section 23.7, the inclusion of this observation type requires a

TABLE 23.2 Coefficients for Linearized Equation (23.14)

$$(s)\ d_1 = \left\{ \frac{\sin \mathrm{Az}_{ib}}{IB \cos v_{ib}} [\cos(\varphi_b - \varphi_i) + \sin \varphi_b \sin(\lambda_b - \lambda_i) / \tan \mathrm{Az}_{ib}] \right\}_0$$

$$(t)\ d_2 = -\left\{ \frac{\cos \mathrm{Az}_{ib}}{IB \cos v_{ib}} [\cos(\lambda_b - \lambda_i) - \sin \varphi_i \sin(\lambda_b - \lambda_i) \tan \mathrm{Az}_{ib}] \right\}_0$$

$$(u)\ d_3 = -\left\{ \frac{\cos \mathrm{Az}_{ib} \cos \varphi_b}{IB \cos v_{ib}} [\sin(\lambda_b - \lambda_i) + (\sin \varphi_i \cos(\lambda_b - \lambda_i) - \cos \varphi_i \tan \varphi_b) \tan \mathrm{Az}_{ib}] \right\}_0$$

$$(v)\ d_4 = \left(\frac{\sin \mathrm{Az}_{if}}{IF \cos v_{if}} - \frac{\sin \mathrm{Az}_{ib}}{IB \cos v_{ib}} \right)_0 \quad (w)\ d_5 = \left(\frac{\cos \mathrm{Az}_{ib}}{IB \cos v_{ib}} - \frac{\cos \mathrm{Az}_{if}}{IF \cos v_{if}} \right)_0 \quad (x)\ d_6 = 0$$

$$(y)\ d_7 = -\left\{ \frac{\sin \mathrm{Az}_{if}}{IF \cos v_{if}} [\cos(\varphi_f - \varphi_i) + \sin \varphi_f \sin(\lambda_f - \lambda_i) / \tan \mathrm{Az}_{if}] \right\}_0$$

$$(z)\ d_8 = \left\{ \frac{\cos \mathrm{Az}_{if}}{IF \cos v_{if}} [\cos(\lambda_f - \lambda_i) - \sin \varphi_i \sin(\lambda_f - \lambda_i) \tan \mathrm{Az}_{if}] \right\}_0$$

$$(aa)\ d_9 = \left\{ \frac{\cos \mathrm{Az}_{if} \cos \varphi_f}{IF \cos v_{if}} [\sin(\lambda_f - \lambda_i) + (\sin \varphi_i \cos(\lambda_f - \lambda_i) - \cos \varphi_i \tan \varphi_f) \tan \mathrm{Az}_{if}] \right\}_0$$

correction for geoidal height differences between the stations and the application of orthometric corrections. Since the adjustment model is nonlinear, the observation equation for elevation differences as given in Equation (12.1) must also be linearized. The prototype equation for differential leveling between stations I and J is

$$1du_j - 1du_i \approx \Delta H_{ij} + \Delta N_{ij} - \Delta h_{ij} \tag{23.13}$$

In Equation (23.13), ΔH_{ij} is the observed elevation difference between the stations; ΔN_{ij} is the difference in geoidal height between the stations; and Δh_{ij} is the change in geodetic height. The approximately equal sign $\approx$ is used in Equation (23.13) since the geodetic height is a geometric distance from the ellipsoid to the point, whereas the orthometric height is the distance to the point from the geoid along a curved plumb line. The difference between the linear distance and that of the curve is very small and can be absorbed easily by the residuals. Thus, Equation (23.13) is shown with an equality sign typically.

23.2.6 Horizontal Distance Observations

As shown in Figure 23.3, the elevation differences of the endpoint stations and converging radii cause the horizontal distances observed from opposite ends of the

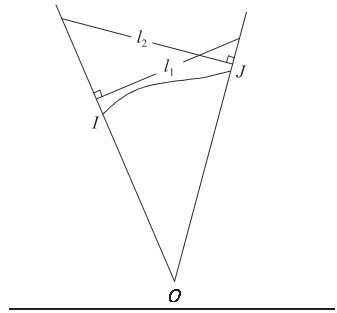


FIGURE 23.3 Comparison of horizontal distances from opposite ends of the line.

lines to start and terminate at different points. Thus, technically l_1 and l_2 are not of the same length. In the local geodetic system, the observation equations for l_1 and l_2 are

$$l_1 = \sqrt{\Delta n_1^2 + \Delta e_1^2}$$
$$l_2 = \sqrt{\Delta n_2^2 + \Delta e_2^2} \tag{23.14}$$

However, for the short distances observed in surveying practice typically, l_1 and l_2, Δn_1 and Δn_2, and Δe_1 and Δe_2 are approximately equal. Letting l equal l_1 and l_2, Δn equal Δn_1 and $-\Delta n_2$, and Δe equal Δe_1 and $-\Delta e_2$, the linearized observation equation in local geodetic system is

$$\left(\frac{\varphi_j - \varphi_i}{IJ} M_i\right)_0 dn_i + \left(\frac{\lambda_i - \lambda_j}{IJ} N_j\right) de_i - \left(\frac{\varphi_j - \varphi_i}{IJ} M_i\right)_0 dn_j - \left(\frac{\lambda_i - \lambda_j}{IJ} N_j\right) de_j = l_{ij} - l_0 \tag{23.15}$$

In Equation (23.15), M_i is the radius in the meridian at the observing station I; N_i is the radius in the normal at the observing station; l_{ij} is the observed horizontal distance; and l_0 is the computed distance using approximate coordinates and Equation (23.14). The radii are computed as

$$M_i = \frac{a(1 - e^2)}{(1 - e^2 \sin^2 \varphi_i)^{3/2}}$$
$$N_i = \frac{a}{\sqrt{1 - e^2 \sin^2 \varphi_i}} \tag{23.16}$$

23.3 MINIMUM NUMBER OF CONSTRAINTS



A three-dimensional geodetic network adjustment requires both horizontal and vertical control. As discussed in Section 16.5, fixing the horizontal part of the adjustment requires one station fixed in position and one line of known direction. This can be accomplished by fixing the latitude and longitude of one station, along with the azimuth of a line or the longitude of a second station. The vertical plane in the adjustment can be fixed with three benchmark stations. Since the adjustment is performed with geodetic heights, the orthometric height of the benchmark stations must be corrected using Equation (23.37).

As discussed in Chapter 20, control can be adjusted or fixed simply by setting the appropriate values in the stochastic model. In the case of vertical control, benchmarks are often given as orthometric heights. Since the geoid model is only known to a few centimeters, the standard deviations of geodetic heights for benchmark stations should not be set any better than a centimeter.

23.4 EXAMPLE ADJUSTMENT

To illustrate a three-dimensional least-squares adjustment, the simple network shown in Figure 23.4 will be used. The standard deviations for the coordinates are shown in Table 23.3 in the local geodetic system. This system was chosen since it is an intuitive system for assignment of realistic uncertainty values in the adjustment, and since the adjustment will be performed in the local geodetic system. As shown in Table 23.3, the network is fixed in horizontal position by overweighting of the northing and easting coordinates of station *A* and rotationally by overweighting the azimuth of line *AB*. The horizontal rotation of the network could have also been fixed by overweighting of the easting of station *B*.

The elevation datum was fixed in position and rotation by overweighting the height components of stations *A*, *B*, and *C*. Since orthometric heights were given

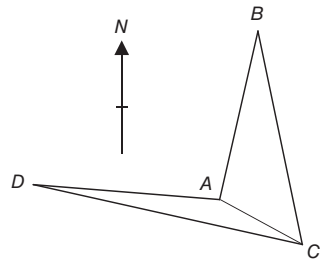


FIGURE 23.4 Example three-dimensional geodetic network.

TABLE 23.3 Data for Figure 23.4

Geodetic Positions

Point	Latitude (φ)	Longitude (λ)	Height (m)	N (m)
A	41°18′26.04850″N	76°00′10.24860″W	372.221	−31.723
B	41°18′40.46660″N	76°00′05.50180″W	351.394	−31.713
C	41°18′22.04010″N	76°00′00.94390″W	362.865	−31.726
D	41°18′27.65860″N	76°00′31.38550″W	370.874	−31.722

Geodetic Azimuth

α_{AB}	13°56′26.9″ ± 0.001″

Slant Distances

Course	Distance	S (m)
AB	458.796	0.005
AC	249.462	0.005
CD	729.122	0.006
DA	494.214	0.005
BC	578.393	0.005

Horizontal Angles

Stations	Angles	S (″)
DAB	98°10′25″	2.8
BAC	105°47′45″	3.5
CAD	156°01′44″	4.1
ABC	335°29′37″	2.4
CBA	24°30′19″	2.4
BCD	294°19′17″	2.3
DCA	15°59′01″	3.3
ACB	49°41′45″	3.4
CDA	352°00′45″	2.2
ADC	7°59′12″	2.2

Zenith Angles

Stations	Angle	S (″)
AC	92°09′01″	2.5
CD	89°22′24″	0.8
DA	89°50′44″	1.2
AB	92°36′12″	1.4
BC	88°52′01″	1.1

TABLE 23.3 (*Continued*)

Elevation Differences

Stations	ΔElev (m)	S (m)
AC	−9.359	0.005

Uncertainties of Positions in the Local Geodetic System

Station	S_n (m)	S_e (m)	S_h (m)
A	0.001	0.001	0.01
B	–	–	0.01
C	–	–	0.01
D	–	–	–

for the stations, the geoid separation was applied following Equation (23.37) to compute geodetic heights for the stations. All the distance observations were observed in metric units. The systematic errors discussed in Section 23.7 were removed from all angular observations. The results of the adjustment are shown in Figure 23.4.

While the adjustment is performed in the local geodetic system, both geodetic and geocentric coordinates are required to compute the coefficients and computed observations. Since traditional observations are taken by an elevated instrument to some elevated target, the geodetic heights of each station must be increased by the setup heights when computing geodetic coordinates. This simple addition to the software removes the need for reducing observations to the station marks. In this example, the setup heights of the instruments and targets are assumed to be zero and do not need to be considered.

23.4.1 Addition of Slant Distances

Following prototype Equation (23.9), each slant distance observation adds one row to the system of equations. As an example for the slant distance AC, the coefficients are computed as

$$-(\cos v_{\mathrm{AB}} \cos \mathrm{Az}_{AB})_0\, dn_A - (\cos v_{AB} \sin \mathrm{Az}_{AB})_0\, de_A - (\sin v_{AB})_0 du_A$$
$$-(\cos v_{BA} \cos \mathrm{Az}_{BA})_0\, dn_B - (\cos v_{BA} \sin \mathrm{Az}_{BA})_0\, de_B - (\sin v_{BA})_0 du_B = s_{AB} - AB_0 \tag{23.17}$$

In Equation (23.17), the values for Az_{AB} and Az_{BA} are computed using Equation (23.6); v_{AB} and v_{BA} are computed using Equation (23.7); and AB

using Equation (23.5). The numerical values, to five decimal places, for Equation (23.17) are

$$-0.96954dn_A - 0.24067de_A + 0.04541du_A - 0.96954dn_B + 0.24069de_B - 0.04534du_B = 0.0017 \tag{23.18}$$

For each slant distance, a similar observation equation is written. The reader should note in Equation (23.18) that unlike plane adjustments as presented in Chapter 14, the coefficients of the occupied and sighted stations vary slightly due to Earth curvature. Four more equations for distances AC, CD, DA, and BC are added to the system of equations.

23.4.2 Addition of Horizontal Angles

The observed angles in the network were corrected for the systematic errors caused by the height of targets and deflection of the vertical, as given in Equations (23.30) through (23.32). Following prototype Equation (23.12), an observation equation is written for each horizontal angle. As an example, the observation equation for angle DAB is

$$\begin{aligned}
&\left\{ \frac{\sin \mathrm{Az}_{AD}}{AD \cos \upsilon_{AD}} \left[\cos(\varphi_D - \varphi_A) + \frac{\sin \varphi_D \sin(\lambda_D - \lambda_A)}{\tan \mathrm{Az}_{AD}} \right] \right\}_0 dn_D \\
&- \left\{ \frac{\cos \mathrm{Az}_{AD}}{AD \cos \upsilon_{AD}} [\cos(\lambda_D - \lambda_A) + \sin \varphi_A \sin(\lambda_D - \lambda_A) \tan \mathrm{Az}_{AD}] \right\}_0 de_D \\
&- \left\{ \frac{\sin \mathrm{Az}_{AD} \cos \varphi_D}{AD \cos \upsilon_{AD}} \left[\sin(\lambda_D - \lambda_A) + \begin{pmatrix} \sin \varphi_A \cos(\lambda_D - \lambda_A) \\ -\cos \varphi_A \tan \phi_D \end{pmatrix} \tan \mathrm{Az}_{AD} \right] \right\}_0 du_D \\
&+ \left(\frac{\sin \mathrm{Az}_{AB}}{AB \cos \upsilon_{AB}} - \frac{\sin \mathrm{Az}_{AD}}{AD \cos \upsilon_{AD}} \right)_0 dn_A \\
&+ \left(\frac{\cos \mathrm{Az}_{AD}}{AD \cos \upsilon_{AD}} - \frac{\cos \mathrm{Az}_{AB}}{AB \cos \upsilon_{AB}} \right)_0 de_A + 0du_A \\
&- \left\{ \frac{\sin Az_{AB}}{AB \cos \upsilon_{AB}} \left[\cos(\varphi_B - \varphi_A) + \frac{\sin \varphi_B \sin(\lambda_B - \lambda_A)}{\tan Az_{AB}} \right] \right\}_0 dn_B \\
&+ \left\{ \frac{\cos \mathrm{Az}_{AB}}{AB \cos \upsilon_{AB}} [\cos(\lambda_B - \lambda_A) + \sin \varphi_A \sin(\lambda_B - \lambda_A) \tan \mathrm{Az}_{AB}] \right\}_0 de_B \\
&+ \left\{ \frac{\sin \mathrm{Az}_{AB} \cos \varphi_B}{AB \cos \upsilon_{AB}} \left[\sin(\lambda_B - \lambda_A) + \begin{pmatrix} \sin \varphi_A \cos(\lambda_B - \lambda_A) \\ -\cos \varphi_A \tan \phi_B \end{pmatrix} \tan \mathrm{Az}_{AB} \right] \right\}_0 du_B \\
&= \theta_{DAB} - \theta_0
\end{aligned} \tag{23.19}$$

In Equation (23.19), the values in the {} are evaluated at their approximate coordinate values, θ_{DAB} is the observed angular value corrected for the systematic

errors discussed in Section 23.7, and θ_0 is the value computed based on the difference in the computed values for the backsight (AD) and foresight (AB) azimuths. Substituting the appropriate values into Equation (23.19) and converting the radian values to units of s/m results in

$$\begin{aligned}&-415.247dn_D - 41.991de_D - 0.00001du_D + 523.669dn_A\\&-394.828de_A + 0du_A - 108.dn_B + 436.790de_B - 0.00003du_B = 11.582''\end{aligned} \tag{23.20}$$

For each angle, a similar observation equation is written resulting in 10 equations in total to be added to the adjustment.

23.4.3 Addition of Zenith Angles

All the zenith angles in this example problem were corrected for deflection of the vertical, refraction, and target height, as discussed in Section 23.7. Following Equation (23.11), the observation equation for the zenith angle from A to C is

$$\begin{aligned}&\left(\frac{\sin v_{AC}\cos \mathrm{Az}_{AC}}{AC}\right)_0 dn_A + \left(\frac{\sin v_{AC}\sin \mathrm{Az}_{AC}}{AC}\right)_0 de_A - \left(\frac{\cos v_{AC}}{AC}\right)_0 du_A\\&\quad+\left(\frac{-\cos\varphi_A\sin\varphi_C\cos(\lambda_C-\lambda_A)+\sin\varphi_A\cos\varphi_C+\sin v_{AC}\cos v_{CA}\cos \mathrm{Az}_{CA}}{AC\cos v_{AC}}\right)_0 dn_C\\&\quad+\left(\frac{-\cos\varphi_A\sin(\lambda_C-\lambda_A)+\sin v_{AC}\cos v_{CA}\sin \mathrm{Az}_{CA}}{AC\cos v_{AC}}\right)_0 de_C\\&\quad+\left(\frac{\sin v_{AC}\sin v_{CA}+\sin\varphi_A\sin\varphi_C+\cos\varphi_A\cos\varphi_C\cos(\lambda_C-\lambda_A)}{AC\cos v_{AC}}\right)_0 du_C\\&\quad= v_{AC} - v_0\end{aligned} \tag{23.21}$$

In Equation (23.21), the coefficients on the left side of the equations are evaluated using the approximate values of the coordinates in Equations (23.5) through (23.7). The numerical values for Equation (23.21) in units of s/m are

$$\begin{aligned}&15.395dn_A - 26.947de_A - 826.229du_A - 15.379dn_C + 26.919de_C + 826.230du_C\\&\quad= 2.971''\end{aligned}$$

For each zenith angle, a similar observation equation is written resulting in five equations in total to be added to the adjustment. These observations are only added to demonstrate their inclusion in an adjustment. Readers are reminded that the inclusion of vertical angles into an adjustment is not recommended due to the difficulties in removing the systematic error caused by refraction.

23.4.4 Addition of Observed Azimuths

As discussed in Section 23.7, observed azimuths must be corrected for wobble of the Earth's spin axis, deflection of the vertical, and height of the target. In this example, the azimuth was already given as its geodetic value. Thus, the azimuth was used to fix the adjustment by overweighting. The observation equation for the geodetic azimuth is

$$\begin{aligned}
&\left(\frac{\sin \mathrm{Az}_{AB}}{AB\cos \upsilon_{AB}}\right)_0 dn_A - \left(\frac{\cos \mathrm{Az}_{AB}}{AB\cos \upsilon_{AB}}\right)_0 de_A + 0du_A \\
&\quad - \left\{\left(\frac{\sin \mathrm{Az}_{AB}}{AB\cos \upsilon_{AB}}\right)[\cos(\varphi_B - \varphi_A) + \sin\varphi_B \sin(\lambda_B - \lambda_A)\cot \mathrm{Az}_{AB}]\right\}_0 dn_B \\
&\quad + \left\{\left(\frac{\cos \mathrm{Az}_{AB}}{AB\cos \upsilon_{AB}}\right)[\cos(\lambda_B - \lambda_A) - \sin\varphi_A \sin(\lambda_B - \lambda_A)\tan \mathrm{Az}_{AB}]\right\}_0 de_B \\
&\quad + \left\{\left(\frac{\cos \mathrm{Az}_{AB}\cos\varphi_B}{AB\cos \upsilon_{AB}}\right)\left[\sin(\lambda_B - \lambda_A) - \begin{pmatrix}\sin\varphi_A\cos(\lambda_B - \lambda_A)\\ -\cos\varphi_A\tan\varphi_B\end{pmatrix}\tan \mathrm{Az}_{AB}\right]\right\}_0 du_B \\
&\quad = \alpha_{AB} - \alpha_0
\end{aligned} \tag{23.22}$$

In Equation (23.22), the coefficients on the left side are evaluated using the approximate coordinate values using Equation (23.5) through Equation (23.7). The numerical values of Equation (23.22), in units of s/m, are

$$\begin{aligned}
&108.42dn_A - 436.79de_A + 0du_A - 108.43dn_B + 436.79de_B - 0.00003du_B \\
&\quad = -0.256''
\end{aligned}$$

23.4.5 Addition of Elevation Differences

Leveling is a process of determining heights above the geoid. The data for this example lists the orthometric and geoidal heights for each station. The difference in orthometric heights requires the application of Equation (23.39) to obtain the geodetic height differences between the stations. The observation equation for the elevation difference between stations A and C is

$$1du_C - 1du_A = \Delta H_{AC} + \Delta N_{AC} - \Delta h_{AC} \tag{23.23}$$

The resulting numerical values for Equation (23.23) are

$$\begin{aligned}
1du_C - 1du_A &= -9.359 + (-31.726 + 31.723) - (331.139 - 340.498) \\
&= -0.003 \text{ m}
\end{aligned}$$

23.4.6 Adjustment of Control Stations

With the addition of the u coordinate, observation equations like those presented in Section 20.2 can be added for each control coordinate. As stated in Section 23.1, the

local geodetic coordinate system has its origin at each instrument station. Thus, it is not a true coordinate system with a single origin. Because of this, the approximate coordinates for each station must be stored in either the geocentric or geodetic coordinate system. In this discussion, it is assumed that the values are stored in the geodetic coordinate system. Again, nonlinear versions of the equations must be written to match the overall nonlinear nature of the adjustment. The observation equations are

$$1dn = N - n \tag{23.24}$$

$$1de = E - e \tag{23.25}$$

$$1du = U - u \tag{23.26}$$

where (N, E, U) are the given control coordinate values; and (n, e, u) are their adjusted values. In the first iteration, the control coordinate values and their adjusted values will be the same. In subsequent iterations, small variations between the control coordinate values and their adjusted counterparts will be observed. The observation equations for Station A in the first iteration are

$$1dn = 0$$

$$1de = 0$$

$$1du = 0$$

Since the control coordinate values and approximate station coordinate values are expressed in terms of geodetic coordinates. The initial K-matrix values given in Equations (23.24)–(23.26) will be expressed as changes in geodetic coordinates. These values must then be transformed into the local geodetic coordinate system. The relationships between changes in the local geodetic coordinate system and the geodetic coordinate system are

$$\begin{bmatrix} dn \\ de \\ du \end{bmatrix} = \begin{bmatrix} M+h & 0 & 0 \\ 0 & (N+h)\cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d\varphi \\ d\lambda \\ dh \end{bmatrix}$$

$$= R_{LG} \begin{bmatrix} d\varphi \\ d\lambda \\ dh \end{bmatrix} \tag{23.27}$$

In Equation (23.27), M represents the radius in the meridian and N is the radius in the normal at a station with latitude φ, as given by Equation (23.16); h is the geodetic height; and R_{LG} is the transformation matrix. The K-matrix values expressed in terms of geodetic coordinate system can be transformed into K-matrix values of the local geodetic coordinate system using Equation (23.27).

23.4.7 Results of Adjustment

A partial listing of the results of the adjustment from the ADJUST software package is shown in Figure 23.5. The solution converged in two iterations. Note that the latitude and longitude of station *A* were held fixed as was the azimuth of line *AB*.

23.4.8 Updating Geodetic Coordinates

At the completion of each iteration, the corrections of *dn*, *de*, and *du* will be determined for each station in the adjustment. However, geodetic coordinates are used to represent station positions. Thus, after each iteration the local geodetic coordinate system corrections of *dn*, *de*, and *du* must be transformed into changes in the geodetic system using the inverse relationship of Equation (23.27), or

$$d\varphi_i = \frac{dn_i}{M_i + h_i}; \quad d\lambda_i = \frac{de_i}{(N_i + h_i)\cos(\varphi_i)}; \quad dh_i = du_i \tag{23.28}$$

In Equation (23.28), the corrections to the latitude, longitude, and geodetic height of station *I* are $d\varphi_i$, $d\lambda_i$, and dh_i, respectively. All other terms are as previously defined in this chapter. Similarly, the uncertainties for each station will be in the local geodetic system. Although these uncertainties can be used to represent northing, easting, and geodetic height errors at each station, Equation (6.13) must be used to transform these uncertainties into the geodetic system as

$$\Sigma_{\varphi,\lambda,h} = R^{-1}\Sigma_{n,e,u}R^{-\mathrm{T}} \tag{23.29}$$

where

$$\Sigma_{\varphi,\lambda,h} = \begin{bmatrix} \sigma_\varphi^2 & \sigma_{\varphi,\lambda} & \sigma_{\varphi,h} \\ \sigma_{\varphi,\lambda} & \sigma_\lambda^2 & \sigma_{\lambda,h} \\ \sigma_{\varphi,h} & \sigma_{\lambda,h} & \sigma_h^2 \end{bmatrix}; \quad R = \begin{bmatrix} M+h & 0 & 0 \\ 0 & (N+h)\cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix};$$

$$\Sigma_{n,e,u} = \begin{bmatrix} \sigma_n^2 & \sigma_{n,e} & \sigma_{n,u} \\ \sigma_{n,e} & \sigma_e^2 & \sigma_{e,u} \\ \sigma_{n,u} & \sigma_{e,u} & \sigma_u^2 \end{bmatrix}.$$

Note that the errors in the coordinates at each station can be computed in using the 3×3 block-diagonal elements from the Q_{xx} matrix of the adjustment.

23.5 BUILDING AN ADJUSTMENT

Since this is a nonlinear adjustment, initial approximations are required for all station parameters. Very good initial values for horizontal coordinates can be determined using the procedures outlined in Chapter 16 with horizontal observations

Adjusted Geodetic Coordinates

Station	Latitude	Longitude	Orth height	N	S-Lat(")	S-Lon(")	S-h
A	41°18'26.04850"N	76°00'10.24860"W	372.2232	−31.72	0.000065	0.000087	0.0119
B	41°18'40.46653"N	76°00'05.50185"W	351.3904	−31.71	0.000231	0.000113	0.0120
C	41°18'22.04015"N	76°00'00.94395"W	362.8664	−31.73	0.000162	0.000270	0.0119
D	41°18'27.65788"N	76°00'31.38547"W	370.8748	−31.72	0.000304	0.000353	0.0124

Station Statistics

Station	Sn	Se	t-ang°	Su	Sv	S-Elev#
A	0.0020	0.0020	45.0000	0.0020	0.0020	0.0119
B	0.0071	0.0026	13.9417	0.0073	0.0020	0.0120
C	0.0050	0.0063	96.0029	0.0063	0.0050	0.0119
D	0.0094	0.0082	175.3588	0.0094	0.0082	0.0124

#-Values do not include uncertainty in geoidal height.

Adjusted Slope Distances

Station Occupied	Station Sighted	Distance	V
A	B	458.792	−0.004
A	C	249.468	0.006
C	D	729.116	−0.006
D	A	494.213	−0.001
B	C	578.399	0.006

Adjusted Mark-to-Mark and Geodetic Distances

Station Occupied	Station Sighted	Mark-to-Mark Distance	Geodetic Distance
A	B	458.792	458.317
A	C	249.468	249.282
C	D	729.116	729.034
D	A	494.213	494.182
B	C	578.399	578.255

Adjusted Angle Observations

Station Backsighted	Station Occupied	Station Foresighted	Angle	V(")
D	A	B	98°10'22.13"	−2.87
B	A	C	105°47'52.54"	7.54
C	A	D	156°01'45.34"	1.34
A	B	C	335°29'34.52"	−2.48
C	B	A	24°30'25.48"	6.48
B	C	D	294°19'09.93"	−7.07
D	C	A	15°59'08.09"	7.09
A	C	B	49°41'41.98"	−3.02
C	D	A	352°00'53.43"	8.43
A	D	C	7°59'06.57"	−5.43

Adjusted Azimuth Observations

Station Occupied	Station Sighted	Azimuth	V
A	B	13°56'26.9"	0.00"

FIGURE 23.5 ADJUST listing of adjustment results for example problem in Figure 23.4.

Adjusted Zenith Angle Observations

Station Occupied	Station Sighted	Zenith Angle	V(")
A	B	92°36'12.2"	−0.19"
A	C	92°09'04.7"	−3.69"
C	D	89°22'25.1"	−1.05"
D	A	89°50'45.6"	−1.61"
B	C	88°52'01.3"	−0.26"

Adjusted Elevation Difference Observations

Station Occupied	Station Sighted	Elevation Diff.	V
A	C	−9.363	−0.0038

Iterations = 2
Redundancies = 15
Reference Variance = 4.075
Reference So = ±2.019

FIGURE 23.5 (*Continued*)

reduced to a map projection surface. Following the adjustment, grid coordinates can be converted to their geodetic equivalents. Likewise, orthometric heights of stations can be determined using the procedures discussed in Section 17.4. The resulting orthometric heights can be converted to geodetic heights using Equation (23.37). Since these are initial approximations, it is possible to use an average geoidal height N for the region when applying Equation (23.37). However, this procedure will result in more uncertainty in the resulting derived heights.

Before combining a large set of three-dimensional observations, it is wise to perform adjustments using the horizontal and vertical functional models presented in Chapters 12 and 16 to isolate potential blunders. For large regional data sets, smaller subsets of data should be adjusted to isolate blunders. After these smaller data sets are cleaned and adjusted, a combined adjustment can be attempted with better initial approximations for the unknowns and the knowledge that most, if not, all large blunders have been removed from the data sets.

23.6 COMMENTS ON SYSTEMATIC ERRORS

When an azimuth is observed, the azimuth derived is with respect to the Earth's instantaneous pole. As was stated earlier, the Earth's axis wobbles, and thus, although very small, the difference between the instantaneous pole and the conventional terrestrial pole should be considered. The International Earth Rotation and Reference System Science (IERS) maintains a website,[2] which provides positions of the Earth's axis with respect to the conventional terrestrial pole. The difference in azimuth can be computed as

$$\mathrm{Az}_{obs} = \mathrm{Az}_{inst} - (x_p \sin\lambda + y_p \cos\lambda)\sec\varphi \tag{23.30}$$

[2]This website can be found at https://hpiers.obspm.fr/eop-pc/index.php?index = bulletin&lang = en.

where Az_{inst} is the astronomic azimuth observed; Az_{obs} is the observed azimuth corrected for the position of the instantaneous spin axis; and x_p and y_p are the positions of the Earth's instantaneous spin axis as given in Bulletin B on the previously noted IERS website.

Usually, small local surveys result in systematic error components that are small enough to be considered negligible. However, as the size of the survey increases, variations in the direction of plumb lines must be considered. This error can be removed from angular observations prior to the adjustment with a priori knowledge of the deflection components η and ξ at each station. The systematic error corrections to the zenith angles, azimuths, and horizontal angles are

$$z_{ij} = z'_{ij} + \xi_i \cos \mathrm{Az}_{ij} + \eta_i \sin \mathrm{Az}_{ij} \tag{23.31}$$

$$\mathrm{Az}_{ij} = \mathrm{Az}_{obs} + \eta_i \tan \phi_i + (\xi_i \sin \mathrm{Az}_{obs} - \eta_i \cos \mathrm{Az}_{obs}) \cot z_{ij} \tag{23.32}$$

$$\begin{aligned} \angle BIF = \angle BIF_{obs} &+ (\xi_i \sin Az_{if} - \eta_i \cos \mathrm{Az}_{if}) \cot z_{if} \\ &- (\xi_i \sin \mathrm{Az}_{ib} - \eta_i \cos \mathrm{Az}_{ib}) \cot z_{ib} \end{aligned} \tag{23.33}$$

where η_i and ξ_i are the deflection of the vertical components at the observation station I; z'_{ij} is the observed zenith angle from station I to J; and z_{ij} is the corrected zenith angle between the same stations; Az_{obs} is the observed astronomical azimuth as corrected by Equation (23.30) between station I and J; and Az_{ij} is the azimuth corrected for deflection of the vertical between the same stations; $\angle BIF_{obs}$ is the observed angle where station B is the backsight station; I is the instrument station; and F is the foresight station; and $\angle BIF$ is the corrected horizontal angle between the same stations.

Additionally, difference in the skewed normals must also be considered for both directions and angles. This is commonly known as the target height reduction. The azimuth of a line corrected for target height is

$$\alpha_{ij} = \mathrm{Az}_{ij} + \frac{e^2}{2b} h \cos^2 \varphi_i \sin(2\mathrm{Az}_{ij}) \tag{23.34}$$

where Az_{ij} is the azimuth as computed from Equation (23.32); φ_i is the latitude of the observation station; b and e are the semimajor axis and eccentricity of the ellipse, respectively; and h is the geodetic height of the target.

Since an angle is simply the difference between the foresight and backsight azimuths, the correction to an angle due to the height of the targets is

$$\theta^C = \theta_{bif} + \frac{e^2}{2b} \cos^2 \varphi_i (h_f \sin 2\mathrm{Az}_{if} - h_b \sin 2\mathrm{Az}_{ib}) \tag{23.35}$$

where θ^C is the corrected angle; θ_{bif} is the observed angle; and Az_{if} and Az_{ib} are the azimuths of the foresight and backsight lines, respectively.

Finally, the azimuth should be reduced from the normal section to the geodesic. This reduction is extremely small, but for completeness is computed as

$$\alpha_g = \alpha - \frac{e^2}{12a^2} S^2 \cos^2 \varphi_i \sin 2\alpha \tag{23.36}$$

where α_g is geodetic azimuth; α is the azimuth computed from Equation (23.34); S is the sight distance of the line between the instrument and target stations; and a and e are the semimajor axis and eccentricity of the ellipse, respectively.

Atmospheric refraction must be considered when including vertical angles in an adjustment. Since the correction for atmospheric refraction is so difficult to model, one can only hope to account for a portion of correction in an adjustment. The remainder of the correction must be modeled in the adjustment or be absorbed in the residuals of the observation. The correction of the first-order effect of refraction can be determined by observing simultaneously vertical/zenith angles at each end of the line IJ. From these observations, the first-order corrected Z_{ij}^c using these observed reciprocal zenith angles is

$$Z_{ij}^c = 0.5(Z_{ij} + Z_{ji}) - 90^\circ - \psi \tag{23.37}$$

In Equation (23.37), Z_{ij} and Z_{ji} are the observed simultaneously reciprocal zenith angles; and ψ is the correction for the deflection of vertical, computed as

$$\psi = -(\xi_i \cos \mathrm{Az}_{ij} + \eta_i \sin \mathrm{Az}_{ij}) \tag{23.38}$$

It is possible to create an adjustment model that includes correction terms for η, ξ, and atmospheric refraction. However, due to the limited number of observations in a typical survey network, there is a danger of overparameterization (Leick, 2004). Thus, it is recommended that these corrections be applied to the angles before carrying out the adjustment.

Unfortunately, to determine the deflection of the vertical components (η and ξ), the latitude and longitude of the station must be known before the completion of the adjustment. Approximate values for latitude and longitude will suffice in most instances. However, in instances where good approximations for the geodetic coordinates are not known, it is possible to correct the original observations after an initial adjustment to ensure that the geodetic coordinates for each station are close to their final values. In this case, after correcting the observations, the adjustment can be run a second time.

A similar problem exists with differential leveling observations. Geodetic height h and orthometric height H differ by the geoid height N, or

$$h \approx H + N \tag{23.39}$$

where h is the geometric distance between the ellipsoid and the point; H is the distance along the plumb line from the geoid to the point; and N is the separation between the ellipsoid and geoid. Equation (23.39) is approximate since the geoid separation differs at the line defined by h and H and since the plumb line is a

curved line between the point and the geoid. These differences are small and thus, the approximately equal symbol is replaced with an equal sign generally in Equation (23.39).

Elevation differences as determined by differential leveling must account for the differences in geoid height between the benchmark stations. This situation can be mathematically written as

$$h_j - h_i = (H_j - N_j) - (H_i - N_i)$$
$$\Delta h = \Delta H + \Delta N \tag{23.40}$$

The correction to leveled height differences should be performed when these observations are included in the three-dimensional geodetic network adjustment. Values for geoid height can be determined using the appropriate geoid modeling software. Similar to the deflection of the vertical parameters, it is possible to include geoid height parameters in the mathematical model of the adjustment. Again, this approach runs a risk of overparameterization in the adjustment.

The orthometric correction O_C for a leveling line can be approximated using a formula derived by Bomford (1980) as

$$O_C = \frac{-H(\Delta\varphi'')(0.0053 \sin 2\varphi)}{\rho} \tag{23.41}$$

where H is the orthometric height at the instrument station; $\Delta\varphi''$ is the change in latitude between the backsight and foresight stations in units of arc-seconds; φ is the latitude of the instrument station; and ρ is the conversion from seconds to radians, which is approximately 206,264.8″/rad. Obviously, Equation (23.41) would be tedious to apply for a substantial north–south leveling circuit. However, for typical surveys involving small regions, this correction can be very small and easily absorbed in the residuals of the adjustment. For instance, a 120-km, north–south leveling line at approximately 42° latitude would result in an approximate orthometric correction of 2 cm.

23.7 SOFTWARE

ADJUST contains an option to adjust three-dimensional geodetic networks. As shown in Figure 23.6, the software allows for the entry of geoidal heights, deflection of the vertical components, and target heights. The help file that accompanies ADJUST describes the file format when these additional features are needed. The data file *Example 23-4.dat* for the example in Section 23.4 is shown in Figure 23.7 and is on the book's companion website. This file has the geoidal heights added to the file. Notice that the observations are listed in the order of slant distances, horizontal angles, azimuths, altitude angles, horizontal distances, and differences in elevation. In this file, only the observations given in Section 23.4 are listed. For those wishing to program the material in this chapter in a higher-level language, the Mathcad® worksheet *X23.xmcd*, which is also available on the companion website, demonstrates the procedures necessary for the adjustment.

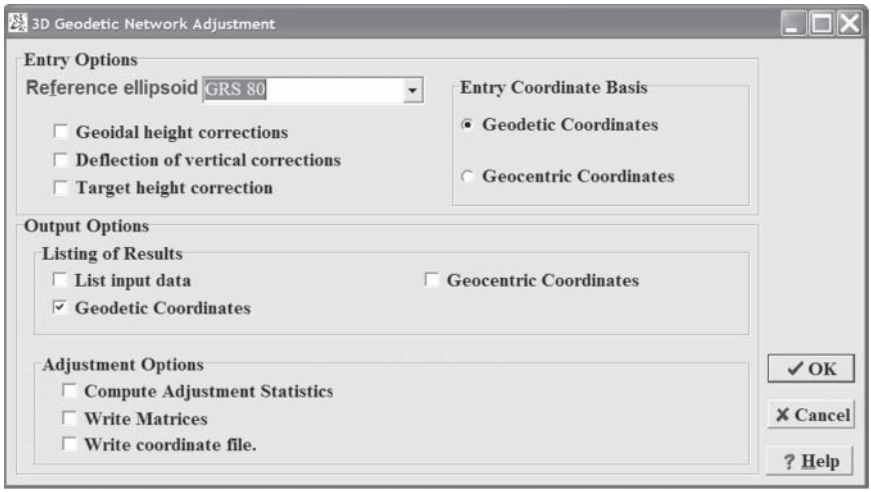


FIGURE 23.6 ADJUST options for a three-dimensional geodetic network adjustment.

```
Example 23.4
5 10 1 5 0 1 4    //sdist, hang, az, vang, hdist, de, sta
A 41 18 26.04850 -76 00 10.24860 340.498 -31.723 0.001 0.001 0.01
B 41 18 40.46660 -76 00 05.50180 319.681 -31.713 0.01
C 41 18 22.04010 -76 00 00.94390 331.139 -31.726 0.01
D 41 18 27.65860 -76 00 31.38550 339.152 -31.722
A B 458.796 0.005
A C 249.462 0.005
B C 578.393 0.005
C D 729.122 0.006
D A 494.214 0.005
D A B 98 10 25 2.8
B A C 105 47 45 3.5
C A D 156 01 44 4.1
A B C 335 29 37 2.4
C B A 24 30 19 2.4
B C D 294 19 17 2.3
D C A 15 59 01 3.3
A C B 49 41 45 3.4
C D A 352 00 45 2.2
A D C 7 59 12 2.2
A B 13 56 26.9 0.001
A B 92 36 12 1.4
A C 92 09 01 2.5
B C 88 52 01 1.1
C D 89 22 24 0.8
D A 89 50 44 1.2
A C -9.359 0.005
```

FIGURE 23.7 ADJUST data file for example in Section 23.4.

PROBLEMS

Note: Unless otherwise specified, use the GRS 80 ellipsoidal parameters of $a = 6.378.137.0$ m and $f = 1/298.257222101$ for the following problems. Partial answers to problems marked with an asterisk can be found in Appendix G.

***23.1** Using the data supplied in Table 23.3, what is the computed value for the slant distance *AB*?

23.2 Same as Problem 23.1, except for slant distance *BC*.

***23.3** Using the data supplied in Table 23.3, what is the computed value, α_0, for the azimuth *AB*?

23.4 Same as Problem 23.3, except for azimuth *BC*?

***23.5** Using the data supplied in Table 23.3, what is the computed value, θ_0, for the horizontal angle *DAB*?

23.6 Same as Problem 23.5, except for horizontal angle *BCD*.

***23.7** Using the data supplied in Table 23.3, what is the computed altitude angle value, v_0, for the v_{AB}?

23.8 Same as Problem 23.7, except for the zenith angle v_{CD}.

***23.9** Develop the observation equation for slant distance *AC* in Table 23.3 (5 decimal places).

23.10 Same as Problem 23.9, except for slant distance *BC*.

***23.11** Develop the observation equation in units of s/m for horizontal angle *BAC* (5 decimal places).

23.12 Same as Problem 23.11, except for horizontal angle *CBA*.

23.13 Develop the observation equation in units of s/m for zenith angle *AB* (5 decimal places).

23.14 The approximate values for the geodetic coordinates of the endpoints of line *AB* are *A*: (42°15′15.40986″N, 121°47′11.93412″W, 1320.394 m) and *B*: (42°15′25.58432″N, 121°47′01.65233″W, 1340.510 m). Determine the computed

(a) slant distance *AB*.

(b) azimuth *AB*.

(c) zenith angle *AB*.

23.15 Similar to Problem 23.14 except the approximate values for the geodetic coordinates are *A*: (34°03′16.70875″N, 117°48′48.69986″W, 216.408 m) and *B*: (34°03′34.83116″N, 117°48′22.55080″W, 216.103 m).

23.16 Develop the observation equation for slant distance *AB* in Problem 23.14 if the observed distance is 393.223 m.

23.17 Develop the observation equation if the observed distance in Problem 23.15 is 872.673 m.

23.18 Develop the azimuth observation equation in units of s/m for Problem 23.14 if the azimuth observation is 36°53′48″. (Assume that all systematic errors are removed from the observation.)

23.19 Develop the azimuth observation equation for Problem 23.15 if the azimuth observation is 50°12′48″. (Assume that all systematic errors are removed from the observation.)

Use the following data and the GRS 80 ellipsoid in Problems 23.20–23.26.

Geodetic Coordinates

Station	Latitude	Longitude	Height (m)	S_n (m)	S_e (m)	S_u (m)
A	44°33′46.09876″N	123°18′39.21056″W	78.943	0.001	0.001	0.001
B	43°33′46.01124″N	123°17′53.02874″W	78.895	–	–	0.001
C	43°34′15.99843″N	123°17′48.23981″W	81.077	–	–	0.001
D	43°34′15.18975″N	123°18′42.71423″W	81.077	–	–	–

Angle Observations

B	I	F	Angle	S (″)	h_b (m)	h_i (m)	h_f (m)
D	*A*	*C*	55°32′35.3″	1.44	1.303	1.295	1.300
C	*A*	*B*	39°31′28.3″	1.40	1.402	1.405	1.398
A	*B*	*D*	39°15′06.0″	1.41	1.305	1.398	1.299
D	*B*	*C*	57°06′16.1″	1.43	1.500	1.500	1.500
B	*C*	*A*	44°07′11.2″	1.43	1.300	1.300	1.300
A	*C*	*D*	38°10′56.6″	1.36	1.425	1.423	1.420
C	*D*	*B*	40°35′35.2″	1.37	1.500	1.500	1.500
B	*D*	*A*	45°40′48.8″	1.44	1.500	1.500	1.500

Slant Distance Observations

From	To	Distance (m)	S (m)	h_i (m)	h_r (m)
A	*B*	1019.148	0.0035	1.400	1.500
B	*C*	931.666	0.0034	1.497	1.595
C	*D*	1202.233	0.0035	1.500	1.500
D	*A*	901.295	0.0034	1.302	1.296
A	*C*	1454.951	0.0036	1.300	1.500
B	*D*	1418.909	0.0036	1.500	1.400

Azimuth Observations

From	To	Azimuth	S ($''$)	h_i (m)	h_r (m)
A	B	90°08′50.6″	1.38	1.500	1.500

Elevation Differences

From	To	ΔElev (m)	S (m)
B	C	5.222	0.055
C	D	0.140	0.061
D	A	−2.242	0.055
A	B	−3.118	0.057

Geoid height and deflection of the vertical components for stations.

Station	N (m)	η''	ξ''
A	−22.580	−4.91	6.69
B	−22.612	−4.51	6.19
C	−22.593	−5.09	6.66
D	−22.555	−5.57	6.92

23.20 Use a computational program to develop the coefficient matrix for the angle observations in the table. List all coefficients in units of s/m.

23.21 Use a computational program to develop the coefficient matrix for the slant distance observations in the table.

23.22 Use a computational program to develop the observation equation for the azimuth observation in the table. List the coefficients in units of s/m.

23.23 Use a computational program to develop the observation equations for the elevation differences in the table.

23.24 Use ADJUST to determine the most probable values for the geodetic coordinates.

23.25 Do Problem 23.25 by removing azimuth AB from the adjustment and overweighting the easting of Station B. The standard deviations of Station B should be $S_n = \pm 1000$ m, $S_e = \pm 0.001$ m, and $S_u = \pm 0.001$ m.

23.26 What are the adjusted orthometric heights of the stations in Problem 23.25?

***23.27** If $\xi = 4.23''$ and $\eta = -1.69''$ are the deflection of the vertical components at Station A in Section 23.4, what is the correction to the zenith angle going from Station A to B?

PROGRAMMING PROBLEMS

23.28 Develop a computational program that corrects vertical angles for deflection of the vertical components.

23.29 Develop a computational program that computes the coefficients for the azimuths, angles, slant distances, and elevation differences. Check your program using data from Problem 23.20 through 23.23.

CHAPTER 24

COMBINING GNSS AND TERRESTRIAL OBSERVATIONS

24.1 INTRODUCTION



Ellipsoids define the mathematical shape of the Earth, or a portion thereof. Ellipsoids are commonly defined by the length of their semimajor axis, a, and the flattening factor, f. Commonly used ellipsoids are the Geodetic Reference System of 1980 (GRS80) and the World Geodetic System of 1984 (WGS84). Table 24.1 lists the length of the semimajor axis and flattening factor for these ellipsoids.

A network of points determined with respect to each other on an ellipsoid is a reference frame, which is also called a datum. These reference frames define the geodetic coordinates of the points, and thus the origin and orientation of the datum. Most reference frames are regional in nature since the network of points only covers a portion of the Earth. These are known as *local reference frames*. For example, the North American Terrestrial Reference Frame of 2022 (NATRF 2022) is a local reference frame consisting of a network of points in Canada, the United States, and Mexico. However, the International Terrestrial Reference Frame (ITRF) is an example of a global reference frame defined by a multitude of points located on all major land masses of the Earth. However, NATRF 2022 is a plate-fixed regional reference frame, which rotates with the North American tectonic plate but is directly tied to ITRF 2020 at epoch 2020.00. A set of Euler pole parameters

TABLE 24.1 Defining Ellipsoidal Parameters

Ellipsoid	a (m)	$1/f$
GRS80	6,378,137.0	298.257222101
WGS84	6,378,137.0	298.257223563

define the rotation between NATRF 2022 and ITRF 2020 for subsequent epochs. These parameters are estimated using a combination of inSAR satellite data and NCN stations located throughout North America. For the purposes of this chapter, a typical survey project site can treat NATRF 2022 as a local datum.

Similarly, the coordinates derived from the GNSS satellites are determined by a global network of tracking stations with coordinates defined in a global datum. Incorrectly stated, GPS is said to be using the WGS84 datum. This statement should be interpreted by the reader as the datum defined by the global network of tracking stations that use the WGS84 reference ellipsoid. The Department of Defense (DoD) has updated this network several times since its creation at the beginning of GPS weeks 730, 873, 1150, 1674, 1762, 2139, and 2296.

Besides their reference ellipsoid, reference frames differ in origin (translation), scale, and rotations about the three cardinal axes. Many modern local reference frames and global reference frames have nearly aligned coordinate axes and differ by only a few meters in their origins. These are known as *Earth-Centered, Earth-Fixed* (*ECEF*) coordinate systems. Thus, it is possible to use a three-dimensional coordinate transformation to transform points from a local datum to a global datum and back again. When using GNSS and absolute positioning techniques, the satellites serve as control points and all points determined by this method are defined in either the ITRF or WGS84 datum, depending on the source of ephemeris. However, this method of surveying is only accurate to the meter level. Thus, relative positioning techniques are generally used. However, if GNSS receivers are placed on control points defined in a regional reference frame such as NATRF 2022, the resultant points are defined in a hybrid of the WGS84 or ITRF and the local reference frame. The differences in these systems are generally at a centimeter level. For lower-order surveys, these differences may not be of much importance. For high-order control surveys, they must be considered.

To combine GNSS and terrestrial observations, the control from two different reference frames must be reconciled. This can be done by transforming the local control coordinates into the GNSS reference datum or by transforming the GNSS established points into the local datum.

As discussed in Chapter 18, a three-dimensional conformal coordinate transformation will take coordinates from one three-dimensional coordinate system into another. However, since the reference frames are nearly aligned, the rotational angles are generally in the millisecond range, which is very small. As will be shown in Section 24.2, the rotational process as well as the entire transformation can be simplified since today's coordinate systems are nearly aligned.

To perform the transformation, coordinates of common points in both systems must be placed into their respective geocentric coordinate systems (see

Section 18.5). Following this conversion, a least-squares adjustment can be performed to determine the transformation parameters between the two systems. Once the transformation parameters are determined, the coordinates of any remaining points can be transformed.

For example, assume that an RTK-GNSS survey is being used to stake out a highway alignment, and that the highway alignment was designed using control points from the State Plane Coordinate System of 2022 (SPCS 2022). The real-time GNSS datum is defined by a set of DoD tracking stations using the WGS 84 ellipsoid, which has been made to closely approximate the *International Terrestrial Reference Frame of 2008* (ITRF 2008). The geodetic control used in the State Plane Coordinate System is based on a series of National Spatial Reference stations in the United States, Canada, and Mexico using the GRS 80 ellipsoid. The local *NEH* coordinates of the highway design stations must be transformed from their local coordinate system and orthometric heights to geodetic coordinates. The geodetic coordinates are then transformed to geocentric coordinates of $(X, Y, Z)_L$ in the local system. These points are occupied using a GNSS receiver and GNSS coordinates $(X, Y, Z)_G$ derived from satellite observations. Once three or more common points are occupied, the transformation parameters can be determined using a *Helmert transformation*, which is a simplification of the three-dimensional conformal coordinate transformation presented in Section 18.7. Following this, all GNSS-derived coordinates are transformed into the local coordinate system, and stakeout of stations can be performed.

24.2 THE HELMERT TRANSFORMATION

A local datum such as NAD 83 is an ECEF coordinate system. This means that the Z-axis is nearly aligned with the Conventional Terrestrial Pole orientation, which was redefined in 1984 by Bureau International de l′Heure (BIH 84), the X-axis with the Greenwich Meridian, and the origin is at the mass-center of the Earth as derived by the datum points used in the definition. International reference frames such as the International Terrestrial Reference Frame use the same definitions for the axes, origin, and ellipsoid, but differ slightly due to the difference in the reference stations used in their determination. Thus, the rotational parameters and translations between two ECEF coordinate systems are usually very small. The scale factor between two reference frames using the same units of measure should be nearly one.

Since the sine of a very small angle is equal to the angle in radians, the cosine of a very small angle is nearly one, and the product of two very small numbers is nearly zero, for two nearly aligned coordinate systems the rotation matrix of the three-dimensional conformal coordinate transformation presented in Equation (18.14) can be simplified as

$$R' = \begin{bmatrix} 1 & \theta_3 & -\theta_2 \\ -\theta_3 & 1 & \theta_1 \\ \theta_2 & -\theta_1 & 1 \end{bmatrix} = I + \begin{bmatrix} 0 & \Delta\theta_3 & -\Delta\theta_2 \\ -\Delta\theta_3 & 0 & \Delta\theta_1 \\ \Delta\theta_2 & -\Delta\theta_1 & 0 \end{bmatrix} = I + \Delta R \qquad (24.1)$$

In Equation (24.1), $\Delta\theta_1$, $\Delta\theta_2$, and $\Delta\theta_3$ are in units of radians and have been separated from the full matrix by the addition of the unit matrix. The introduction of "Δ" indicates that these values are differentially small.

The transformation of coordinates from one local datum to another datum is performed as

$$X_{LD} = sR'X_{GD} + T \tag{24.2}$$

where s is the scale factor; X_{GD} is the (x, y, z) coordinates from the global data (G) to be transferred into the local datum (L); and T is the xyz translations necessary to make the origins of the two reference frames coincide. Similarly, the scale factor, s, and translation parameters, T, can be modified as

$$s = 1 + \Delta s \tag{24.3}$$

$$T = T_0 + \Delta T \tag{24.4}$$

In Equation (24.4), the approximate shift vector, T_0, can be computed as

$$T_0 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_L - \begin{bmatrix} x \\ y \\ z \end{bmatrix}_G \quad \text{and} \quad \Delta T = \begin{bmatrix} \Delta Tx \\ \Delta Ty \\ \Delta Tz \end{bmatrix} \tag{24.5}$$

Since Equation (24.2) is nonlinear, a single common point or an average of all the common points can be used in Equation (24.5) to obtain initial approximations. For a single station, i, the linearized model for the corrections is

$$X_{L_i} - X_{G_i} - T_0 = J_i dx \tag{24.6}$$

where

$$J_i = \begin{bmatrix} x_i & 0 & -z_i & y_i & 1 & 0 & 0 \\ y_i & z_i & 0 & -x_i & 0 & 1 & 0 \\ z_i & -y_i & x_i & 0 & 0 & 0 & 1 \end{bmatrix}; \quad dx = \begin{bmatrix} \Delta s \\ \Delta\theta_1 \\ \Delta\theta_2 \\ \Delta\theta_3 \\ \Delta Tx \\ \Delta Ty \\ \Delta Tz \end{bmatrix} \tag{24.7}$$

Example 24.1 The geocentric coordinates in the North American Terrestrial Reference Frame of 2022 (NATRF 22) and, from a GNSS adjustment using a precise ephemeris, from an International Terrestrial Reference Frame (ITRF) datum are shown below. Determine the transformation parameters to transform the additional station into the local NATRF 2022 datum.

	NATRF 2022			ITRF		
Sta	x (m)	y (m)	z (m)	x (m)	y (m)	z (m)
A	1,160,604.924	−4,655,917.607	4,188,338.994	1,160,604.374	−4,655,916.161	4,188,338.876
E	1,160,084.380	−4,655,635.664	4,188,722.464	1,160,083.830	−4,655,634.217	4,188,722.346
F	1,160,787.133	−4,655,565.726	4,188,637.156	1,160,786.583	−4,655,564.279	4,188,637.038
G	1,160,648.640	−4,656,192.596	4,188,040.975	1,160,648.090	−4,656,191.149	4,188,040.857
B				1,160,636.707	−4,655,589.358	4,188,659.259

SOLUTION *First Iteration*: Using station A's coordinate values, T_0 is

$$T_0 = \begin{bmatrix} 1,160,604.924 - 1,160,604.374 \\ -4,655,635.664 + 4,655,634.217 \\ 4,188,338.994 - 4,188,338.876 \end{bmatrix} = \begin{bmatrix} 0.550 \\ -1.446 \\ 0.118 \end{bmatrix}$$

The J and K matrices are

$$J = \begin{bmatrix} 1160604.374 & 0.000 & -4188338.876 & -4655916.161 & 1 & 0 & 0 \\ -4655916.161 & 4188338.876 & 0.000 & -1160604.374 & 0 & 1 & 0 \\ 4188338.876 & 4655916.161 & 1160604.374 & 0.000 & 0 & 0 & 1 \\ 116,0083.830 & 0 & -4,188,722.346 & -4,655,634.217 & 1 & 0 & 0 \\ -4,655,634.217 & 4,188,722.346 & 0 & -1,160,083.830 & 0 & 1 & 0 \\ 4,188,722.346 & 4,655,634.217 & 1,160,083.830 & 0 & 0 & 0 & 1 \\ 1,160,786.583 & 0 & -4,188,637.038 & -4,655,564.279 & 1 & 0 & 0 \\ -4,655,564.279 & 4,188,637.038 & 0 & -1,160,786.583 & 0 & 1 & 0 \\ 4,188,637.038 & 4,655,564.279 & 1,160,786.583 & 0 & 0 & 0 & 1 \\ 1,160,648.090 & 0 & -4,188,040.857 & -4,656,191.149 & 1 & 0 & 0 \\ -4,656,191.149 & 4,188,040.857 & 0 & -1,160,648.090 & 0 & 1 & 0 \\ 4,188,040.857 & 4,656,191.149 & 1,160,648.090 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$K = \begin{bmatrix} 0.000 \\ 0.000 \\ 0.000 \\ 0.000 \\ -0.001 \\ 0.000 \\ 0.000 \\ -0.001 \\ 0.000 \\ 0.000 \\ -0.001 \\ 0.000 \end{bmatrix}$$

Solving Equation (11.37) using the above matrices and adding X to the initial values results in

$$X = \begin{bmatrix} -0.0000001095 \\ -0.0000001448 \\ -0.0000000383 \\ -0.0000001220 \\ -0.6013 \\ -0.0458 \\ 1.1773 \end{bmatrix} \quad \begin{aligned} s &= 1 - 0.0000001095 = 0.9999998905 \\ \theta_1 &= -0°00'00.029867'' \\ \theta_2 &= -0°00'00.007896'' \\ \theta_3 &= -0°00'00.025165'' \\ Tx &= 0.550 - 0.6013 = -0.0513 \\ Ty &= -1.446 - 0.0458 = -1.4918 \\ Tz &= 0.118 + 1.1773 = 1.2953 \end{aligned}$$

The next iteration resulted in negligible changes in the unknowns, and thus the transformation parameters are listed. Using Equation (24.2), transformation parameters and the GNSS-derived ITRF coordinates for station *B*, the NATRF 2022 coordinates[1] for station *B* are

Geocentric coordinates = (1,160,637.2568, −4,655,590.8052, 4,188,659.3772)

Geodetic coordinates = (41°18′40.46653″ N, 76°00′05.50185″ W, 319.677 m)

PA Northeast zone coordinates = (93,025.774 m, 383,474.444 m)

The reader should note that the NATRF 2022 coordinates for Station *B* are the same as those listed in Figure 23.5. However, slight differences in coordinates can be expected due to random errors in the transformation.

Example 24.1 demonstrates the mathematical relationship between the two reference frames for this limited set of points. This process should always be considered when combining traditional observations with GNSS coordinates for higher order surveys. That is, if GNSS-derived coordinates are to be entered into an adjustment, they should first be transformed into the local datum.

The process of transforming points from global reference frames to local reference frames is important when performing real-time satellite stakeout surveys or trying to match coordinates from an earlier survey, which was performed using a different reference frame. A satellite survey implicitly uses the points located in a global datum. Since engineering plans are generally developed in a local datum such as NATRF 2022, the GNSS-derived coordinate values must be transformed into the local coordinate system. This process is known as *localization* (also called *site calibration*) by manufacturers. From a design point of view, it is important to recognize that the best results will be obtained if points common in both reference frames surround the project area. After entering in the local datum coordinates, the GNSS receiver should occupy each station. The software then computes the transformation parameters and uses these to determine the coordinate values for the points to be surveyed or laid out. Section 24.5 discusses an alternative method used by manufacturers to accomplish this transformation.

[1]Equations to convert geocentric coordinates to geodetic coordinates can be found in Section 17.5.

24.2.1 Accounting for Plate Tectonics

The transformation of Equation (24.2) assumes a rigid body transformation between two reference frames. However, coordinates of points on the Earth's surface will change with time due to crustal plate motion associated with plate tectonics. The 7-parameter transformation can be extended to a 14-parameter transformation by adding time-dependent parameters. Note that the 14-parameter transformation assumes linear crustal plate motion. The 14-parameter transformation is given as (Soler and Snay, 2004)

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{LD} = \begin{bmatrix} T_X + \dot{T}_X(t - t_0) \\ T_Y + \dot{T}_Y(t - t_0) \\ T_Z + \dot{T}_Z(t - t_0) \end{bmatrix} + \{1 + s + \dot{s}(t - t_0)\} R_{14} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{GD} \tag{24.8}$$

where the dotted parameters designate the velocity of the transformation parameter, t_0 is the reference epoch date that the parameters and their velocities refer to, t is the desired epoch date, and R_{14} is the rotation matrix for the 14-parameter transformation given as

$$R_{14} = \begin{bmatrix} 1 & \theta_3 + \dot{\theta}_3(t - t_0) & -[\theta_2 + \dot{\theta}_2(t - t_0)] \\ -[\theta_3 + \dot{\theta}_3(t - t_0)] & 1 & \theta_1 + \dot{\theta}_1(t - t_0) \\ \theta_2 + \dot{\theta}_2(t - t_0) & -[\theta_1 + \dot{\theta}_1(t - t_0)] & 1 \end{bmatrix} \tag{24.9}$$

Transformation parameters and their velocities for the various ITRF solutions can be derived from International Earth Rotation and Reference Systems Service (IERS). Professionals looking to obtain transformation parameters and velocities between ITRF and local reference frames (e.g., NATRF, ETRF) have to reach out to their local geodetic agencies.

24.3 ROTATIONS BETWEEN COORDINATE SYSTEMS

GNSS uses the geocentric coordinate system and provides baseline vectors in this system. Often, it is preferable to obtain the coordinate value changes in terms of geodetic coordinates. From geodesy, the relationship between changes in the geodetic coordinate system and the geocentric coordinate system is

$$dx = \begin{bmatrix} dX \\ dY \\ dZ \end{bmatrix} = \begin{bmatrix} -(M + h)\sin\varphi\cos\lambda & -(N + h)\cos\varphi\sin\lambda & \cos\varphi\cos\lambda \\ -(M + h)\sin\varphi\sin\lambda & -(N + h)\cos\varphi\cos\lambda & \cos\varphi\sin\lambda \\ (M + h)\cos\varphi & 0 & \sin\varphi \end{bmatrix} \begin{bmatrix} d\varphi \\ d\lambda \\ dh \end{bmatrix}$$

$$= R_{XG} \begin{bmatrix} d\varphi \\ d\lambda \\ dh \end{bmatrix} \tag{24.10}$$

where dx represents the changes in the geocentric coordinates; M represents the radius in the meridian; N represents the radius in the normal at latitude φ as given by Equation (17.6); h represents the geodetic height of the point; and R_{XG} represents the transformation matrix.

The transformation between changes in the geodetic coordinate system and the local geodetic coordinate system are given in Equation (23.27), which is repeated here for convenience.

$$\begin{bmatrix} dn \\ de \\ du \end{bmatrix} = \begin{bmatrix} M+h & 0 & 0 \\ 0 & (N+h)\cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d\varphi \\ d\lambda \\ dh \end{bmatrix}$$
$$= R_{LG} \begin{bmatrix} d\varphi \\ d\lambda \\ dh \end{bmatrix} \tag{24.11}$$

where R_{LG} is the rotation matrix between the geodetic and local geodetic coordinate systems.

24.4 COMBINING GNSS BASELINE VECTORS WITH TRADITIONAL OBSERVATIONS

 As discussed in Chapter 23, the three-dimensional geodetic adjustment allows the adjustment of all traditional surveying observations. If the control is known in the local coordinate system, then Equation (24.2) can be included in adjustment to account for datum differences.

Baseline vectors are the geocentric coordinate differences between two points. Thus, the translation component of Equation (24.2) is removed leaving only the scaling and rotational parameters. Therefore, the last three columns of the J_i matrix in Equations (24.6) and (24.7) can be eliminated leaving

$$J_i = \begin{bmatrix} x_i & 0 & -z_i & y_i \\ y_i & z_i & 0 & -x_i \\ z_i & -y_i & x_i & 0 \end{bmatrix}; \quad dx = \begin{bmatrix} \Delta s \\ \Delta\theta_1 \\ \Delta\theta_2 \\ \Delta\theta_3 \end{bmatrix} \tag{24.12}$$

Equation (24.12) can be used in Equation (24.6). However, since the three-dimensional geodetic network adjustment is performed in the local geodetic coordinate system, the rotational elements of Equation (24.12) can be transformed about a single station in this system. Dropping the Δ symbol, the rotation about the single station in the local geodetic system becomes

$$R = R_3^T(\lambda_0)R_2^T(90-\varphi_0)R_3(\theta_u)R_2(\theta_e)R_1(\theta_n)R_2(90-\varphi_0)R_3(\lambda_0) \tag{24.13}$$

where θ_n is a rotation about the north axis of the local geodetic coordinate system; θ_e is a rotation about the east axis of the local geodetic coordinate system; θ_u is a rotation about the up axis of the local geodetic coordinate system; R_1, R_2, and R_3 are the rotation matrices defined in Section 18.7; and φ_0 and λ_0 are the geodetic coordinates of the rotational point. This point should be picked near the center of the project area.

Again, since in nearly aligned coordinate systems the rotations are small, the above rotations can be simplified to

$$R = \theta_n R_n + \theta_e R_e + \theta_u R_u + I \tag{24.14}$$

where

$$R_u = \begin{bmatrix} 0 & \sin\varphi_0 & -\cos\varphi_0 \sin\lambda_0 \\ -\sin\varphi_0 & 0 & \cos\varphi_0 \cos\lambda_0 \\ \cos\varphi_0 \sin\lambda_0 & -\cos\varphi_0 \cos\lambda_0 & 0 \end{bmatrix}$$

$$R_e = \begin{bmatrix} 0 & 0 & -\cos\lambda_0 \\ 0 & 0 & -\sin\lambda_0 \\ \cos\lambda_0 & \sin\lambda_0 & 0 \end{bmatrix}$$

$$R_n = \begin{bmatrix} 0 & -\cos\varphi_0 & -\sin\varphi_0 \sin\lambda_0 \\ \cos\varphi_0 & 0 & \sin\varphi_0 \cos\lambda_0 \\ \sin\varphi_0 \sin\lambda_0 & -\sin\varphi_0 \cos\lambda_0 & 0 \end{bmatrix}$$

I is a three-dimensional identity matrix; θ_n, θ_e, and θ_u are in radian units; and φ_0 and λ_0 are the geodetic coordinates of the rotational point.

Finally, the transformation going from the observed GNSS vector to its local geodetic equivalent between stations I and J is

$$\begin{bmatrix} \Delta X_{IJ} \\ \Delta Y_{IJ} \\ \Delta Z_{IJ} \end{bmatrix}_2 = (1+s)R \begin{bmatrix} \Delta X_{IJ} \\ \Delta Y_{IJ} \\ \Delta Z_{IJ} \end{bmatrix}_1 \tag{24.15}$$

where s represents the differential scale change between system 1 and 2; R is defined in Equation (24.14); $[]_1$ represents the GNSS baseline vector components in the local coordinate system; and $[]_2$ represents the GNSS observed baseline vector components between stations I and J.

In the preceding chapter, the three-dimensional geodetic network adjustment was developed in the local geodetic coordinate system. Thus, the addition of the GNSS baseline vectors into this adjustment requires that the coefficient matrix be rotated into the same system. Recall from Chapter 17 that the coefficient matrix (A) for each GNSS baseline vector consisted of three rows containing -1, 0, and 1. The matrix, as presented in Chapter 17, must be rotated into the local geodetic

system. Thus, for the baseline vector IJ, the new coefficient matrix values are derived as

$$\begin{aligned}&\text{For station } I\text{: } -(1+s)R \cdot R_{XG}(\varphi_i, \lambda_i) \cdot R_{LG}(\varphi_i)^{-1}\\ &\text{For station } J\text{: } (1+s)R \cdot R_{XG}(\varphi_j, \lambda_j) \cdot R_{LG}(\varphi_j)^{-1}\end{aligned} \tag{24.16}$$

where s is the change in scale between the two systems; rotation matrix R is defined in Equation (24.14); rotation matrix R_{XG} is defined in Equation (24.10); rotation matrix R_{LG} is defined in Equation (24.11); and (φ_i, λ_i) and (φ_j, λ_j) are the geodetic coordinates from stations I and J, respectively.

When developing the matrices for a least-squares adjustment, the unknown parameters for scale and rotation should be set to zero. After the first iteration, these values will be modified and updated. At the end of the adjustment, these parameters can be checked for statistical significance, as was described in Section 18.8.

The elements of the coefficient matrix for the unknown rotation angles and scale are

$$J = \begin{bmatrix} X_J - X_I & r_{n_1} & r_{e_1} & r_{u_1} \\ Y_J - Y_I & r_{n_2} & r_{e_2} & r_{u_2} \\ Z_J - Z_I & r_{n_3} & r_{e_3} & r_{u_3} \end{bmatrix} \tag{24.17}$$

where

$$X^{\mathrm{T}} = [\Delta s \;\; \Delta\theta_n \;\; \Delta\theta_e \;\; \Delta\theta_u],$$

$$r_n = R_n \begin{bmatrix} X_J - X_I \\ Y_J - Y_I \\ Z_J - Z_I \end{bmatrix}, \quad r_e = R_e \begin{bmatrix} X_J - X_I \\ Y_J - Y_I \\ Z_J - Z_I \end{bmatrix}, \quad r_u = R_u \begin{bmatrix} X_J - X_I \\ Y_J - Y_I \\ Z_J - Z_I \end{bmatrix}.$$

Example 24.2 The local datum coordinates for station B are given in Example 24.1 and the GNSS-derived and transformed approximate NATRF 2022 coordinates for station E are (41°18′43.9622″N, 76°00′29.0384″W, 292.354 m). Assuming initial values of zero for the scale and rotation parameters in Equation 24.15 and using point A as the rotational point, develop the coefficient (J) and constant's (K) matrices for the transformation parameter and the following GNSS-observed baseline.

Baseline	ΔX (m)	ΔY (m)	ΔZ (m)
EB	552.8771	44.8606	−63.0862

SOLUTION From Example 24.1, the approximate NATRF 2022 geocentric and geodetic coordinates for station E and B are

Station	φ	λ	h (m)
E	41°18′43.9622″	−76°00′29.0384″	292.354
B	41°18′40.4665″	−76°00′05.5018″	319.677
A	41°18′26.0485″	−76°00′10.2486″	–

By Equation (24.10), R_{XG} for station B is

$$R_{XG}^{B} = \begin{bmatrix} -1,016,184.4652 & 4,655,590.8055 & 0.1817 \\ 4,076,156.4565 & -1,160,637.2580 & -0.7288 \\ 4,779,908.9967 & 0.0000 & 0.6601 \end{bmatrix}$$

By Equation (24.10), R_{XG} for station E is

$$R_{XG}^{E} = \begin{bmatrix} -1,015,734.7310 & 4,655,634.2161 & 0.1816 \\ 4,076,334.1770 & -1,160,083.8309 & -0.7288 \\ 4,779,818.0865 & 0.0000 & 0.6602 \end{bmatrix}$$

By Equation (24.11), R_{LG} for station B is

$$R_{LG}^{B} = \begin{bmatrix} 6,363,584.8656 & 0.0000 & 0.0000 \\ 0.0000 & 4,798,083.4291 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 \end{bmatrix}$$

By Equation (24.11), R_{LG} for station E is

$$R_{LG}^{E} = \begin{bmatrix} 6,363,558.6197 & 0.0000 & 0.0000 \\ 0.0000 & 4,797,991.7099 & 0.0000 \\ 0.0000 & 0.0000 & 1.0000 \end{bmatrix}$$

Since the initial approximations for the three differential rotations and scale are initially 0, the R matrix in Equation (24.14) reduces to a 3×3 identity matrix for the first iteration. In subsequent iterations, this matrix will change.

Combining the data from Table (23.3) with the GNSS baseline vectors of Example 24.1 yields a J matrix with 25 columns where the unknown parameters are

$$\begin{bmatrix} dX_A & dY_A & dZ_A & dX_B & dY_B & dZ_B & dX_C & \cdots & dZ_D & dX_E & dY_E & dZ_E & s & \theta_n & \theta_e & \theta_u \end{bmatrix}^{\mathrm{T}}$$

The J coefficient matrix for baseline EB is

$$J = \begin{bmatrix} 0 & 0 & 0 & a_{11} & a_{12} & a_{13} & 0 & \cdots & 0 & a_{14} & a_{15} & a_{16} & 0 & \cdots & 0 & b_{11} & b_{12} & b_{13} & b_{14} \\ 0 & 0 & 0 & a_{21} & a_{22} & a_{23} & 0 & \cdots & 0 & a_{24} & a_{25} & a_{26} & 0 & \cdots & 0 & b_{21} & b_{22} & b_{23} & b_{24} \\ 0 & 0 & 0 & a_{31} & a_{32} & a_{33} & 0 & \cdots & 0 & a_{34} & a_{35} & a_{36} & 0 & \cdots & 0 & b_{31} & b_{32} & b_{33} & b_{34} \end{bmatrix} \tag{24.18}$$

where, using Equation (24.16), the coefficients for A are

$$A = \begin{bmatrix} -0.1597 & 0.9703 & 0.1817 & 0.1596 & -0.9703 & -0.1816 \\ 0.6405 & -0.2419 & -0.7288 & -0.6406 & 0.2418 & 0.7288 \\ 0.7511 & 0.0000 & 0.6601 & -0.7511 & 0.0000 & -0.6602 \end{bmatrix}$$

and, using Equation (24.17), the coefficients for the transformation parameters, B, are

$$B = \begin{bmatrix} 553.4271 & -72.9412 & 15.2308 & -17.2421 \\ 43.4106 & 405.6700 & -61.1002 & -376.7564 \\ -62.9699 & -361.3992 & 91.7378 & -411.2673 \end{bmatrix}$$

With the aid of Equations (18.2) through (18.4), the geocentric coordinates for stations E and B are

Station	X (m)	Y (m)	Z (m)
E	1,160,083.8309	−4,655,634.2161	4,188,722.3464
B	1,160,637.2580	−4,655,590.8055	4,188,659.3765

The values for the constant matrix (K) are

$$K = \begin{bmatrix} 552.8771 - (1,160,637.2580 - 1,160,083.8309) \\ 44.8606 - (-4,655,590.8055 + 4,655,634.2161) \\ -63.0862 - (4,188,659.3765 - 4,188,722.3464) \end{bmatrix} = \begin{bmatrix} -0.5500 \\ 1.4500 \\ -0.1163 \end{bmatrix}$$

Notice that the coefficient matrix for each GNSS baseline vector component is no longer a matrix of −1, 0, and 1, but rather contains noninteger values. Also, note that the coefficients for the differential changes in scale and rotations are populated for each baseline vector component. The reader may wish to review the Mathcad worksheet *C24.xmcd* on the book's companion website to explore the complete set of matrix operations.

24.5 ANOTHER APPROACH TO TRANSFORMING COORDINATES BETWEEN REFERENCE FRAMES



The Helmert transformation discussed in Section 24.2 is the most rigorous method of converting between reference frames. However, many GNSS software manufacturers have broken this problem into their horizontal and vertical components. This method is especially useful when the local coordinates are in some arbitrarily created temporal coordinate system. To effect this transformation, the satellite-derived geodetic coordinates must be transformed into a two-dimensional

Cartesian coordinates using a map projection system. Often, a stereographic map projection is used to convert the GNSS-derived latitude and longitude of stations into local coordinate system values of *NE*.

The oblique stereographic map projection uses a plane as its developable surface. It is defined by its grid origin and scale factor. Typically, the average of the control station's latitudes and longitudes are used to define the origin of the projection. To minimize the scaling differences between the distances observed conventionally with those derived from satellites, the map projection coordinate system is brought to the surface using the average height of the project control. This is accomplished using a defining scaling factor for the map projection system of

$$k_0 = 1 + \frac{H_{avg}}{R_e} \tag{24.19}$$

where k_0 is the scale factor used to define the oblique stereographic map projection; H_{avg} is the average orthometric height of the project; and R_e is the average radius of the Earth at the origin of the coordinate system.[2]

The transformation process involves observing stations using a GNSS receiver that already has coordinates that are defined in the arbitrary/local coordinate system. The satellite-derived coordinates are then used to define the map projection system. Following this, the satellite-derived map projection coordinate values are transformed into the local coordinate system using a two-dimensional conformal coordinate transformation from Section 18.2. However, the general least-squares approach may provide a better transformation since both systems of coordinates will contain observational errors. Using either method, the satellite-derived plane coordinates are brought into the local horizontal coordinate system. Subsequently, these transformation parameters are used to transform the remaining satellite-derived map projection coordinates into the arbitrary/local coordinate system.

Once the horizontal transformation is completed, the satellite-derived heights must also be transformed into local elevations. To do this, at least three points with known local elevations must be observed to determine their satellite-derived geodetic heights. These geodetic heights are then converted into orthometric heights using an appropriate geoid model and Equation (23.39) rearranged as

$$H = h - N \tag{24.20}$$

where H is the station's orthometric height; h the geodetic height; and N the geoidal separation determined from a geoid model for that station. These three points determine a level surface that is oblique to the surface as determined by the same points using their local orthometric height values. This obliquity is caused by deflection

[2]The Gaussian mean radius is defined as $R = \sqrt{MN} = \frac{a(1-e^2)^{1/2}}{1-e^2\sin^2\varphi}$, where M is the radius of the meridian and N the radius of the normal defined in Equation (23.18).

of the vertical (see Section 24.6) and is corrected by applying two rotations in the cardinal directions. A translation between the two systems completes the transformation. The linear transformation is computed as

$$r_e N_{GNSS} + r_n E_{GNSS} + T = H_{Local} - H_{GNSS} + v \tag{24.21}$$

where T is the translation between the two level surfaces; r_e and r_n are the rotations in the east and north directions; N_{GNSS}, E_{GNSS}, and H_{GNSS} are the satellite-derived local map projection coordinates of the stations having orthometric heights as determined by Equation (24.20) of the local orthometric height of the station, H_{GNSS}, H_{Local}; and v is the residual error determined after the transformation.

Example 24.3 Four stations with state plane coordinates and orthometric heights determined by conventional surveying are also occupied with a GNSS receiver to determine their geodetic coordinate values. The satellite-derived geodetic heights are converted to orthometric heights using Equation (24.20). Using the average of the satellite-derived coordinate values, an oblique stereographic map projection system is defined with a grid origin of (41°18′44.28403″ N, 76°00′18.69234″ W) and a scaling factor of 1.000050598 where a Gaussian mean radius[2] of the Earth at 41°18′44.28403″ N latitude is used as the radius of the Earth. The satellite geodetic coordinate values are then transformed into a map projection coordinate system. The resulting coordinates are shown in the following table. What are the transformation parameters for the localization and the residuals for the control points?

	Satellite Coordinates			Local Coordinates		
Station	X (m)	Y (m)	H (m)	E (m)	N (m)	H (m)
A	49.916	−141.948	339.845	441.616	1807.534	340.500
E	−387.185	410.711	291.690	223.893	2022.852	292.354
F	311.626	279.250	312.076	532.227	2003.068	312.731
G	25.655	−547.996	350.268	452.616	1631.044	350.935

SOLUTION Following Equation (18.5) and the procedures outlined in Sections 18.2 through 18.4, a two-dimensional conformal coordinate transformation is performed. The resulting transformation parameters are $a = 0.43139044$, $b = 0.05283832$, $T_E = 412.587$ m, and $T_N = 1866.123$. These transformation parameters in conjunction with satellite-derived positions can be used to transform all other observed points into the local coordinate system. Following this, the transformation between the level surfaces is developed using Equation (24.21) as

$$\begin{bmatrix} -141.948 & 49.916 & 1 \\ 410.711 & -387.185 & 1 \\ 279.250 & 311.626 & 1 \\ -547.996 & 25.655 & 1 \end{bmatrix} \begin{bmatrix} r_e \\ r_n \\ T \end{bmatrix} = \begin{bmatrix} 340.500 - 339.845 \\ 292.354 - 291.690 \\ 312.731 - 312.076 \\ 350.935 - 350.268 \end{bmatrix}$$

Using Equation (11.32), $r_e = -1.574''$, $r_n = -3.205''$, and $T = 0.660$ m. The resulting residuals for both transformations are

Station	N (m)	E (m)	H (m)
A	−0.009	0.004	0.006
E	−0.010	−0.035	−0.001
F	−0.014	0.038	−0.002
G	0.034	−0.007	−0.003

Following these two adjustments, the residuals should be examined for blunders. It should be remembered when doing this that both sets of coordinates have observational errors. For example, in a real-time satellite survey, the estimated accuracy of the coordinates is between 1 and 2 cm. Additionally, the application of the geoid model in Equation (24.20) can add two or more centimeters of error to the orthometric heights derived using this equation. Similarly, the local coordinates contain errors that are dependent on the field procedures followed, the quality of the instruments, and abilities of the observer. Thus, the residuals shown above may be considered acceptable for the survey procedures.

In Example 24.3, an unweighted adjustment model was used since the stochastic model for the conventional and satellite surveys were unknown. However, if the stochastic model is known, then a weighted adjustment would be appropriate. Since GNSS receivers provide estimates on the accuracies of their coordinates, the weighted adjustment given in Equation (12.11) would more appropriately be used. However, if estimates on the accuracies of the local coordinates are also known, then a general least-squares approach, as discussed in Chapter 22, would be more appropriate and would account for the errors in both systems of observations.

On the companion website is the Mathcad worksheet® *C24-2.xmcd*, which demonstrates the solution to Example 24.3. It uses matrices to perform the computations and reads a data file, which is shown next to the initial data read. Additionally, Example 24.3 is solved in the spreadsheet file *Example 24-3.xls*.

24.6 OTHER CONSIDERATIONS WHEN LOCALIZING A SURVEY

When converting a global reference frame such as any ITRF or GNSS system to a local reference frame such as NATRF 2022 or NAD 83, the location of the control should encompass the project area as shown in Figure 24.1. Additionally, any points that are important to the engineering design of the project such as benchmarks on bridges and so on should also be part of the control for the transformation. The rationale for encompassing the project area is due to error propagation. For instance, assume, the survey crews are working for a department of transportation and that the project includes a bridge. If all the control is along the highway,

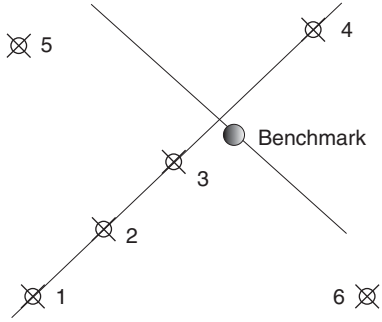


FIGURE 24.1 Location of control in alignment project.

for example points 1 through 4 in Figure 24.1, the control will be linear and the transformation from one system to the other will be rotationally weak across the alignment. That is, if points 5 and 6 in Figure 24.1 are not part of the control for the transformation, the entire transformation can rotate around the alignment. Any points surveyed away from the alignment, such as points 5 and 6, will likely show large propagational errors. Additionally, if the project involves a cross alignment, such as another road, any transformation of points along the cross alignment will suffer from uncertainty in rotation. These errors will increase as one moves farther from the alignment. Additionally, it is reasonable to assume that a bridge will have a benchmark that engineers will use to design a new bridge. If this benchmark is not included in the transformation but is used in the design process, any errors in the benchmark will not become evident until the construction phase of the project, which will be too late to correct for the error. In fact, the horizontal part of the transformation would be well served using only points 1, 4, 5, and 6 along with the benchmark on the bridge. Points 2 and 3 could then be used as checks on the quality of the transformation. However, for large regions, additional points dispersed throughout the project area will result in a better transformation for the project.

Using procedures like those shown in Chapters 24 and Example 24.2, a combined adjustment of both terrestrial and GNSS baseline vectors can be performed. If GNSS-derived station coordinates are to be held, they must first be transformed in the local datum to ensure consistency with any local control stations. However, if GNSS-derived points are the only control in the adjustment, then the entire adjustment can be performed using the global datum that was used to reduce the GNSS observations. However, it is likely that passive marks have coordinates and elevations in some local reference frame, and thus the different reference frames must be considered.

In both chapters, the adjustments are performed in the local geodetic coordinate system. This system was chosen since defining standard deviations for control stations in the (n, e, u) system is more intuitive to surveyors than either the geodetic or geocentric coordinate systems. References in the bibliography at the end of this book contain procedures for combining GNSS and terrestrial observations using either the geocentric or geodetic systems.

24.7 USING TOTAL STATION PSEUDO-OBSERVATIONS



In 2021, Tadeusz Gargula published another approach to combining total station observations with GNSS baseline vectors. In his paper, he outlined a method for converting total station horizontal distance and angle observations into what he termed *pseudo-observations*. Pseudo-observations are mark-to-mark distances between stations generated with total station observed quantities. The method used horizontal distances, both horizontal and zenith angles, and instrument and target heights. It has been altered here to use slant/slope distances instead of horizontal distances.

The method assumes that for every horizontal angle observation, there is a backsight and foresight distance with corresponding zenith angle observations. Thus, there will be three pseudo-observations for each instrument setup where two pseudo-observations are from the backsight and foresight distance observations and one from the observed horizontal angle observation. One of the drawbacks of this method is the use of the observed zenith angles, which as stated in Chapter 23, suffer from the fact that for long sight distances, the effects of refraction are difficult to model. However, for typical surveys, sight distances are relatively short, and the time between measuring a foresight zenith angle and its counterpart as a backsight zenith angle for the same line is typically short. Thus, the effect of refraction on the angle is typically small to insignificant. Another disadvantage is that post-adjustment residuals and observational uncertainties are for the pseudo-observations and not the original total station observations. However, this method provides a straightforward method for adding total station observations where the differences between the local coordinate system and the geodetic coordinate system are removed.

Figure 24.2 depicts a survey where baseline vectors were used to establish positions of station A through E. However, two stations, F and G, could not be observed using GNSS methods, so the traverse $B–F–G–C–E–B$ was created to generate coordinates on F and G. The observation data is shown in Figure 24.3. To create pseudo-observations from the total station observations, the mark-to-mark distances must be generated from the total station data for backsight and foresight distances, as well as the pseudo-observation connecting the backsight and foresight stations for each observed horizontal angle.

Figure 24.4 depicts the backsight or foresight slant distance observations created by a total station. The difference in height, Δh, between the instrument station and target station is computed using the observed zenith angle, z, and slant distance, S, along with the height of the instrument, hi, and height of the target, ht as

$$\Delta h = S\cos z + hi - ht \tag{24.22}$$

Similarly, the horizontal distance is computed as

$$D = S\sin z \tag{24.23}$$

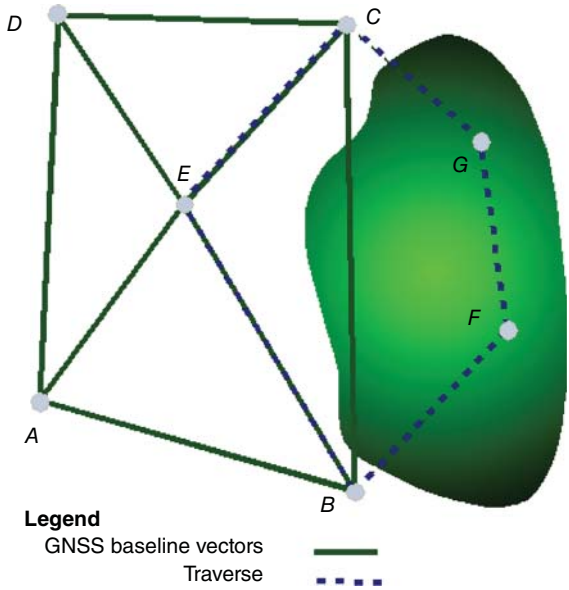


FIGURE 24.2 An integrated survey.

Station	X	Y	Z
A	1160360.804	-4656388.174	4187939.703
B	1161561.719	-4656562.759	4187444.144
C	1161098.348	-4655636.456	4188599.069
D	1160583.160	-4655767.430	4188598.439
E	1160940.673	-4656096.356	4188117.651
F	1161830.538	-4656070.885	4187924.237
G	1161648.864	-4655763.985	4188294.946

GNSS Baseline Vectors

From	To	dX	dY	dZ	Covariance matrix elements				
B	E	-621.050	466.399	673.509	3.6E-6	1.3E-7	1.6E-6	3.8E-6	3.7E-8	3.6E-6
B	C	-463.372	926.301	1154.923	3.7E-6	2.3E-7	-7.6E-9	3.7E-6	1.0E-7	3.8E-6
B	A	-1200.917	174.584	495.559	3.6E-6	8.1E-8	4.6E-8	3.7E-6	9.7E-8	3.8E-6
E	C	157.676	459.899	481.418	3.6E-6	4.8E-8	1.7E-8	3.6E-6	1.5E-7	3.7E-6
E	D	-357.513	328.928	480.787	3.7E-6	4.0E-8	1.0E-7	3.6E-6	6.5E-8	3.7E-6
E	B	621.047	-466.400	-673.507	3.6E-6	9.0E-8	4.6E-8	3.5E-6	1.8E-7	3.6E-6
E	A	-579.867	-291.816	-177.950	3.7E-6	6.9E-8	-3.4E-7	3.5E-6	1.5E-7	3.7E-6
D	C	515.186	130.973	0.627	3.7E-6	6.5E-8	9.0E-8	3.7E-6	-4.8E-9	3.7E-6
D	A	-222.358	-620.746	-658.739	5.0E-6	1.0E-6	6.5E-8	3.6E-6	5.4E-9	3.7E-6

Total Station Observations

BS	Occ	FS	Hor. Ang.	BS Z.Ang	BS S.Dist
hi	ht_BS	ht_FS		FS Z.Ang	FS S.Dist
E	B	F	59°25'36.0"	90°26'51.4"	1,028.024
1.670	2.000	2.000		89°24'55.5"	738.035
B	F	G	137°34'54.0"	90°35'04.5"	738.037
1.670	2.000	2.000		91°18'17.1"	514.394
F	G	C	139°46'56.0"	88°41'42.9"	514.407
1.670	2.000	2.000		89°16'53.5"	641.736
G	C	E	74°23'07.0"	90°43'42.9"	641.730
1.670	2.000	2.000		90°54'47.7"	684.196
C	E	B	128°49'31.0"	89°05'12.3"	684.201
1.670	2.000	2.000		89°33'08.6"	1,028.028

FIGURE 24.3 Data for the integrated survey shown in Figure 24.2.

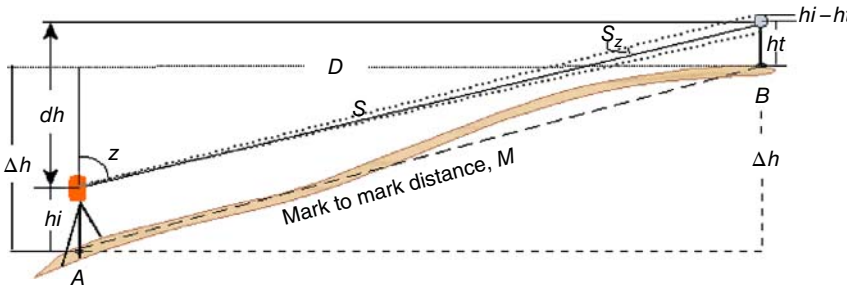


FIGURE 24.4 Analysis of a total station distance observation.

Using the Pythagorean theorem, the mark-to-mark distance, M, between the stations is

$$M = \sqrt{D^2 + \Delta h^2} = \sqrt{(S \sin z)^2 + (S \cos z + hi - ht)^2} \tag{24.24}$$

Example 24.4 A slant distance of 1028.024 m is observed with a zenith angle of 90°26′51.4″. The height of the instrument was 1.670 m, and the height of the target was 2.000 m. What is the mark-to-mark distance of the line?

SOLUTION By Equation (24.24):

$$\begin{aligned} M &= \sqrt{(1028.024 \ \sin z)^2 + (1028.024 \cos z + 1.670 - 2.000)^2} \\ &= 1028.0266 \text{ m} \end{aligned}$$

Figure 24.5 shows the mark-to-mark distance, M_3, opposite to the observed horizontal angle, θ. The horizontal distance between the backsight station, b, and foresight station, f, is shown as D_3. Using the cosine law, its length is computed as

$$D_3 = \sqrt{D_1^2 + D_2^2 - 2D_1 D_2 \cos\theta} \tag{24.25}$$

where D_1, D_2, and D_3 are the horizontal distances in the horizontal plane of the instrument to and between the backsight and foresight stations, respectively. The horizontal distances D_1 and D_2 distances are computed using trigonometry as

$$D_1 = S_b \sin z_b \quad \text{and} \quad D_2 = S_f \sin z_f$$

where S_b and z_b are the observed slant distance and zenith angle to the backsight target of the angle; and S_f and z_f are the observed slant distance and zenith angle

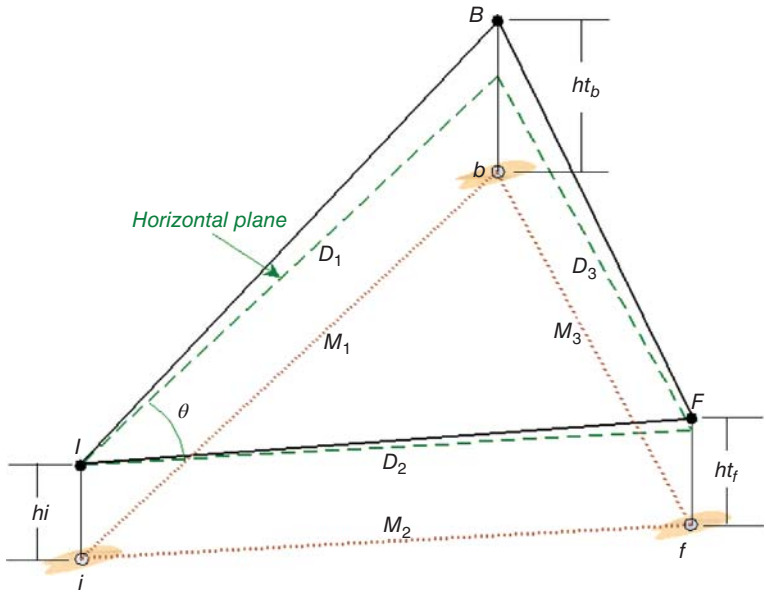


FIGURE 24.5 Analysis of a total station angle observation.

to the foresight target. Substituting these into Equation (24.25) yields

$$D_3 = \sqrt{(S_b \sin z_b)^2 + (S_f \sin z_f)^2 - 2(S_b \cos z_b)(S_f \cos z_f)\cos\theta} \qquad (24.26)$$

Figure 24.6 depicts a method for determining the elevation difference between the backsight and foresight stations based on the elevation differences for the backsight and foresight observations along with the difference in target heights at the backsight and foresight stations. Thus, the difference in height between the

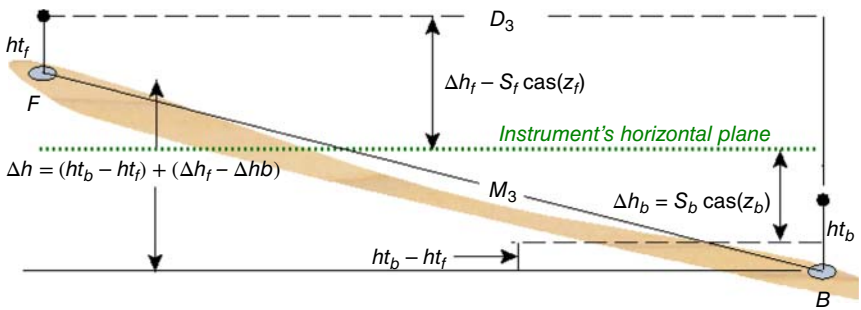


FIGURE 24.6 Analysis of elevation difference between backsight and foresight stations.

backsight and foresight stations is computed as

$$\Delta h_{bf} = (ht_b - ht_f) + (\Delta h_f - \Delta h_b) \tag{24.27}$$

where Δh_{bf} is the difference in height between the backsight and foresight stations; ht_b and ht_f are the height of the backsight and foresight targets, respectively; and Δh_b andΔh_f are the difference in height from the horizontal plane of the instrument to the backsight and foresight targets, respectively. The difference in heights, Δh_b andΔh_f can be computed using the observed slant distances to the targets along with their respective zenith angles. Substituting these values into Equation (24.27) yields

$$\Delta h_{bf} = (ht_b - ht_f) + (S_f \cos z_f - S_b \cos z_b) \tag{24.28}$$

Now the mark-to-mark distance, M_3, is computed using the Pythagorean theorem and Equations (24.26) and (24.28) yielding

$$M_3 = \sqrt{D_3^2 + \Delta h_{bf}^2} \tag{24.29}$$

Example 24.5 An angle is observed with a total station. The backsight slant distance was 1028.024 along a line of sight with a zenith angle of 90°26′51.4″. The foresight slant distance was 738.035 m along a line of sight with a zenith angle of 89°24′55.5″. The height of the instrument was 1.670 m. The height of both backsight and foresight targets were 2.000 m. The observed horizontal angle, θ, between the backsight target and foresight target is 59°25′36″. What is the mark-to-mark distance between the backsight and foresight targets?

SOLUTION Substituting $S_b = 1028.024$ m, $z_b = 90°26'51.4''$, $S_f = 738.035$ m, and $z_f = 89°24'55.5''$ into Equation (24.26), yields

$$\begin{aligned} D_3 &= \sqrt{(S_b \sin z_b)^2 + (S_f \sin z_f)^2 - 2(S_b \cos z_b)(S_f \cos z_f) \cos 59°25'36''} \\ &= \sqrt{(1027.9926)^2 + (737.9966)^2 - 2(1027.9926)(737.9966) \cos 59°25'36''} \\ &= 910.8469 \text{ m} \end{aligned}$$

Substituting the appropriate values into Equation (24.28) yields

$$\begin{aligned} \Delta h_{bf} &= (2.000 - 2.000) + (738.035 \cos z_f - 1028.024 \cos z_b) \\ &= 0 + 7.5300 - (-8.0311) = 15.5611 \text{ m} \end{aligned}$$

Thus, by Equation (24.29), the mark-to-mark distance, M_3, is

$$M_3 = \sqrt{910.8469^2 + 15.5611^2} = 910.9798 \text{ m}$$

24.8 BUILDING A STOCHASTIC MODEL

The stochastic model for the proposed adjustment can be developed by propagating the estimated errors of the observations in Equations (24.24) and (24.29). Error sources such as the uncertainty in the slant distances and angles were covered in Chapter 7. Additionally, these two equations include the uncertainty in the measurement of the height of the instrument and target. Thus, as shown in Figure 24.4, the zenith angle is not only affected by the ability of the point and reading the angle to the target but also by the uncertainty in the heights of the target and instrument. Thus, the uncertainty in the zenith angle can be expressed as

$$s_z = \sqrt{\left(\frac{2\sigma_{ISO}}{\sqrt{n}}\right)^2 + \left(\frac{\sqrt{\sigma_{hi}^2 + \sigma_{ht}^2}}{S}\rho\right)^2} \tag{24.30}$$

where σ_{ISO} is the instrument's ISO 17123-3 standard; n is the number of repetitions of the angle; σ_{hi} and σ_{ht} are the uncertainties in the height of the instrument and target, respectively; S is the slant distance between the instrument and target; and ρ is a conversion factor to keep the units consistent in the equation. It is assumed that all these error sources are independent. Thus, the ability to point and read an angle and slant distance is not affected by the uncertainties to center the instrument and target, or the uncertainties determine the height of the instrument or target. Thus, Equation (6.16), SLOPOV, can be used to determine the uncertainty in the observed zenith angle.

From Equation (24.24), the following error terms for each of the error sources are derived as

$$\frac{\partial M}{\partial S}\sigma_S = \frac{S\sin^2 z + (S\cos z + hi - ht)\cos z}{M}\sigma_S = \frac{S\sin^2 z + \Delta h\cos z}{M}\sigma_S \tag{24.31a}$$

$$\frac{\partial M}{\partial z}\sigma_z = \frac{S\sin z(S\cos z + hi - ht) - S^2\sin z\cos z}{M}\sigma_z = \frac{S\sin z\,\Delta h - S^2\sin z\cos z}{M}\sigma_z \tag{24.31b}$$

$$\frac{\partial M}{\partial hi}\sigma_{hi} = \frac{S\cos z + hi - ht}{M}\sigma_{hi} = \frac{\Delta h}{M}\sigma_{hi} \tag{24.31c}$$

$$\frac{\partial M}{\partial ht}\sigma_{ht} = \frac{ht - S\cos z - hi}{M}\sigma_{ht} = -\frac{\Delta h}{M}\sigma_{ht} \tag{24.31d}$$

Using Equation (6.16), the error in the pseudo-observation from the instrument to the target is

$$\sigma_M = \sqrt{\left(\frac{\partial M}{\partial S}\sigma_s\right)^2 + \left(\frac{\partial M}{\partial z}\sigma_z\right)^2 + \left(\frac{\partial M}{\partial hi}\sigma_{hi}\right)^2 + \left(\frac{\partial M}{\partial ht}\sigma_{ht}\right)^2} \tag{24.32}$$

When using Equation (24.32) to estimate the uncertainty in the pseudo-observation, M, it is important to consider unit consistency. Thus, all the linear units for S, hi, and ht must be the same. Additionally, the uncertainty for the zenith angle, σ_z, must be in radians.

Example 24.6 The manufacturer's uncertainty for the EDM observation in Example 24.4 is listed as $\pm(3\,\text{mm} + 3\,\text{ppm})$, and its ISO 17123-3 standard is listed as $\pm 3''$. The observations were repeated four times. The centering uncertainty for the instrument and target is estimated to be ± 0.0015 m. The uncertainty in the height of the instrument is estimated to be ± 0.0018 m. The estimated uncertainty in the height of the target is ± 0.001 m. What is the estimated uncertainty in the pseudo-observation M?

SOLUTION Recall from Example 24.4 the pseudo-observation, M, is 1028.0266 m and the observed zenith angle is $90°26'51.4''$. Equation (24.30) is used to determine the uncertainty in the zenith angle since the zenith angles are nearly 90°. If this was not the case, Equation (7.24) should also be included in determining uncertainty in the zenith angle. Using Equation (24.30), the uncertainty in the zenith angle is estimated as

$$\sigma_z'' = \sqrt{\left(\frac{2(3'')}{\sqrt{4}}\right)^2 + \left(\frac{\sqrt{0.0018^2 + 0.001^2}}{1028.024}\rho\right)^2} = \pm 3.03''$$

Using Equation (7.38), the estimated uncertainty in the slant distance is

$$\sigma_S = \sqrt{0.0015^2 + 0.0015^2 + 0.003^2 + \left(1028.024\,\frac{3}{10^6}\right)^2} = \pm 0.0048\ \text{m}$$

Additionally, note that Equations (24.31a) through (24.31d) all contain the term for the difference in the heights between the instrument and backsight target station, which is

$$dh = 1028.024\cos z + 1.670 - 2.000 = -8.3611$$

Using Equation (24.31a), yields

$$\frac{\partial M}{\partial S}\sigma_s = \frac{1028.024\sin^2 z + (-8.3611)\cos z}{1028.0266}\sigma_S = \pm 0.0048\ \text{m}$$

Using Equation (24.31b), yields

$$\frac{\partial M}{\partial z}\sigma_z = \frac{1028.024\sin z(-8.3611) - 1028.024^2\sin z\cos z}{1028.0266}\frac{3.03''}{\rho} = -5.0 \times 10^{-6}$$

Using Equation (24.31*c*), yields

$$\frac{\partial M}{\partial hi}\sigma_{hi} = \frac{-8.3611}{1028.0266}0.0018 = -1.5 \times 10^{-5}$$

Using Equation (24.31*d*), yields

$$\frac{\partial M}{\partial ht}\sigma_{ht} = \frac{8.3611}{1028.0266}0.001 = 8.1 \times 10^{-6}$$

Now the uncertainty in the pseudo-observation, M, can be found using Equation (6.16) which yields

$$\begin{aligned}\sigma_M &= \sqrt{0.0048^2 + (-5.0 \times 10^{-6})^2 + (-1.5 \times 10^{-5})^2 + (8.1 \times 10^{-6})^2} \\ &= \pm 0.0048 \text{ m}\end{aligned}$$

Note in this example that only Equation (24.31*a*) made a significant contribution to the uncertainty in the pseudo-observation. This is typically the case, and thus using only Equation (24.31*a*) is acceptable for use in the stochastic model typically. However, if this operation is programmed, it should include all the error sources.

For the uncertainty in M_3, the uncertainties in the observations must be propagated through Equation (24.29). In other words, we must investigate the contributions of each observation's error source to the overall error in M_3. Thus, we must consider the errors in the backsight slant distance, S_b, foresight slant distance, S_f, backsight, z_b, and foresight, z_f, zenith angles, horizontal angle, θ, and backsight, ht_b, and foresight, ht_f, target heights. Table 24.2 lists the derivatives for each observation present in Equation (24.29).

Example 24.7 Compute the uncertainty for M_3 in Example 24.5.

SOLUTION Using the values from Example 24.5 and uncertainties from Example 24.6 were used to compute the values in the last column of Table 24.2. As can be seen in this column, the major contributors to the error in M_3 are the uncertainties in the backsight and foresight slant distances and the uncertainty in the horizontal angle θ with the uncertainty in the horizontal angle being the largest contributor. This is to be expected, since any change in the horizontal angle will directly affect M_3. The uncertainty for the horizontal angle, σ_θ, is computed using Equation (6.16) with components from Equations (7.7), (7.10), and (7.21), which results in

$$\sigma_\theta = \sqrt{\left(\frac{2\sigma_{ISO}}{\sqrt{n}}\right)^2 + \left(\frac{\sqrt{D_b^2 + D_f^2}}{D_b D_f}\sigma_t \rho\right)^2 + \left(\frac{D_3}{D_b D_f}\frac{\sigma_i}{\sqrt{2}}\rho\right)^2}$$

TABLE 24.2 List the Partial Derivatives for Each of the Error Terms Present in Equation (24.30)

Component	Component Derivatives to be used in Equation (6.16)	Value
	Let $dh = ht_b - ht_f - S_b \cos z_b + S_f \cos z_f$	15.56111
$\frac{\partial M_3}{\partial S_b}\sigma_{S_b}$	$-\frac{S_f \sin z_f \sin z_b \cos\theta - S_b \sin^2 z_b + \cos z_b\, dh}{M_3}\sigma_{S_b}$	0.00344
$\frac{\partial M_3}{\partial S_f}\sigma_{S_f}$	$\frac{S_f \sin^2 z_f - S_b \sin z_b \sin z_f \cos\theta + \cos z_f\, dh}{M_3}\sigma_{S_f}$	0.00101
$\frac{\partial M_3}{\partial z_b}\sigma_{z_b}$	$\frac{S_b^2 \cos z_b \sin z_b - S_b S_f \cos z_b \sin z_f\, cos\theta + S_b \sin z_b\, dh}{M_3}\frac{\sigma_{z_b}}{\rho}$	0.00017
$\frac{\partial M_3}{\partial z_f}\sigma_{z_f}$	$-\frac{S_f \sin z_f\, dh - S_f^2 \cos z_f \sin z_f + S_b S_f \cos z_f \sin z_b \cos\theta}{M_3}\frac{\sigma_{z_f}}{\rho}$	−0.00016
$\frac{\partial M_3}{\partial \theta}\sigma_\theta$	$\frac{S_b S_f \sin z_b \sin z_f \sin\theta}{M_3}\frac{\sigma_\theta}{\rho}$	0.01062
$\frac{\partial M_3}{\partial ht_b}\sigma_{ht_b}$	$\frac{dh}{M_3}\sigma_{ht_b}$	0.00002
$\frac{\partial M_3}{\partial ht_f}\sigma_{ht_f}$	$-\frac{dh}{M_3}\sigma_{ht_f}$	−0.00002

Thus, the uncertainty in the horizontal angle is

$$\sigma_\theta = \sqrt{\left[\frac{2(3'')}{\sqrt{4}}\right]^2 + \left[\frac{\sqrt{1027.9926^2 + 737.9966^2}}{1027.9926(737.9966)}0.0015\rho\right]^2 + \left[\frac{910.9798}{1027.9926(737.9966)}\frac{0.0015}{\sqrt{2}}\rho\right]^2}$$
$$= \pm 3.06''$$

Similarly, the uncertainty in each zenith angle is computed using Equation (24.30). The uncertainty for the backsight zenith angle was determined in Example 24.6. Remember to convert the units of these uncertainties to radians when computing their contributions to the uncertainty in the mark-to-mark distance, M_3. Using the values in the last column of Table 24.2 and Equation (6.16), the estimated uncertainty for M_3 is

$$\sigma_{M_3} = \sqrt{0.0034^2 + 0.0010^2 + 0.0002^2 + (-0.0002)^2 + 0.01062^2 + 2(0.00002^2)}$$
$$= \pm 0.0112 \text{ m}$$

Note that in this analysis, the contribution of the errors in the zenith angle observations and the uncertainties in the height errors of the instrument and targets are

insignificant to the overall uncertainty in the derived mark-to-mark distance, M_3. Thus, these error contributors can typically be ignored. However, software should always compute these values, nonetheless.

24.9 LEAST-SQUARES ADJUSTMENT USING PSEUDO-OBSERVATIONS

In Chapter 17, it was noted that the adjustment of GNSS baseline vectors was a linear adjustment. However, in an integrated adjustment, the inclusion of pseudo-observations from a total station involves nonlinear equations, and thus the integrated adjustment is nonlinear. This means that initial approximations for all the stations are required for the adjustment. These values can be obtained by first adjusting the GNSS observations and the total station observations independently. This should also be done to remove any blunders in the data prior to the combined adjustment. Thus, the observations used in the integrated adjustment should be free from blunders since identifying blunders once the data is combined is difficult.

The linearization of the GNSS equations results in the same coefficients used in the J matrix as was used in the A matrix in Chapter 17. However, the K matrix values for the GNSS observations are

$$K_{dx} = \Delta X_{obs} - (X_t - X_f) \tag{24.33a}$$

$$K_{dy} = \Delta Y_{obs} - (Y_t - Y_f) \tag{24.33b}$$

$$K_{dz} = \Delta Z_{obs} - (Z_t - Z_f) \tag{24.31c}$$

where K_{dx}, K_{dy}, and K_{dz} are the K matrix values for each baseline vector, ΔX_{obs}, ΔY_{obs}, and ΔZ_{obs}, respectively; and $(X, Y, Z)_f$ and $(X, Y, Z)_t$ are the station coordinates of the "from," f, and "to," t, endpoint stations.

For the pseudo-observation IJ, the observation equation is

$$\sqrt{(x_j - x_i)^2 + (y_j - y_i)^2 + (z_j - z_i)^2} = IJ + v_{IJ} \tag{24.34}$$

where (x_i, y_i, z_i) and (x_j, y_j, z_j) are the geocentric coordinates of stations I and J where I is the instrument station and J is the target station for pseudo-observation IJ with residual error v_{ij}. From Equation (24.34), we can derive the partial derivatives and express its linearized form as

$$\begin{aligned}\left(\frac{x_i - x_j}{IJ}\right)_0 dx_i + \left(\frac{y_i - y_j}{IJ}\right)_0 dy_i + \left(\frac{z_i - z_j}{IJ}\right)_0 dz_i \\ + \left(\frac{x_j - x_i}{IJ}\right)_0 dx_j + \left(\frac{y_j - y_i}{IJ}\right)_0 dy_j + \left(\frac{z_j - z_i}{IJ}\right)_0 dz_j\end{aligned} = k_{IJ} + v_{IJ} = (IJ - IJ_0) + v \tag{24.35}$$

where items in $()_0$ are derived using the approximate coordinate values for stations I and J before each iteration; IJ_0 is the pseudo-observation derived from the approximate coordinate values using Equation (24.34); and IJ is the derived pseudo-observation as determined in Section 24.7.

```
                    Reference Variance = 0.6614
                Standard Deviation of Unit Weight = ±0.8132

            Passed X² test at 95.0% significance level!
                        X² lower value = 12.40
                        X² upper value = 39.36
            A priori value of 1 used for reference variance
                     in computations of statistics.

*********************
Adjusted GNSS Vectors
*********************
      From        To          dX          dY          dZ        Vx        Vy        Vz
=============================================================================================
         B         E     -621.048     466.400     673.508    0.0016    0.0011   -0.0008
         B         C     -463.373     926.300    1154.924   -0.0009   -0.0007    0.0012
         B         A    -1200.917     174.584     495.558    0.0004   -0.0005   -0.0007
         E         C      157.675     459.900     481.416   -0.0006    0.0012   -0.0020
         E         D     -357.511     328.928     480.788    0.0016    0.0002    0.0014
         E         B      621.048    -466.400    -673.508    0.0014   -0.0001   -0.0012
         E         A     -579.868    -291.817    -177.950   -0.0012   -0.0006    0.0001
         D         C      515.187     130.972       0.628    0.0009   -0.0009    0.0007
         D         A     -222.357    -620.745    -658.738    0.0012    0.0013    0.0007

****************************
Adjusted Pseudo-Observations
****************************

      From        To         Mark-2-Mark         V
=================================================
         B         E           1,028.029    0.0022
         B         F             738.035    0.0034
         E         F             910.997    0.0171
         F         B             738.035   -0.0054
         F         G             514.400   -0.0013
         B         G           1,170.251    0.0026
         G         F             514.400    0.0007
         G         C             641.733    0.0014
         F         C           1,086.378   -0.0047
         C         G             641.733   -0.0009
         C         E             684.201   -0.0002
         G         E             802.148   -0.0034
         E         C             684.201    0.0053
         E         B           1,028.029    0.0035
         C         B           1,551.321   -0.0076

********************
Adjusted Coordinates
********************
   Station            X               Y               Z        Sx        Sy        Sz
=============================================================================================
         A   1,160,360.804   -4,656,388.174   4,187,939.703
         B   1,161,561.721   -4,656,562.758   4,187,444.145    0.0013    0.0013    0.0013
         C   1,161,098.348   -4,655,636.457   4,188,599.069    0.0016    0.0015    0.0016
         D   1,160,583.161   -4,655,767.429   4,188,598.441    0.0015    0.0014    0.0014
         E   1,160,940.672   -4,656,096.357   4,188,117.653    0.0013    0.0013    0.0013
```

FIGURE 24.7 Partial listing of adjustment results for data shown in Figure 24.3.

Again, Figure 24.2 shows the integrated survey where a total station polygon traverse was performed. The accompanying observation data is shown in Figure 24.3. The traverse data from the total station survey was converted to pseudo-observations as previously outlined. The total station has an ISO 17123-3 uncertainty of 3″ for the angles observed using four repetitions and an EDM uncertainty of 3 mm + 3 ppm. The estimated centering uncertainties are ±1.5 mm for the instrument and target. The uncertainty for the instrument height is ±1.8 mm and the target height is ±1 mm. Figure 24.7 shows partial listing of the results from the adjustment.

24.10 SOFTWARE

From the foregoing, it is evident that the methods discussed in Sections 24.7–24.9 require extensive computations to develop the pseudo-observations, stochastic model, and perform a least-squares adjustment. Additionally, the total station horizontal and vertical angle observations should be geodetic values and not their observed astronomical values; that is, the angles should have geodetic corrections applied to them as discussed in Section 23.6. However, these corrections are typically small, and small changes to astronomical (field observed) values typically cause very small changes to the pseudo-observations. Thus, the residuals in the adjustment can easily absorb these very small differences.

While a limited number of computations can be performed with hand calculators or spreadsheets, it is recommended that these procedures be programmed. The software ADJUST has the computations in Section 24.7–24.9 in the Programs menu under the GNSS Adjustment/Integrated Survey. Figure 24.8 shows the ADJUST dialog presented to the user when attempting to perform a GNSS – total station integrated survey adjustment. Note, the user needs to enter the angle, EDM, and set up uncertainties for the total station observations. The software will use these values along with the equations presented in Section 24.8 to compute the stochastic model for the total station observations. Following this, it will perform the combined adjustment as presented in Section 24.9. It is important to remember that both the GNSS and total station observations should be adjusted separately, and any observations with blunders be fixed or removed from the data sets before trying a combined adjustment.

Additionally, this material is also implemented in the MATHCAD handbook file "C24c.XMCD" titled "Combining GNSS and Total Station Data – Pseudo-Observations" for those wishing to analyze a coded version of the adjustment. In the file immediately following the reading of the total station observations are the total station uncertainties. Thus, after the data file is created with a text editor such as the one in ADJUST, the user needs to modify these values as appropriate. They are highlighted in aqua at the top of page four in the file. Of course, these uncertainties could become part of the data file, but this

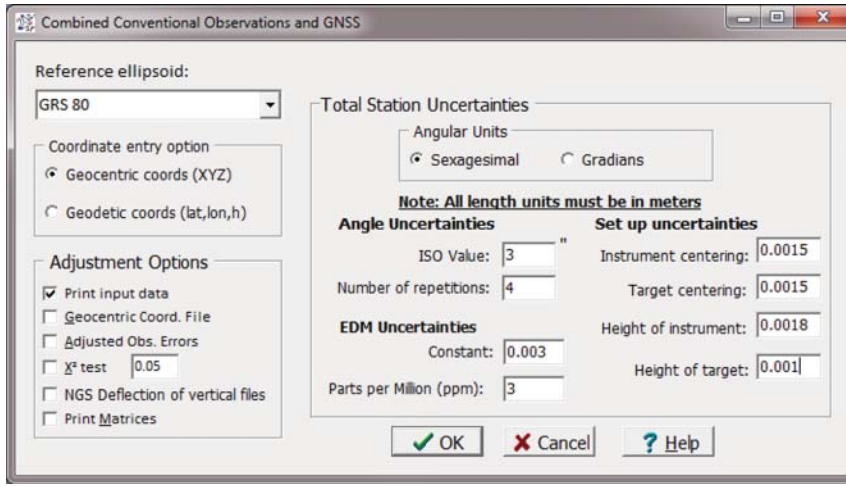


FIGURE 24.8 The ADJUST dialog for performing a GNSS-total station integrated adjustment.

will be left to the user to implement, should they desire this improvement in the software. This code can be used to create a similar program in a higher level computer language such as ADJUST.

PROBLEMS

Note: Partial answers to problems marked with an asterisk * can be found in Appendix G.

24.1 Discuss what is meant by a local reference frame.

***24.2** Discuss what is meant by a global reference frame.

24.3 How do local and global reference frames differ?

24.4 What is meant by the term *localization* or *site calibration*?

***24.5** Why is it important to localize a GNSS survey before staking out a highway alignment?

***24.6** Using the Helmert transformation parameters derived in Example 24.1, derive the NAD83 geocentric coordinates (in meters) for a point having ITRF coordinates of (1,160,652.008, −4,655,693.197, 4,188,423.986).

24.7 Same as Problem 24.6, except for a station having geocentric coordinates of (1,160,598.346, −4,655,899.999, 4,188,487.086).

24.8 Using the accompanying data, compute the Helmert transformation parameters to take the coordinate values from WGS84 to NAD83.

	WGS 84			NATRF 2022		
Sta	*X* (m)	*Y* (m)	*Z* (m)	*X* (m)	*Y* (m)	*Z* (m)
A	−2,499,794.418	−3,803,997.340	4,452,902.983	−2,499,794.872	−3,803,997.918	4,452,902.888
B	−2,541,365.016	−3,869,143.566	4,373,055.547	−2,541,365.471	−3,869,144.145	4,373,055.451
C	−2,540,927.093	−3,868,672.586	4,373,729.704	−2,540,927.548	−3,868,673.165	4,373,729.608
D	−2,541,958.169	−3,868,015.773	4,373,711.626	−2,541,958.624	−3,868,016.352	4,373,711.529

24.9 Using the accompanying data, compute the Helmert transformation parameters to take the coordinate values from WGS 84 to NATRF 2022.

	WGS 84			NATRF 2022		
Sta	*X* (m)	*Y* (m)	*Z* (m)	*X* (m)	*Y* (m)	*Z* (m)
A	−720,604.916	−5,604,512.413	2,948,272.770	−720,604.995	−5,604,512.421	2,948,272.735
B	−720,348.563	−5,604,348.824	2,948,642.551	−720,348.642	−5,604,348.832	2,948,642.517
C	−719,941.424	−5,604,486.186	2,948,480.672	−719,941.503	−5,604,486.194	2,948,480.638
D	−720,253.133	−5,604,661.495	2,948,072.711	−720,253.212	−5,604,661.503	2,948,072.677

***24.10** Why is it important to have control points dispersed about the perimeter of a project during a stakeout survey?

24.11 List the steps outlined in Section 24.5 to localize satellite-derived geocentric coordinates into a local coordinate system.

24.12 Using the procedures discussed in Section 24.5 and the following data determine the following:

(a) Transformation parameters for the horizontal and vertical transformations.

(b) Residuals of the transformed coordinates.

	Satellite-Derived Coordinates			Local Coordinates		
Station	*X* (m)	*Y* (m)	*h* (m)	*E* (m)	*N* (m)	*H* (m)
A	−314.0165	−107.1171	7.315	260.6725	370.2071	29.5300
B	−80.6074	310.8978	6.706	295.5440	847.7097	28.9194
C	340.7201	128.3654	6.096	754.5382	860.3314	28.3169
D	53.9009	−332.1368	5.486	689.2208	321.7670	27.6966

***24.13** Using the transformation parameters found in Problem 24.11, what are local coordinate values for a station with satellite-derived map projection coordinates of 9.062, −9.321, 6.205) in units of meters?

24.14 Repeat Problem 24.12 with the following data.

	Satellite-derived Coordinates			Local Coordinates		
Station	*X* (m)	*Y* (m)	*h* (m)	*N* (ft)	*E* (ft)	*H* (m)
A	−155.9973	−51.2075	353.873	102,188 637	80.384.044	385.858
B	−15.5672	−177.9981	355.397	102,725.777	80,072.955	387.380
C	185.5816	37.0210	337.718	103,224.623	80,900.183	369.709
D	8.1114	179.2322	351.739	102,558.102	81,235.510	383.720

24.15 Using the transformation parameters found in Problem 24.14, what are local coordinate values for Station *E* with satellite-derived oblique stereographic map projection coordinates of (−22.1311, 12.9566, 351.434)?

24.16 Using the procedures discussed in Section 24.5 and the following data determine the following:

(a) Transformation parameters for the horizontal and vertical transformations.

(b) Residuals of the transformed coordinates.

(c) Local coordinates for Station 5.

	Satellite-Derived Coordinates			Local Coordinates		
Station	*E* (m)	*N* (m)	*h* (ft)	*E* (ft)	*N* (ft)	*h* (ft)
1	−302.3329	67.0951	777.00	2,547,669.56	395,517.43	872.13
2	−363.6479	−271.5678	777.00	2,548,105.98	394,476.04	872.12
3	363.7930	−234.2808	705.00	2,550,040.94	395,878.46	800.13
4	239.8926	162.9304	797.00	2,548,990.27	396,750.02	892.12
5	−119.1956	128.4144	788.00			
6	33.1608	112.3895	776.00			
7	148.3272	35.0431	768.00			

24.17 A slant distance of 956.873 m is observed with a zenith angle of 89°06′01.8″. The height of the instrument was 1.500 m, and the height of the target was 2.000 m. What is the pseudo-observation for the line?

24.18 Same as Problem 24.17 with the following observations of a slant distance of 1256.874 m along a line of sight at a zenith angle of 87°08′46.8. The height of the foresight target is 1.750 m.

24.19 Using the data in Problem 24.17 as the backsight observations for a horizontal angle of 36°16′28″ and the data in Problem 24.18 as the foresight observations, what is the pseudo-observation for the line connecting the backsight and foresight stations?

PROGRAMMING PROBLEMS

24.20 Develop software that solves Problem 24.8.

24.21 Develop software that solves Problem 24.16.

CHAPTER 25

ANALYSES OF ADJUSTMENTS

25.1 INTRODUCTION

The initial adjustment of data is not the end of the adjustment. Rather, it is the beginning of the analysis process. Throughout this book, the mechanics of performing a least-squares adjustment properly have been discussed. Additionally, statistical methods were introduced to analyze the quality of observations. This chapter will explore the procedures used in analyzing the residuals of the observations after the adjustment.

25.2 BASIC CONCEPTS, RESIDUALS, AND THE NORMAL DISTRIBUTION

 The normal distribution and statistical testing were introduced in the early chapters of this book using simple sets of data. These basic concepts also apply in the analysis of data after an adjustment. When viewing Figure 25.1, four guiding principles are used in analyzing observations from normally distributed data:

1. Data tends to be clustered around a single value.
2. Errors tend to be distributed equally about this value.

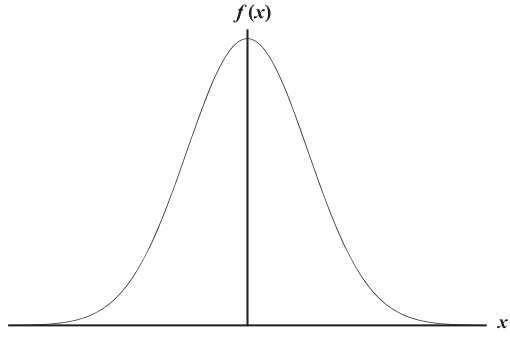


FIGURE 25.1 The normal distribution.

3. Errors are distributed in sign equally.
4. Most errors tend to be small in magnitude, and errors large in magnitude seldom occur.

At the conclusion of a least-squares adjustment, the residuals in the observations should be scanned and analyzed to locate outliers and blunders in the data. However, terms such as *large in magnitude* must be defined statistically. This is accomplished by comparing the observation's a priori standard deviation against its residual using a *t* distribution multiplier. For example, if a distance observation has a residual of +0.25 ft when all other residuals in the adjustment are below 0.10 ft, this observation may be viewed as large. However, if the observation had an estimated standard deviation of only ±0.5 ft, then its residual is well within its estimated range.

As an example, the results of the adjustment from Example 16.2 will be analyzed. A quick review of the residuals listed for the distance observations in Figure 25.2 shows that the residual for distance *QR* is at least twice the size of the other distance residuals and might be viewed as large. However, the a priori standard deviation of this observation was ±0.026 ft from an adjustment with 13 redundant observations. Thus, the 99% confidence interval for this observational residual is computed as

$$v_{99\%} = \pm t_{0.005,13} S = \pm 3.012(0.026) = \pm 0.078 \text{ ft}$$

From this, it can be shown that the residual of −0.038 ft for distance *QR* is well within the 99% confidence interval of ±0.078 ft.

When analyzing the angles, it can be seen that *QTR* has a residual of 18.52″. This residual is more than three times larger than the next largest residual of 5.23″. Also note that this angle opposes distance *QR*. Therefore, it may explain why distance *QR* had the largest distance residual. The a priori estimated standard deviation for

```
******************************
Adjusted Distance Observations
******************************
   Station   Station
  Occupied  Sighted       Distance          V          S
==========================================================
         Q         R      1,639.978    -0.0384     0.0236
         R         S      1,320.019     0.0176     0.0228
         S         T      1,579.138     0.0155     0.0235
         T         Q      1,664.528     0.0039     0.0250
         Q         S      2,105.953    -0.0087     0.0232
         R         T      2,266.045     0.0104     0.0241

***************************
Adjusted Angle Observations
***************************
    Station   Station     Station
  Backsight  Occupied   Foresight          Angle        V        S"
==========================================================================
         R         Q           S     38°48'52.8"     2.1"     2.59
         S         Q           T     47°46'13.1"     0.7"     2.89
         T         Q           R    273°24'54.1"    -2.4"     3.56
         Q         R           S    269°57'34.0"     0.6"     3.36
         R         S           T    257°32'57.3"     0.5"     3.71
         S         T           Q    279°04'34.5"     3.3"     3.45
         S         R           T     42°52'52.3"     1.3"     2.70
         S         R           Q     90°02'26.0"    -0.7"     3.36
         Q         S           R     51°08'41.3"    -3.7"     2.94
         T         S           Q     51°18'21.4"     5.2"     3.02
         Q         T           R     46°15'20.5"    18.5"     2.70
         R         T           S     34°40'05.0"    -0.7"     2.55
```

FIGURE 25.2 Adjusted distances and angles from Example 16.2.

angle *QTR* is $\pm 4.0''$. Using the same *t*-distribution critical value for a 99% confidence interval yields

$$v_{95\%} = 3.012(4.0'') = \pm 12.0''$$

The actual residual for angle *QTR* is well outside the range of $\pm 12''$. In this case, the residual for angle *QTR* fits the definition of "large" and is a candidate for reobservation or removal from the adjustment.

Using the procedures outlined in Chapter 21, blunder detection techniques confirm that angle *QTR* is a detectable blunder. This angular observation can be removed from the data set and the adjustment rerun. The results of this readjustment are shown in Figure 25.3. Note that the residual for distance *QR* was reduced by about half of its previous value after angle *QTR* was removed. This demonstrates the interrelationship between the distance *QR* and angle *QTR*. Also note that all the residuals are within a single standard deviation of their a priori estimated values. In fact, all the distance residuals are less than a third of their a

Adjusted stations

Station	X	Y	Error ellipse confidence level at 0.950 Sx	Sy	Su	Sv	t
R	1,003.06	2,640.01	0.000	0.006	0.017	0.000	0.11[e]
S	2,323.06	2,638.47	0.005	0.007	0.019	0.014	156.28[e]
T	2,661.74	1,096.09	0.006	0.007	0.021	0.015	26.18[e]

Adjusted Distance Observations

Station Occupied	Station Sighted	Distance	V	S	Std.Res.	Red.#
Q	R	1,640.008	-0.0081	0.0060	-0.409	0.576
R	S	1,320.006	0.0054	0.0055	0.295	0.579
S	T	1,579.133	0.0099	0.0056	0.510	0.597
T	Q	1,664.514	-0.0097	0.0060	-0.495	0.569
Q	S	2,105.966	0.0039	0.0056	0.162	0.702
R	T	2,266.034	-0.0014	0.0058	-0.057	0.700

Adjusted Angle Observations

Station Backsighted	Station Occupied	Station Foresighted	Angle	V	S	Std.Res.	Red.#
R	Q	S	38°48'50.2"	-0.45"	0.64	-0.13	0.795
S	Q	T	47°46'11.7"	-0.73"	0.69	-0.21	0.757
T	Q	R	273°24'58.1"	1.58"	0.89	0.44	0.672
Q	R	S	269°57'34.7"	1.31"	0.80	0.32	0.767
R	S	T	257°32'56.9"	0.11"	0.88	0.03	0.716
S	T	Q	279°04'30.3"	-0.91"	0.87	-0.24	0.700
S	R	T	42°52'52.6"	1.58"	0.64	0.41	0.821
S	R	Q	90°02'25.3"	-1.41"	0.80	-0.36	0.746
Q	S	R	51°08'44.5"	-0.53"	0.73	-0.14	0.767
T	S	Q	51°18'18.6"	2.43"	0.74	0.71	0.722
R	T	S	34°40'04.3"	-1.37"	0.61	-0.38	0.814

Adjusted Azimuth Observations

Station Occupied	Station Sighted	Azimuth	V	S"	Std.Res.	Red.#
Q	R	0°06'24.5"	-0.00"	0.00"	0.0	0.000

Adjustment Statistics

Iterations = 2
Redundancies = 12

Reference Variance = 0.1243
Reference So = ±0.35

Failed to pass X^2 test at 95.0% significance level!
X^2 lower value = 4.40
X^2 upper value = 23.34

FIGURE 25.3 Readjustment of data in Example 16.2 after removing angle *QTR*.

priori estimated errors, and all the angle residuals are less than half their a priori estimates.

There are 18 observations in this adjustment. Thus, about half of the residuals (9) should be positive, while the others are negative in sign. In this adjustment, there are exactly nine positive residuals and nine negative residuals. This follows the second guideline for normally distributed data. In this example, the residuals are not skewed as discussed in Section 2.4.

Example 25.1 A baseline vector determined by the rapid-static surveying method (5 mm + 1 ppm) has a length of about 10.5 km. The instrument setups were estimated to be within ±3 mm of the station's true location. Its combined *XYZ* residual is 0.021 m. If the number or redundancies in the adjustment is 20, at a 99% level of confidence, should this residual be considered too large?[1]

SOLUTION The a priori estimated error in this baseline using the rapid-static method is

$$\sigma = \sqrt{3^2 + 3^2 + 5^2 + \left(\frac{10,500,000}{1,000,000}\right)^2} = \pm 12.4 \text{ mm} \quad \text{and} \quad t_{0.005,20} = 2.84$$

The acceptable 99% range is $R = 2.84(12.4) = \pm 35.2$ mm. The residual was 21 mm. This is inside of the 99% confidence interval and is an acceptable size for a residual from a baseline of this length observed using the rapid-static surveying method.

25.3 GOODNESS-OF-FIT TEST



The results of the adjustment in Figure 25.3 appear to follow the normal distribution, and the sizes of the residuals indicate that the data appears to be consistent. However, the adjustment failed to pass the goodness-of-fit test. This demonstrates a weakness in the χ^2 test. That is, passing or failing the test is not a good indicator of the quality of the data or the presence of blunders. Nor does the test reveal the exact problem in the data when the test fails. The χ^2 test should be viewed as a warning flag for an adjustment that requires further analysis and not an indicator of problems with the data.

As discussed in Chapters 5 and 16, the χ^2 test compares the a priori reference variance, set equal to 1, against the reference variance computed from the data. An analysis of Equation (12.15) shows that the test will fail if the residuals are too large or too small compared to the weights of the observations. For example, an observation with a small residual should have a high weight, and a large residual should have a low weight. If the residuals tend to be smaller than their a priori standard deviations, then the resulting reference variance will be less than 1, most likely. This is an example of an incorrect stochastic model, which occurred in Example 16.2 discussed previously. If the residuals tend to be larger than their a priori standard deviations, the reference variance will be greater than 1, and the χ^2 test will fail in the upper bounds of the distribution.

[1]Readers are encouraged to rerun the data in Example 16.2 with the a priori standard deviations reduced by a factor of one-half to compare the adjustment results.

The original set of data in Example 16.2 failed the goodness-of-fit test. In the first adjustment, it failed because the computed reference variance was greater than 1. This was probably caused by the large residual for angle *QTR*. After angle *QTR* was removed from the data, the χ^2 test failed because the reference variance was statistically less than 1. Since the residuals appear to be much smaller than their a priori standard deviations, an incorrect stochastic model is the cause for the failure. In this case, the distance residuals were about three times less than the estimated standard deviations, and the angles were one-half of their estimated standard deviations. Thus, to pass the χ^2 test with this adjustment, the stochastic model needs to be modified.

In developing the stochastic model for this problem, it was originally believed that the field crew had used a total station with an ISO 17123-3 accuracy of $\pm 4''$ and EDM accuracy of $5\,\text{mm} + 5\,\text{ppm}$. It was later discovered that the crew used an instrument having an ISO 17123-3 accuracy of $\pm 2''$ and EDM accuracy of $2\,\text{mm} + 2\,\text{ppm}$. Estimated standard deviations using this new information resulted in the adjustment shown in Figure 25.4. This adjustment passed the goodness-of-fit test. However, by closely examining the two adjustments, this did not change the adjustment results significantly. That is, the adjusted coordinate values and adjusted observations are nearly the same in the adjustments shown in Figures 25.3 and 25.4. What did change in Figure 25.4 were the statistical results. Since the statistical results of the adjustment are of little value to the surveyor once analyzed, the adjustment shown in Figure 25.3 is sufficient for most applications. However, this does point out the importance of getting a realistic stochastic model and the importance of using statistical analysis, as discussed in Chapters 7 and 9.

Practitioners often view the goodness-of-fit test as an indicator of a possible problem. If the goodness-of-fit test failed because the computed reference variance was too small, the test result is often ignored since correcting it does not change the adjusted coordinates significantly. In this case, some practitioners will say that the adjustment failed the goodness-of-fit test on the "good" side. That is, the reference variance was too small because the residuals are "too" small. However, if the computed reference variance is greater than 1, the results should be analyzed for possible blunders in the observations using methods discussed in this chapter and Chapter 21. After the identifiable blunders have been eliminated from the set of data, the stochastic model might still exhibit problems. In this case, if the computed reference variance is greater than 1, the estimated standard deviations for the observations should be increased. In the preceding example, the computed reference variance is less than 1. To correct this, the estimated standard deviations for the observations were decreased. By doing this, the surveyor can achieve a correctly weighted least-squares adjustment.

This adjustment brings about another important discussion point. Had all the a priori standard deviations been reduced by a single scale factor, the results of the

Adjusted stations

Station	X	Y	Error ellipse confidence level at 0.950 Sx	Sy	Su	Sv	t
R	1,003.06	2,640.00	0.000	0.009	0.024	0.000	0.11°
S	2,323.06	2,638.47	0.008	0.009	0.028	0.021	151.27°
T	2,661.74	1,096.09	0.008	0.011	0.031	0.022	26.14°

Adjusted Distance Observations

Station Occupied	Station Sighted	Distance	V	S	Std.Res.	Red.#
Q	R	1,640.008	-0.0083	0.0086	-0.830	0.580
R	S	1,320.006	0.0053	0.0081	0.527	0.603
S	T	1,579.133	0.0100	0.0081	0.971	0.615
T	Q	1,664.515	-0.0089	0.0087	-0.895	0.568
Q	S	2,105.966	0.0039	0.0079	0.341	0.673
R	T	2,266.034	-0.0011	0.0082	-0.097	0.658

Adjusted Angle Observations

Station Backsighted	Station Occupied	Station Foresighted	Angle	V	S	Std.Res.	Red.#
R	Q	S	38°48'50.2"	-0.46"	0.95	-0.24	0.808
S	Q	T	47°46'11.7"	-0.73"	1.02	-0.38	0.776
T	Q	R	273°24'58.1"	1.59"	1.29	0.87	0.667
Q	R	S	269°57'34.7"	1.27"	1.14	0.64	0.751
R	S	T	257°32'56.8"	0.03"	1.27	0.02	0.696
S	T	Q	279°04'30.4"	-0.79"	1.25	-0.43	0.684
S	R	T	42°52'52.6"	1.55"	0.94	0.77	0.820
S	R	Q	90°02'25.3"	-1.37"	1.14	-0.69	0.751
Q	S	R	51°08'44.4"	-0.57"	1.06	-0.29	0.772
T	S	Q	51°18'18.7"	2.53"	1.09	1.34	0.751
R	T	S	34°40'04.3"	-1.42"	0.90	-0.72	0.827

Adjusted Azimuth Observations

Station Occupied	Station Sighted	Azimuth	V	S"	Std.Res.	Red.#
Q	R	0°06'24.5"	-0.00"	0.00"	0.0	0.000

Adjustment Statistics
****
Iterations = 2
Redundancies = 12

Reference Variance = 0.4572
Reference So = ±0.68

Passed X^2 test at 95.0% significance level!
X^2 lower value = 4.40
X^2 upper value = 23.34

FIGURE 25.4 Readjusted data from Example 16.2 with a different stochastic model.

adjustment would be the same.[2] That is, if the a priori standard deviations for all the observations had been cut in half, this would have resulted in increasing all the

[2]It should be realized that most baseline vectors are determined from several epochs of data involving a multitude of satellites. In these cases, the number of redundant observations is typically over 30, and the critical t from the normal distribution can be used to determine the estimated error in a baseline vector.

weights by a factor of four. Since weights are relative, performing a scalar reduction in the standard deviations does not change the adjustment.

This example demonstrates the importance of estimating valid a priori standard deviations for the observations, as discussed in Chapters 7 and 9. Estimated standard deviations for observations cannot be selected from the recesses of one's mind. Doing this not only affects how the errors in the observations are distributed, but also how the results of the adjustment are analyzed. For traditional surveys, it is always best to select a stochastic model that reflects the estimated accuracies in the observations using the procedures discussed in Chapters 7 and 9. Good estimates of setup errors will typically result in a sound stochastic model.

However, the network adjustment of baseline vectors derives its stochastic model from the least-squares reduction of each baseline. As discussed in Chapter 17, the stochastic model is part of the printout from the baseline reduction. Assuming proper field procedures, if the network adjustment fails to pass the χ^2 test, there is little one can do to modify the model. In fact, if field procedures are consistent, scaling the entire stochastic model might be the only option. While this procedure may result in a "passed" χ^2 test, it will not change the coordinates in the solution since weights are relative. In fact, it will only change the a posteriori statistics. Thus, scaling the entire stochastic model by a single factor is only important if the statistics are important. Since this is seldom true, there is little value in scaling the stochastic model.

If the commercial software employed the multipoint solution technique, better baseline vector adjustments would be obtained. This solution takes the individual pseudoranges from the satellites to compute baseline vectors and the unknown coordinates of the network stations simultaneously. Current software offerings compute the baseline vectors from the pseudoranges and then the coordinates of the stations from these computed vectors. This is a two-step process. The multipoint solution is like the three-dimensional geodetic network and the photogrammetric block-bundle adjustment in that all unknowns are computed in one adjustment. However, it is important to remove any large blunders before attempting a multipoint solution. Thus, the two-step solution process must be maintained to ferret out problems in the data. Since the main advantage of the multipoint solution lies in the development of the stochastic model, only a few examples of this solution technique exist.

25.4 COMPARISON OF GNSS RESIDUAL PLOTS

Massive quantities of data are collected and reduced during the reduction of carrier-phase observations typically. Some software manufacturers plot pseudorange residuals against a timeline. Examples of this are shown in Figure 25.5, where plot Figure 25.5(*a*) is for satellite vehicle 24 (SV 24) and Figure 25.5(*b*) is for satellite vehicle 28 (SV 28). Notice that the residuals in Figure 25.5(*b*) have a slight downward slope at the beginning of the session. In fact, except for the first

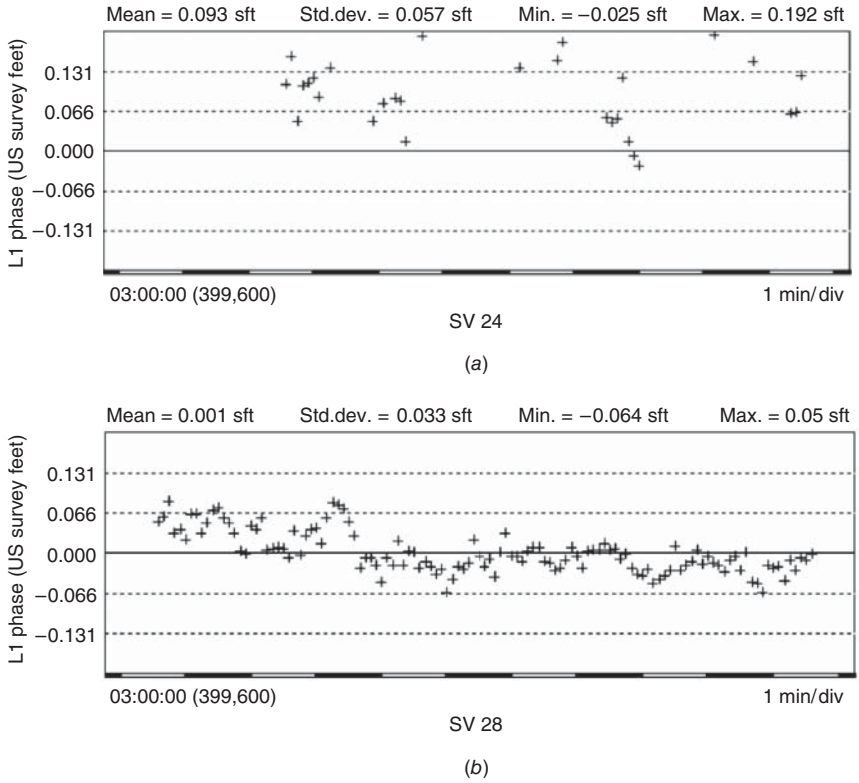


FIGURE 25.5 Pseudorange residuals plots from satellites 24 (*a*) and 28 (*b*).

3 minutes, the residuals are uniform even though most are negative in sign. During the first 3 minutes of the session, satellite 28 was just clearing the set mask angle and suffered some loss of lock problems. This skewed the data.

The first 3 minutes of data from this satellite could be removed, and the baseline reprocessed to correct this problem. However, this might not be practical given the shortness of the overall observation session. Since much field time is lost to travel, setup, and tear down, it is always wise to collect more data than is needed to resolve the position of a GNSS receiver. This is similar to observing distances and angles more than once.

Note that the residuals in Figure 25.5(*a*) are mostly positive. This could have several causes, including loss of lock problems. However, these types of plots are typically seen when the broadcast ephemeris is used in the reduction. The broadcast ephemeris is a near-future prediction of the location of the satellite. For various reasons, satellites sometimes stray from their predicted paths. When this happens, the coordinates for the satellite's position at the time of broadcast are in error. It is also possible that the integer ambiguity for this satellite's pseudorange was in error.

These problems result in plots like Figure 25.5(*a*), where most of the residuals have one sign, indicating a skewed set of data. Broadcast ephemeris errors can sometimes be corrected by downloading and processing with a precise ephemeris. Ambiguity-resolution problems after the data collection is complete can only be corrected by removing the satellite from the dataset. In general, the best solutions can always be obtained with a precise ephemeris. Since the ultra-rapid ephemeris is available within a few hours of data collection, it is wise to wait for this ephemeris before processing the data. Of course, better results can be obtained by waiting for later precise ephemerides, which provide even better locations for the satellites at the time of the survey.

Figure 25.5 also depicts a problem with real-time kinematic (RTK) GNSS surveys. That is, by their very nature, the broadcast ephemeris must be used in processing the data. Thus, the option of downloading a precise ephemeris is not available. RTK surveys have their place in stake-out and mapping surveys. However, they should never be used to establish anything other than low-order control. The RTK survey is the equivalent of a radial total station survey, leaving only minimal checks in the quality of the data. Since all GNSS surveys can be seriously affected by solar activity, ephemeral errors, and multipathing, the RTK survey leaves the surveyor open to poor results without any indication at the receiver. Checks on these surveys can be obtained only by reobserving the stations using a different satellite geometry or by post-processing the observations. In the latter case, this is no longer an RTK survey.

25.5 USE OF STATISTICAL BLUNDER DETECTION



When more than one blunder exists in a set of data, statistical blunder detection methods presented in Chapter 21 help isolate the offending observations. These methods, along with graphical techniques, were presented in that chapter.

The problem of having multiple blunders usually exists in large control surveys. When adjusting large sets of data, it is wise to break the data into smaller subsets that can be checked for blunders before attempting the larger adjustment. By breaking a larger adjustment into smaller subsets, problems can often be isolated, corrected, or removed from the observations.

GNSS surveys provide similar examples. When performing a large baseline vector adjustment, it is often wise to download and process the data by session initially and then by day. Since a GNSS campaign typically involves several observational sessions and possibly several days of data, trying to combine all sessions into one large adjustment generally takes more time than individually processing the data by session and then day. By processing baseline vector data by session and then day, it becomes easier to isolate weaker and trivial baselines. After processing each day's sessions satisfactorily, a combined network adjustment can be attempted and analyzed for blunders. It may be important at this point to reanalyze and possibly reprocess baselines to obtain the best solution. Again,

statistical blunder detection will help analyze possible blunders in baselines. If the observation sessions are long enough, it is possible to eliminate data from the session to remove possible problems caused by obstructions and loss of lock. This, again, indicates the need to collect more data than is required to obtain a solution. A few extra minutes used to observe additional data can often result in improved solutions without returning to the field to reobserve a baseline.

PROBLEMS

Note: Partial answers to problems marked with an asterisk can be found in Appendix G.

25.1 Following an adjustment, how is the term *large* defined in relation to residuals?

25.2 What are the basic concepts of a normal distribution, and how are these concepts used to analyze the results of a least-squares adjustment?

***25.3** An adjustment has 22 observations. Only 10 of these residuals are negative in sign. Should this cause concern during the analysis?

25.4 Discuss why the goodness-of-fit test is not always a reliable indicator of a blunder in data.

25.5 What does it mean to "fail" the χ^2 test?

25.6 Discuss possible methods that can be used to develop a proper stochastic model in a horizontal network survey.

25.7 Discuss why the scaling of the stochastic model does not affect the adjusted parameters.

25.8 What are the possible reasons for the residual plot in Figure 25.5(*a*)?

25.9 Discuss the importance of collecting more than the minimum amount of data in a GNSS survey.

25.10 Why should the RTK survey method not be used to establish high-order control stations coordinates?

25.11 Why should baseline vectors be processed by session rather than by job?

***25.12** A baseline vector determined by rapid-static method ($3\,\text{mm} + 0.5\,\text{ppm}$) has a length of about 4.5 km. The instrument setups were estimated to be within ± 3 mm of the true station location. Its combined *XYZ* residual is 2.5 cm. At a 99% confidence level, should this residual be considered too large if the number of redundant observations in the adjustment is 15?

25.13 Similar to Problem 25.12, except the length of the baseline vector is 20 km and the size of the *XYZ* residual is 3.0 cm. At a 99% confidence level, should this residual be considered too large if the number of redundant observations in the adjustment is 25?

***25.14** A distance in a horizontal network survey with 16 degrees of freedom has a residual of −0.14 ft. Its a priori estimated error is ±0.034 ft. At a 99% level of confidence, is this residual too large?

25.15 A distance in a horizontal network survey with 24 degrees of freedom has a residual of −0.05 ft. Its a priori estimated error is ±0.024 ft. At a 99% level of confidence, is this residual too large?

In Problems 25.16 through 25.26, analyze the residuals from the least-squares adjustment using ADJUST. Indicate if there are any areas of concern in the data or stochastic model. When possible, isolate and remove questionable data and stochastic models at a 99% level of confidence. After the changes, rerun the adjustment and compare the results. (*Note*: Not all of the problems have questionable data.)

25.16 Problem 12.14.

25.17 Problem 12.15.

25.18 Problem 15.10.

25.19 Problem 15.11.

25.20 Problem 15.14.

25.21 Problem 16.8.

25.22 Problem 16.9.

25.23 Problem 16.10.

25.24 Assuming a rapid static survey method was used with a manufacturer's stated accuracy of (3.0 mm + 0.5 ppm), and the setup errors were estimated to be ±3 mm at each setup. Analyze the data in Problem 17.5 at a 99% level of confidence.

25.25 Assuming a rapid static survey method was used with a manufacturer's stated accuracy of (5.0 mm + 1 ppm), and the setup errors were estimated to be ±3 mm at each setup. Analyze the data in Problem 17.10 at a 99% level of confidence.

CHAPTER 26

TERRESTRIAL LASER SCANNING AND UNCERTAINTY ESTIMATION

26.1 INTRODUCTION

As technology progresses, new geospatial technologies emerge for data collection. One of the major technologies that has revolutionized surveying is laser scanning, which utilizes light detection and ranging (lidar) to rapidly generate 3D point clouds. Laser scanning has found diverse uses in several professions including, but not limited to, surveying, engineering, construction, architecture, geology, and more (Olsen, 2022). Laser scanners use angle and range measurements similar to conventional total station instruments; however, a spinning mirror deflects the laser beam at fixed angular increments along each scanline across a 360-degree field of view to collect hundreds of thousands or even millions of points per second. In contrast to total stations, accurate target aiming for a single point is not possible, and therefore direct repetitions of angle and range measurements cannot be achieved. Instead, the advantage lies in the abundance of points (point clouds) with the ability to deliver high-resolution 3D datasets with unprecedented detail. Repeat scans can be completed from these same vantage points to provide repeat measurements for improved accuracy or filter out inconsistencies from moving objects in this scene. However, double scanning is not common practice as the optimal results are typically achieved by collecting data from multiple vantage points to minimize occlusions throughout the entire scene. Another fundamental difference is that many scanners utilize high-precision, multi-axis inclination sensors, which allow the instrument to be set up in an unlevel, but still stable, condition to acquire

data (Silvia and Olsen, 2012). Nevertheless, there are some higher precision scanners that utilize level compensation technology like that used in a total station. Lastly, many scanners cannot be directly set up over a control point, requiring resection to carefully placed targets or cloud-to-cloud surface matching to determine its location.

This chapter explains the primary uncertainties associated with terrestrial laser scanning (TLS) measurements, demonstrates their propagation in the scanning coordinates, provides numerical examples, and discusses practical considerations for survey planning with laser scanning technology. Although terrestrial or ground-based laser scanning is the focus, many aspects of this chapter apply to other platforms. For simplicity, the chapter will show computations for a single point in most examples. Realistically, a single scan can contain tens to hundreds of millions of points; hence, efficient computation techniques are needed to apply these computations to the entire point cloud. Additionally, only fundamental measurement errors will be considered. Other error sources are not considered such as ranging error from saturation or blooming errors resulting from highly reflective surfaces, multipath errors from reflective surfaces such as mirrors or water bodies, mixed pixels from objects spaced too close together, or misclassification. Surveyors performing laser scans should be aware of these errors and how to minimize their impact on the resulting products by applying filters in software.

26.2 LASER SCANNER COORDINATES



Terrestrial laser scanning generates point clouds in Cartesian coordinates from spherical coordinate observations as follows:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{ls} = \begin{bmatrix} r\cos\alpha\cos\theta \\ r\cos\alpha\sin\theta \\ r\sin\alpha \end{bmatrix} \tag{26.1}$$

where x_{ls}, y_{ls}, z_{ls} are the local laser scanner Cartesian coordinates, *ls* denotes the local scan (unregistered) coordinates; r is the laser scanner range; α is the vertical angle; and θ is the horizontal angle. Notice in Equation (26.1) to derive a laser scanner point, a triplet of scanner measurements is required: the range, horizontal angle, and vertical angle. The vertical angle (α) is defined as 0° when aligned with the scanner xy plane, 90° when aligned with the positive z axis, and −90° when aligned with the negative z axis. The horizontal angle (θ) is defined as 0° when aligned with the positive x axis and measured counterclockwise up to 360°. The origin coordinates of the scanner are initially (0, 0, 0) for each local scan

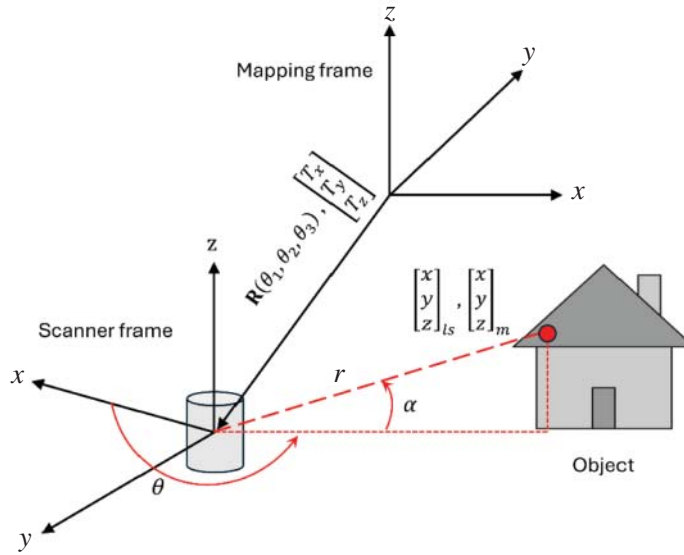


FIGURE 26.1 Terrestrial laser scanner measurement geometry and registration.

without registration, which represents the scanner's optical center that the measurements are referenced to. The range is then computed based on the two-way travel time of the laser pulse to the object or by observing the phase-shift of the signal. Figure 26.1 demonstrates the terrestrial laser scanning measurement geometry.

Example 26.1 Suppose the following measurements were obtained for a point $r = 312.350\,\text{m}$, $\theta = 260°05'22''$, $\alpha = 42°33'12''$ from a precisely leveled laser scanner. Find the local coordinates of the point in the scanner coordinate system.

SOLUTION The local scanner coordinates, using the scanner origin of (0, 0, 0), are found using Equation (26.1). Therefore,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{ls} = \begin{bmatrix} r\cos\alpha\cos\theta \\ r\cos\alpha\sin\theta \\ r\sin\alpha \end{bmatrix} = \begin{bmatrix} 312.350\cos 42°33'12''\cos 260°05'22'' \\ 312.350\cos 42°33'12''\sin 260°05'22'' \\ 312.350\sin 42°33'12'' \end{bmatrix} = \begin{bmatrix} -39.601 \\ -226.659 \\ 211.235 \end{bmatrix}$$

Manufacturers and software developers typically only provide the Cartesian coordinates x_{ls}, y_{ls}, z_{ls} for a scan. For error propagation, the initial scanner observations are needed and can be obtained from the Cartesian coordinates using

the following equations:

$$\text{Range: } r = \sqrt{x_{ls}{}^2 + y_{ls}{}^2 + z_{ls}{}^2} \tag{26.2}$$

$$\text{Horizontal angle: } \theta = \text{atan2}(x_{ls}, y_{ls}) = \tan^{-1}\left(\frac{y_{ls}}{x_{ls}}\right) + \text{C} \tag{26.3}$$

$$\text{Vertical angle: } \alpha = \tan^{-1}\frac{z_{ls}}{\sqrt{x_{ls}{}^2 + y_{ls}{}^2}} \tag{26.4}$$

Note that the equations above assume that the scanner origin coordinates are (0, 0, 0). In the case of a registered point cloud, the scanner origin coordinates must first be subtracted from the point cloud of each scan. Additionally, θ is measured counterclockwise from the x axis and the correct quadrant must be determined by using the sign of the numerator and denominator. Use of the atan2 function simplifies the computation by considering the signs, but the atan2 definition can vary depending on the computation platform and language. For example, Microsoft Excel uses atan2(x_{ls}, y_{ls}), while C++ uses atan2(y_{ls}, x_{ls}).

Example 26.2 For the given laser scanner coordinates $x_m = 745{,}106.308$ m, $y_m = 127{,}947.168$ m, $z_m = 606.560$ m, compute the laser scanner measurements, assuming scanner origin coordinates of $x_0 = 745{,}145.909$ m, $y_0 = 128{,}173.827$ m, $z_0 = 395.325$ m.

SOLUTION First, subtract the scanner origin from the laser scanner coordinates to find the local scan coordinates:

$$x_{ls} = x_m - x_0 = 745{,}106.308 - 745{,}145.909 = -39.601$$
$$y_{ls} = y_m - y_0 = 127{,}947.168 - 128{,}173.827 = -226.659$$
$$z_{ls} = z_m - z_0 = 606.560 - 395.325 = 211.235$$

From Equations (26.2) to (26.4),

$$r = \sqrt{x_{ls}{}^2 + y_{ls}{}^2 + z_{ls}{}^2} = \sqrt{-39.601^2 + -226.659^2 + 211.235^2} = 312.350\text{ m}$$

$$\theta = \text{atan2}(x_{ls}, y_{ls}) = \text{atan2}(-39.601,\ -226.659) = -99°54'38'' + 360°$$
$$= 260°05'22''.$$

$$\text{or } \theta = \tan^{-1}\frac{y_{ls}}{x_{ls}} + C = \tan^{-1}\frac{-226.659}{-39.601} + 180° = 260°05'22''$$

$$\alpha = \tan^{-1}\frac{z_{ls}}{\sqrt{x_{ls}{}^2 + y_{ls}{}^2}} = \tan^{-1}\frac{211.235}{\sqrt{-39.601^2 + -226.659^2}} = 42°33'12''$$

26.3 BEAM DIVERGENCE AND INCIDENCE ANGLE EFFECTS

The laser beam diverges (grows) as it travels through the atmosphere, resulting in additional uncertainty of the angular measurement. This uncertainty originates from the assumption that the angle and range will be recorded along the apparent centerline of the laser beam; however, the actual location of the return is uncertain and can be located anywhere within the beam footprint (Glennie, 2007). This gives rise to additional angular, $\sigma_{BWangle}$, and range uncertainties, $\sigma_{BWrange}$ resulting from the beam divergence (Figure 26.2).

The laser power across the entire beamwidth diameter typically follows a Gaussian-like power distribution (Glennie, 2007; Hartzell et al., 2015). Many instrument manufacturers provide the diameter of the beam divergence as the point where the power has fallen to a fraction of $1/e^2$ of the peak, which coincides with the $\pm 2\sigma$ interval. The angular uncertainty due to beam divergence is defined as

$$\sigma_{BWangle} = \frac{\gamma}{4} \tag{26.5}$$

where γ is the beam divergence angle.

The effect the laser's beamwidth has on the range uncertainty is also a function of the surface orientation to the scanner unit, termed incidence angle. The beam footprint elongates when scanning at incidence angles other than 0°. The range uncertainty at any point as a function of a local planar estimate of the surface is computed as

$$\sigma_{BWrange} = r\frac{\gamma}{4}\tan I_a \tag{26.6}$$

where I_a is the incidence angle determined as the angle between the local terrain or object normal vector and the laser direction.

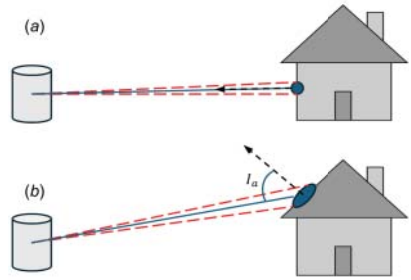


FIGURE 26.2 Beam divergence and incidence angle effect. (*a*) Beam footprint when laser direction and the normal of the object's surface are aligned ($I_a = 0$). (*b*) Beam footprint when laser direction and the normal of the object's surface are at an angle ($I_a > 0$).

Example 26.3 A scanner has a beam divergence specification of 0.33 mrad[1] defined as $1/e^2$. Scanner measurements are planned for a wall at an incidence angle of 30° and range of 20 m. Estimate the angular and range effects on the uncertainty resulting from the beam divergence.

SOLUTION The angular uncertainty due to beam divergence is estimated from Equation (26.5) as

$$\sigma_{BWangle} = \frac{\gamma\rho}{4} = \frac{0.33 \times 10^{-3}\rho}{4} = \frac{68.07''}{4} = \pm 17.0''$$

Note that the multiplication of 10^{-3} converts from mrad to radians and the multiplication with ρ converts from radians to seconds.

The range uncertainty due to the beam footprint is calculated from Equation (26.6)

$$\sigma_{BWrange} = r\frac{\gamma}{4}\tan I_a = 20\left(\frac{0.33 \times 10^{-3}}{4}\right)\tan 30^\circ = \pm 0.0010$$

The result is ±0.0010 m or ±1.0 mm.

Equation (26.6) assumes that the laser beam expands linearly. However, for short ranges, the laser beam expands nonlinearly and transitions to linear behavior for ranges greater than 40 m (the actual threshold depends on the laser wavelength and beam waist or exit radius) (O'Banion et al., 2018b). For airborne systems, the linear laser beam expansion model is sufficient, but for terrestrial laser scanning and short-range scanners, the nonlinear laser beam expansion model should be used. This difference in the laser beam radius using a linear expansion model versus a nonlinear one can be up to about 20% at the shortest range of 1 m (O'Banion et al., 2018b). Considering this nonlinear behavior, the range uncertainty equation can be modified as

$$\sigma_{BWrange} = \frac{1}{2}w_0\sqrt{1 + \left(\frac{\lambda r}{\pi w_0^2}\right)^2}\tan I_a \tag{26.7}$$

where w_0 is the laser beam exit radius; and λ is the laser wavelength. The term $w_0\sqrt{1 + \left(\frac{\lambda r}{\pi w_0^2}\right)^2}$ is the beam expansion equation. The radius is used instead of the diameter, resulting in a division by 2 instead of by 4. Figure 26.3 demonstrates an example of range error due to beam divergence, computed using Equation (26.7) for various incidence angle cases.

[1]One mrad (milliradian) is 0.001 rad.

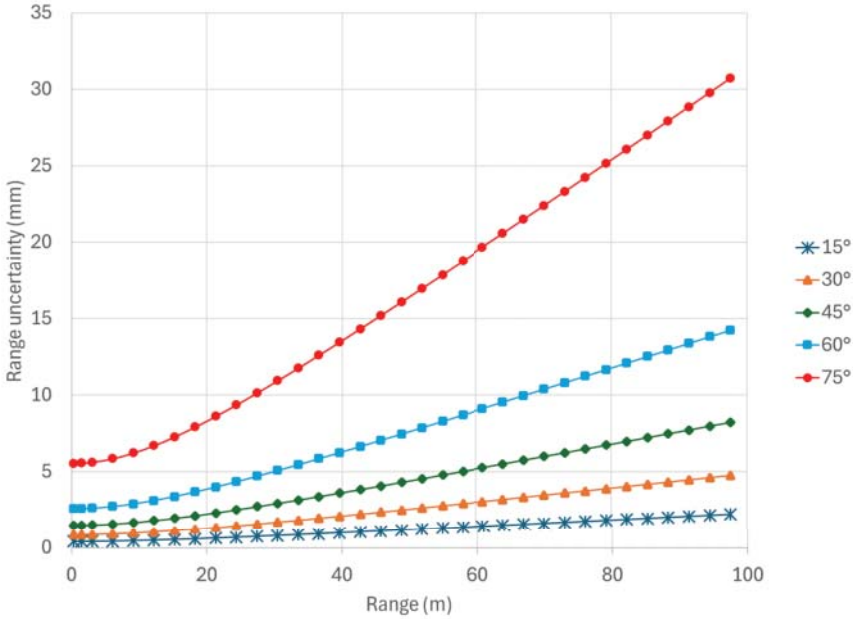


FIGURE 26.3 Range uncertainty (in mm) due to beam divergence and incidence angle computed using Equation (26.7). A beam diameter at the front window of 3.5 mm (FWHM) was used.

Some instrument manufacturers use a different measure of beam divergence such as the full width at half maximum (FWHM). To convert the FWHM beam diameter to the $1/e^2$ definition, users need to multiply the FWHM diameter by 1.699. In addition, the $1/e^2$ beam divergence (full angle) can be calculated as

$$\gamma = \frac{2\lambda}{\pi n w_0} \tag{26.8}$$

where n is the refractive index of the beam's medium of propagation. For standard air at standard pressure, the refractive index is approximately 1.000293. Note that Equation (26.8) provides the beam divergence in radians.

Example 26.4 Using the same scanner measurements of Example 26.3, calculate the range uncertainty due to the beam footprint. Consider the nonlinear behavior of the range uncertainty assuming a beam diameter at the front window of 3.5 mm (FWHM definition) and a scanner wavelength of 1550 nm.

SOLUTION The beam exit diameter is given in the FWHM definition. This is converted to the $1/e^2$ definition by multiplying by 1.699; therefore, $1.699(3.5) = 5.95$ mm. The laser beam exit radius becomes $w_0 = \frac{5.95}{2} = 2.98$ mm.

In using Equation (26.7), meters are used for consistency. The range uncertainty is estimated as

$$\sigma_{BWrange} = \frac{1}{2} w_0 \sqrt{1 + \left(\frac{\lambda r}{\pi w_0^2} \right)^2} \tan I_a$$

$$= \frac{1}{2} 2.98 \times 10^{-3} \sqrt{1 + \left(\frac{1550 \times 10^{-9} \times 20.000}{\pi (2.98 \times 10^{-3})^2} \right)^2} \tan 30^\circ = \pm 0.0013$$

Therefore, the result is ± 0.0013 m or ± 1.3 mm. Examples 26.3 and 26.4 demonstrate that in short ranges, in this case 20 m, the linear expansion slightly underestimates the effect of the beam footprint on the range uncertainty, and the robust nonlinear expansion equation should be used.

Example 26.5 Suppose a surveyor is using a laser scanner with a wavelength of 1550 nm and a beam diameter at the front window of 3.5 mm (FWHM definition). Calculate the beam divergence in mrad.

SOLUTION Convert the beam diameter from the FWHM definition to the $1/e^2$. Beam diameter $(1/e^2) = 1.699 \times$ beam diameter (FWHM) $= 1.699 \times 3.5$ mm $=$ 5.95 mm. Then, the laser beam exit radius becomes $w_0 = \frac{5.95}{2} = 2.98$ mm. Using Equation (26.8) and a refractive index of 1.000293, the beam divergence in mrads is found as

$$\gamma = \frac{2\lambda}{\pi n w_0} = \frac{2 \times 1550 \times 10^{-9}}{\pi \times 1.000293 \times 2.98 \times 10^{-3}} = 3.3 \times 10^{-4}$$

Converting to milliradians (mrad), the result is multiplied by 10^3, which equates to 0.33 mrad.

26.4 RANGE AND ANGULAR UNCERTAINTY

Several factors, such as atmospheric conditions, instrument calibration, and object surface interaction with the laser beam, affect the quality of laser scanner measurements. The equations in this chapter assume only random uncertainties related to the laser ranging uncertainty, horizontal and vertical angle encoding uncertainties, inclination sensor uncertainty, and laser beamwidth uncertainties for the horizontal and vertical angles and range. This reduction facilitates practical uses as manufacturers often quote their expected accuracy in terms of these components. It is assumed that systematic errors, which may result from mechanical imperfections or improper atmospheric parameters, are removed by proper instrument calibration.

Three primary error sources: (1) the angular resolution of the laser scanner encoder, (2) uncertainty due to beam divergence, and (3) inclination sensor uncertainty, are used to compute the angular uncertainty of the horizontal and vertical angles as

$$\sigma_\theta = \sqrt{\sigma_{\theta_{en}}^2 + \sigma_{BWangle}^2 + \sigma_{inc}^2} \tag{26.9}$$

$$\sigma_\alpha = \sqrt{\sigma_{\alpha_{en}}^2 + \sigma_{BWangle}^2 + \sigma_{inc}^2} \tag{26.10}$$

where σ_θ is the uncertainty of the laser scanner horizontal angle; σ_α is the uncertainty of the laser scanner vertical angle; $\sigma_{\theta_{en}}$ is the horizontal angle encoding uncertainty; $\sigma_{\alpha_{en}}$ is the vertical angle encoding uncertainty; $\sigma_{BWangle}$ is the beamwidth-related uncertainty for the horizontal and vertical angles; and σ_{inc} is the inclination sensor uncertainty. Modern laser scanners use a dual-axis compensator or inclination sensor capable of precise leveling; however, the contribution of the σ_{inc} factor can vary depending on the specific scanner model (Silvia and Olsen, 2012). Longer range scanners tend to use higher quality level compensators, while short-range scanners tend to use inclination sensors capable of faster data acquisitions.

Range uncertainty considers the constant and scalar error discussed in Chapter 7, and the range uncertainty due to beam divergence, which results in

$$\sigma_r = \sqrt{a^2 + (r \times b\,ppm)^2 + \sigma_{BWrange}^2} \tag{26.11}$$

where σ_r is the laser scanner range uncertainty in the observed range r; a and b are the constant and scale factors listed in the manufacturer specifications for the instrument; and $\sigma_{BWrange}$ is the beamwidth-related uncertainty for the range.

The constant and scale factor parameters also depend on the reflectivity of the target surface. In the absence of a prism to reflect the emitted laser beam to the instrument, the scanner relies on the reflective properties of the scanned object to receive the laser beam and perform a range measurement. Manufacturer specifications estimate the constant and scale factors under lab conditions with reflectivity of the test surface often being at around 80%. A degradation of the range measurement performance should be expected for scanned objects with reflectivity less than what the manufacturer used to estimate the range uncertainty of the scanner (Bolkas and Martinez, 2018). In addition to the degradation in ranging uncertainty, scanning objects with low reflectivity (black and gray surfaces) results in a reduced maximum range of the scanner unit. Some manufacturers may provide information about the level of ranging measurement degradation that can be considered in error propagation for practical applications. For example, the maximum range of some scanners can drop from several hundred meters to about 120–150 m for dark gray targets, and to about a few tens of meters for black targets. The range noise can significantly increase from about 0.2 mm for white targets to about 0.5 mm for dark-gray targets, and even exceed the 1 mm noise

level for black targets, at a scanning distance of 25 m. The range noise can even reach the level of several millimeters to a centimeter when scanning dark-gray targets at longer ranges of about 100 m.

Example 26.6 Suppose a laser scanner has the specifications shown in the following table. The angular effect of beam divergence was determined to be ±17.0″, and the range effect of beam divergence was calculated to be ±2.9 mm. Estimate the angular and range uncertainty at a point located 60.960 m from the scanner.

Horizontal encoder uncertainty	±12″
Vertical encoder uncertainty	±12″
Inclination sensor uncertainty	±2.0″
Range uncertainty	1.5 mm + 10 ppm

SOLUTION To find the horizontal and vertical angle uncertainty, we must combine the angular error sources of the scanner encoder, beam divergence, and inclination as

$$\sigma_\theta = \sqrt{\sigma_{\theta_{en}}^2 + \sigma_{BWangle}^2 + \sigma_{inc}^2} = \sqrt{12^2 + 17.0^2 + 2.0^2} = \pm 20.9''$$

$$\sigma_\alpha = \sqrt{\sigma_{\alpha_{en}}^2 + \sigma_{BWangle}^2 + \sigma_{inc}^2} = \sqrt{12^2 + 17.0^2 + 2.0^2} = \pm 20.9''$$

Neglecting the inclination uncertainty yields a similar angular uncertainty of ±20.8″, which demonstrates that when the laser scanner instrument is equipped with a high-quality dual-axis compensator, this uncertainty factor is typically negligible. However, this can vary based on the instrument used. Users should check their instrument specifications and estimate whether the inclination uncertainty should be included or not in their calculations.

For the range uncertainty, we must combine the constant and scale manufacturer values along with the beam divergence effect on the range uncertainty yielding

$$\begin{aligned}\sigma_r &= \sqrt{a^2 + (r \times b\,ppm)^2 + \sigma_{BWrange}^2} \\ &= \sqrt{0.0015^2 + \left(\frac{60.960 \times 10}{10^6}\right)^2 + 0.0029^2} \\ &= \pm 0.0033 \text{ m}\end{aligned}$$

The range uncertainty is then found to be ±0.0033 m or ±3.3 mm.

26.5 ERROR PROPAGATION OF LASER SCANNER MEASUREMENTS



The main error components of laser scanning measurements, established in the previous sections, can be propagated to the laser scanner coordinates using Equation (26.1) and derive the uncertainty of each point of the point cloud. The covariance matrix of the scanner measurements is given as

$$\Sigma_{TLS(noise)} = \begin{bmatrix} \sigma_r^2 & 0 & 0 \\ 0 & \sigma_\theta^2 & 0 \\ 0 & 0 & \sigma_\alpha^2 \end{bmatrix} = \begin{bmatrix} a^2 + (r \times b\,ppm)^2 + \sigma_{BWrange}^2 & 0 & 0 \\ 0 & \sigma_{\theta_{en}}^2 + \sigma_{BWangle}^2 + \sigma_{inc}^2 & 0 \\ 0 & 0 & \sigma_{\alpha_{en}}^2 + \sigma_{BWangle}^2 + \sigma_{inc}^2 \end{bmatrix} \tag{26.12}$$

where $\Sigma_{TLS(noise)}$ is the 3×3 covariance matrix of the laser scanner measurements. A diagonal matrix is used assuming that there are no correlations or dependencies between measurements.

The Jacobian matrix of the partial derivatives of the laser scanner coordinates with respect to the laser scanner measurements is

$$J_{ls} = \begin{bmatrix} \frac{\partial x_{ls}}{\partial r} & \frac{\partial x_{ls}}{\partial \theta} & \frac{\partial x_{ls}}{\partial \alpha} \\ \frac{\partial y_{ls}}{\partial r} & \frac{\partial y_{ls}}{\partial \theta} & \frac{\partial y_{ls}}{\partial \alpha} \\ \frac{\partial z_{ls}}{\partial r} & \frac{\partial z_{ls}}{\partial \theta} & \frac{\partial z_{ls}}{\partial \alpha} \end{bmatrix} = \begin{bmatrix} \cos\alpha\cos\theta & -r\cos\alpha\sin\theta & -r\sin\alpha\cos\theta \\ \cos\alpha\sin\theta & r\cos\alpha\cos\theta & -r\sin\alpha\sin\theta \\ \sin\alpha & 0 & r\cos\alpha \end{bmatrix} \tag{26.13}$$

Using Equation (6.13), the covariance matrix of the laser scanner coordinates is derived as

$$\Sigma_{xyz_{ls}} = J_{ls}\Sigma_{TLS(noise)}J_{ls}^{\mathrm{T}} = J_{ls}\begin{bmatrix} \sigma_r^2 & 0 & 0 \\ 0 & \sigma_\theta^2 & 0 \\ 0 & 0 & \sigma_\alpha^2 \end{bmatrix}J_{ls}^{\mathrm{T}} = \begin{bmatrix} \sigma_{x_{ls}}^2 & \sigma_{x_{ls},y_{ls}} & \sigma_{x_{ls},z_{ls}} \\ \sigma_{y_{ls},x_{ls}} & \sigma_{y_{ls}}^2 & \sigma_{y_{ls},z_{ls}} \\ \sigma_{z_{ls},x_{ls}} & \sigma_{z_{ls},y_{ls}} & \sigma_{z_{ls}}^2 \end{bmatrix} \tag{26.14}$$

where $\sigma_{x_{ls}}$, $\sigma_{y_{ls}}$, $\sigma_{z_{ls}}$ are the local TLS coordinate uncertainties due to measurement noise; and $\sigma_{x_{ls},y_{ls}} = \sigma_{y_{ls},x_{ls}}$, $\sigma_{x_{ls},z_{ls}} = \sigma_{z_{ls},x_{ls}}$, $\sigma_{y_{ls},z_{ls}} = \sigma_{z_{ls},y_{ls}}$ are the covariance elements.

Example 26.7 Suppose a laser scanner has the specifications shown in the following table. The scanner has measured a point with the following observations: $r = 152.400$ m, $\theta = 86°55'42''$, $\alpha = 12°30'31''$. The measurement took place at an

incidence angle of 25°. Estimate the coordinates of the laser scanner point in the scanner frame and its coordinate uncertainties.

Horizontal encoder uncertainty	$\pm 12''$
Vertical encoder uncertainty	$\pm 12''$
Inclination sensor uncertainty	$\pm 2.0''$
Beam divergence ($1/e^2$ definition)	0.33 mrad
Range uncertainty	1.5 mm + 10 ppm

SOLUTION First, the laser scanner coordinates are computed by using Equation (26.1):

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{ls} = \begin{bmatrix} 152.400 \cos 12°30'31'' \cos 86°55'42'' \\ 152.400 \cos 12°30'31'' \sin 86°55'42'' \\ 152.400 \sin 12°30'31'' \end{bmatrix} = \begin{bmatrix} 7.973 \\ 148.569 \\ 33.008 \end{bmatrix}$$

The range and angular uncertainties are calculated from Equations (26.5), (26.6), (26.9), (26.10), (26.11). The linear range effect due to the beam footprint is used. The beam divergence is already provided in the $1/e^2$ definition, so no conversion is necessary. The range uncertainty in meters is calculated as follows:

$$\begin{aligned} \sigma_r &= \sqrt{a^2 + (r \times b\, ppm)^2 + \sigma_{BWrange}^2} \\ &= \sqrt{0.0015^2 + (152.400 \times 10 \times 10^{-6})^2 + \left(152.400\, \frac{0.33 \times 10^{-3}}{4} \tan 25°\right)^2} \\ &= \pm 0.0062 \text{ m} \end{aligned}$$

The angular uncertainties in radians are calculated as

$$\begin{aligned} \sigma_\theta &= \sqrt{\sigma_{\theta_{en}}^2 + \sigma_{BWangle}^2 + \sigma_{inc}^2} = \sqrt{\left(\frac{12''}{\rho}\right)^2 + \left(\frac{0.33 \times 10^{-3}}{4}\right)^2 + \left(\frac{2.0''}{\rho}\right)^2} \\ &= \pm 10.14 \times 10^{-5} \text{rad or } \pm 20.9'' \\ \sigma_\alpha &= \sqrt{\sigma_{\alpha_{en}}^2 + \sigma_{BWangle}^2 + \sigma_{inc}^2} \\ &= \sqrt{\left(\frac{12''}{\rho}\right)^2 + \left(\frac{0.33 \times 10^{-3}}{4}\right)^2 + \left(\frac{2.0''}{\rho}\right)^2} = \pm 10.14 \times 10^{-5} \text{ rad} \\ &= \pm 20.9'' \end{aligned}$$

The covariance matrix of the laser scanner measurements from Equation (26.12) is

$$\Sigma_{TLS(noise)} = \begin{bmatrix} \sigma_r^2 & 0 & 0 \\ 0 & \sigma_\theta^2 & 0 \\ 0 & 0 & \sigma_\alpha^2 \end{bmatrix} = \begin{bmatrix} 0.0062^2 & 0 & 0 \\ 0 & (10.14 \times 10^{-5})^2 & 0 \\ 0 & 0 & (10.14 \times 10^{-5})^2 \end{bmatrix}$$

The Jacobian matrix of the partial derivatives of the laser scanner coordinates with respect to the laser scanner measurements is

$$\begin{aligned} J_{ls} &= \begin{bmatrix} \frac{\partial x_{ls}}{\partial r} & \frac{\partial x_{ls}}{\partial \theta} & \frac{\partial x_{ls}}{\partial \alpha} \\ \frac{\partial y_{ls}}{\partial r} & \frac{\partial y_{ls}}{\partial \theta} & \frac{\partial y_{ls}}{\partial \alpha} \\ \frac{\partial z_{ls}}{\partial r} & \frac{\partial z_{ls}}{\partial \theta} & \frac{\partial z_{ls}}{\partial \alpha} \end{bmatrix} \\ &= \begin{bmatrix} \cos\alpha\cos\theta & -r\cos\alpha\sin\theta & -r\sin\alpha\cos\theta \\ \cos\alpha\sin\theta & r\cos\alpha\cos\theta & -r\sin\alpha\sin\theta \\ \sin\alpha & 0 & r\cos\alpha \end{bmatrix} \\ &= \begin{bmatrix} 0.0523 & -148.5688 & -1.7687 \\ 0.9749 & 7.9725 & -32.9603 \\ 0.2166 & 0.0000 & 148.7826 \end{bmatrix} \end{aligned}$$

Using Equation (26.14), the covariance matrix of the local laser scanner coordinates can be found:

$$\begin{aligned} \Sigma_{xyz_{ls}} &= J_{ls} \begin{bmatrix} \sigma_r^2 & 0 & 0 \\ 0 & \sigma_\theta^2 & 0 \\ 0 & 0 & \sigma_\alpha^2 \end{bmatrix} J_{ls}^{\mathrm{T}} \\ &= J_{ls} \begin{bmatrix} 0.0062^2 & 0 & 0 \\ 0 & (10.14 \times 10^{-5})^2 & 0 \\ 0 & 0 & (10.14 \times 10^{-4})^2 \end{bmatrix} J_{ls}^{\mathrm{T}} \\ &= \begin{bmatrix} 22.72 \times 10^{-5} & -0.96 \times 10^{-5} & -0.23 \times 10^{-5} \\ -0.96 \times 10^{-5} & 4.88 \times 10^{-5} & -4.22 \times 10^{-5} \\ -0.23 \times 10^{-5} & -4.22 \times 10^{-5} & 22.95 \times 10^{-5} \end{bmatrix} \end{aligned}$$

Taking the square root of the diagonal elements in matrix $\Sigma_{xyz_{ls}}$, the point uncertainty is computed as $\sigma_{x_{ls}} = \sqrt{22.72 \times 10^{-5}} = 0.015$ m, $\sigma_{y_{ls}} = \sqrt{4.88 \times 10^{-5}} = 0.007$ m, $\sigma_{z_{ls}} = \sqrt{22.95 \times 10^{-5}} = 0.015$ m.

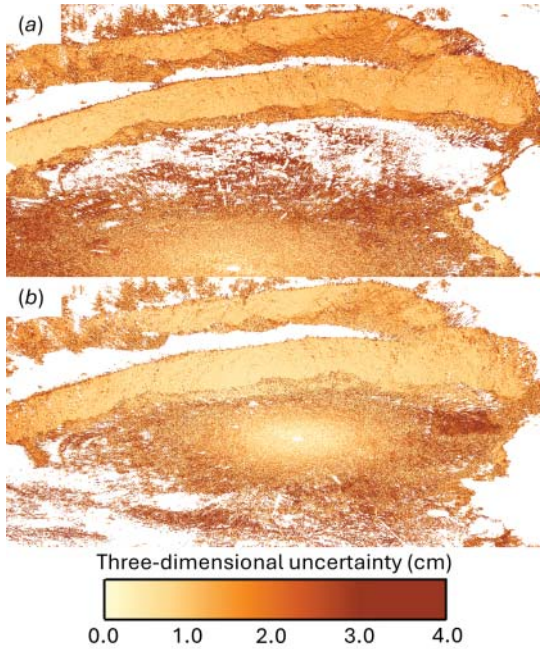


FIGURE 26.4 Three-dimensional uncertainty visualization (*a*) using a Leica ScanStation P50 (Leica Geosystems 2019) at the center of the outcrop and (*b*) using a Leica RTC360 (Leica Geosystems 2021), which is closer to the outcrop by approximately 40 m.

Figure 26.4 shows an example of three-dimensional point cloud uncertainty at a quarry with data collected with two laser scanners. The three-dimensional uncertainty was computed using a MATLAB script, which had the manufacturer specifications, and implementing the methods presented in this section. Note that the three-dimensional uncertainty is up to approximately 4 cm for this dataset because of the surface variability and scanning geometry. The uncertainty is at the level of 1 cm for points close to the scanners and increases for points on the ground away from the scanner, mostly affected by the higher incidence angle. The outcrop, being approximately perpendicular to the laser's direction, results in lower incidence angles and thus lower uncertainties than points on the ground at the same location. Furthermore, the Figure 26.4(*b*) scan was captured 40 m closer to the outcrop, resulting in lower uncertainty than the Figure 26.4(*a*) scan. This example visually demonstrates that point cloud uncertainty can vary substantially for spatially complex surfaces, depending on the scanner used, the scan distance, and incidence angle. The importance of having robust estimates of point cloud uncertainty should not be underestimated, as they can become crucial depending on the desired application.

26.6 REGISTRATION AND ITS UNCERTAINTY

Laser scanning is a line-of-sight method. To derive complete 3D datasets, multiple scans are needed from different viewpoints. Individual scans need to be registered into a common coordinate system (local or global). Terrestrial, static (non-kinematic) scans can be registered by using one of the following methods: (1) direct georeferencing (by setting up and back sighting on known points), (2) target-based (rigid body transformation, resection, traversing), (3) cloud-based methods (cloud-to-cloud and cloud-to-surface), and (4) mixed approaches of target-based and cloud-based methods. In all cases, the aim is to estimate a rigid-body transformation (three translations and rotations) through least-squares adjustment to align the local scanner origin with the target coordinate system. In the target-based approach, at least three common targets must be observed to register each setup. The rigid-body transformation is given as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_m = \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} + R(\theta_1, \theta_2, \theta_3) \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{ls} \tag{26.15}$$

where x_m, y_m, z_m are the registered scanner coordinates, m denotes that a registration operation has taken place and coordinates are in the desired mapping frame, T_x, T_y, T_z are the translations for each axis to transform the scanner origin to the scanner location in the new coordinate system, and $R(\theta_1, \theta_2, \theta_3)$ is a 3×3 rotation matrix to rotate and align the axes of the local Cartesian system of the laser scanner to the new coordinate system. A scale is often not included as that is often neglected for short range scanners or directly applied on the instrument for long range scanning to account for atmospheric effects (temperature, pressure, and humidity).

Modern scanners have inclination sensors which measure the misalignment of the scanner's z axis with the local level line, expressed as roll and pitch angles, corresponding to the scanner's x and y axes, respectively. In these cases, the six-parameter transformation reduces to a four-parameter transformation (three translations and a single rotation about the z axis). Thus, a target-based registration approach requires only two common targets to be observed in each setup, and the registration equation becomes

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_m = \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} + R(\theta_3) \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{ls} = \begin{bmatrix} T_x + \cos(\theta_3) x_{ls} - \sin(\theta_3) y_{ls} \\ T_y + \sin(\theta_3) x_{ls} + \cos(\theta_3) y_{ls} \\ T_z + z_{ls} \end{bmatrix} \tag{26.16}$$

However, generally it is recommended to have additional targets to provide redundancy.

The registration step, being an estimation of unknown position and orientation information, is not error-free; hence, its uncertainty and impact on the quality of the point cloud should be considered. In the case where the transformation parameters and their uncertainties are known, then the registration uncertainty can

be estimated by applying Equation (6.13), and the general law of propagation of variances (GLOPOV) on Equation (26.15) or Equation (26.16). When GLOPOV is applied to the six-parameter transformation of Equation (26.15), the registration uncertainty is computed from

$$\Sigma_{xyz_{reg}} = J_{reg}\Sigma_{6-param.}J_{reg}^{\mathrm{T}} = J_{reg}\begin{bmatrix} \sigma_{T_x}^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_{T_y}^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_{T_z}^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_{\theta_1}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_{\theta_2}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_{\theta_3}^2 \end{bmatrix} J_{reg}^{\mathrm{T}}$$

$$= \begin{bmatrix} \sigma_{x_{reg}}^2 & \sigma_{x_{reg},y_{reg}} & \sigma_{x_{reg},z_{reg}} \\ \sigma_{y_{reg},x_{reg}} & \sigma_{y_{reg}}^2 & \sigma_{y_{reg},z_{reg}} \\ \sigma_{z_{reg},x_{reg}} & \sigma_{z_{reg},y_{reg}} & \sigma_{z_{reg}}^2 \end{bmatrix} \tag{26.17}$$

where $\Sigma_{6-param}$ is the 6×6 covariance matrix of the transformation parameters; σ_{T_x}, σ_{T_y}, σ_{T_z} are the translation parameter uncertainties; and σ_{θ_1}, σ_{θ_2}, σ_{θ_3} are the rotational parameter uncertainties. $\Sigma_{xyz_{reg}}$ the 3×3 registration covariance matrix for each scan, $\sigma_{x_{reg}}$, $\sigma_{y_{reg}}$, $\sigma_{z_{reg}}$ are the propagated registration uncertainties for a point of the point cloud; and $\sigma_{x_{reg},y_{reg}} = \sigma_{y_{reg},x_{reg}}$, $\sigma_{x_{reg},z_{reg}} = \sigma_{z_{reg},x_{reg}}$, $\sigma_{y_{reg},z_{reg}} = \sigma_{z_{reg},y_{reg}}$ the corresponding covariance elements. J_{reg}, the 3×6 matrix of partial derivatives of Equation (26.15) with respect to the transformation parameters, which are shown in Equation (18.16), but should be implemented without considering the scale factor.

When GLOPOV is applied to the four-parameter transformation of Equation (26.16), the registration uncertainty is computed from

$$\Sigma_{xyz_{reg}} = J_{reg}\Sigma_{4-param.}J_{reg}^{\mathrm{T}} = J_{reg}\begin{bmatrix} \sigma_{T_x}^2 & 0 & 0 & 0 \\ 0 & \sigma_{T_y}^2 & 0 & 0 \\ 0 & 0 & \sigma_{T_z}^2 & 0 \\ 0 & 0 & 0 & \sigma_{\theta_3}^2 \end{bmatrix} J_{reg}^{\mathrm{T}}$$

$$= \begin{bmatrix} \sigma_{x_{reg}}^2 & \sigma_{x_{reg},y_{reg}} & \sigma_{x_{reg},z_{reg}} \\ \sigma_{y_{reg},x_{reg}} & \sigma_{y_{reg}}^2 & \sigma_{y_{reg},z_{reg}} \\ \sigma_{z_{reg},x_{reg}} & \sigma_{z_{reg},y_{reg}} & \sigma_{z_{reg}}^2 \end{bmatrix} \tag{26.18}$$

where $\Sigma_{4-param}$ is the 4×4 covariance matrix of the transformation parameters; and J_{reg} is the 3×4 matrix of partial derivatives of Equation (26.16) with respect to the transformation parameters given as

$$J_{reg.} = \begin{bmatrix} \frac{\partial x_m}{\partial T_x} & \frac{\partial x_m}{\partial T_y} & \frac{\partial x_m}{\partial T_z} & \frac{\partial x_m}{\partial \theta_3} \\ \frac{\partial y_m}{\partial T_x} & \frac{\partial y_m}{\partial T_y} & \frac{\partial y_m}{\partial T_z} & \frac{\partial y_m}{\partial \theta_3} \\ \frac{\partial z_m}{\partial T_x} & \frac{\partial z_m}{\partial T_y} & \frac{\partial z_m}{\partial T_z} & \frac{\partial z_m}{\partial \theta_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -\sin(\theta_3)x_{ls} - \cos(\theta_3)y_{ls} \\ 0 & 1 & 0 & \cos(\theta_3)x_{ls} - \sin(\theta_3)y_{ls} \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (26.19)$$

In most cases, there will be limited or no information about the uncertainty of the transformation parameters because it is not provided in most commercial post-processing software. Ordinarily, one will only have access to the variance elements ($\sigma^2_{x_{reg}}$, $\sigma^2_{y_{reg}}$, $\sigma^2_{z_{reg}}$); therefore, off-diagonal elements can be considered zero ($\sigma_{x_{reg},y_{reg}} = \sigma_{y_{reg},x_{reg}} = \sigma_{x_{reg},z_{reg}} = \sigma_{z_{reg},x_{reg}} = \sigma_{y_{reg},z_{reg}} = \sigma_{z_{reg},y_{reg}} = 0$), resulting in a diagonal matrix.

Example 26.8 Suppose a point has the following local scanner coordinates $x_{ls} =$ 7.973 m, $y_{ls} = 148.569$ m, $z_{ls} = 33.008$ m. The scanner origin coordinates, also serving as the translation parameters, are $T_x = 745{,}144.450$ m, $T_y =$ 128,173.570 m, $T_z = 395.324$ m. The instrument is leveled, so only the rotation of the scanner frame about the z axis is needed, which is $\theta_3 = 45°00'00''$. The uncertainty of the scanner origin is ± 0.005 m for each coordinate, and the rotation angle uncertainty is $\sigma_{\theta_3} = \pm 5''$. Estimate the propagated registration uncertainty for this point.

SOLUTION The covariance matrix of the four-parameter transformation is

$$\Sigma_{4-param.} = \begin{bmatrix} \sigma^2_{T_x} & 0 & 0 & 0 \\ 0 & \sigma^2_{T_y} & 0 & 0 \\ 0 & 0 & \sigma^2_{T_z} & 0 \\ 0 & 0 & 0 & \sigma^2_{\theta_3} \end{bmatrix} = \begin{bmatrix} 0.005^2 & 0 & 0 & 0 \\ 0 & 0.005^2 & 0 & 0 \\ 0 & 0 & 0.005^2 & 0 \\ 0 & 0 & 0 & \left(\frac{5''}{\rho}\right)^2 \end{bmatrix}$$

The Jacobian matrix with the partial derivatives with respect to the transformation parameters is computed using Equation (26.19)

$$J_{reg} = \begin{bmatrix} \frac{\partial x_m}{\partial T_x} & \frac{\partial x_m}{\partial T_y} & \frac{\partial x_m}{\partial T_z} & \frac{\partial x_m}{\partial \theta_3} \\ \frac{\partial y_m}{\partial T_x} & \frac{\partial y_m}{\partial T_y} & \frac{\partial y_m}{\partial T_z} & \frac{\partial y_m}{\partial \theta_3} \\ \frac{\partial z_m}{\partial T_x} & \frac{\partial z_m}{\partial T_y} & \frac{\partial z_m}{\partial T_z} & \frac{\partial z_m}{\partial \theta_3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -\sin(\theta_3)x_{ls} - \cos(\theta_3)y_{ls} \\ 0 & 1 & 0 & \cos(\theta_3)x_{ls} - \sin(\theta_3)y_{ls} \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & -110.6919 \\ 0 & 1 & 0 & -99.4164 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The covariance matrix of the registration uncertainty for point this point is

$$\Sigma_{xyz_{reg}} = J_{reg}\Sigma_{4-param.}J_{reg}^{\mathrm{T}} = \begin{bmatrix} 3.22\times10^{-5} & 0.65\times10^{-5} & 0 \\ 0.65\times10^{-5} & 3.08\times10^{-5} & 0 \\ 0 & 0 & 2.50\times10^{-5} \end{bmatrix}$$

Computing the square root of the diagonal elements, the propagated registration uncertainties are $\sigma_{x_{reg}} = \sqrt{3.22\times10^{-5}} = 0.006$ m, $\sigma_{y_{reg}} = \sqrt{3.08\times10^{-5}} = 0.006$ m, $\sigma_{z_{reg}} = \sqrt{2.50\times10^{-5}} = 0.005$ m.

26.7 POINT CLOUD UNCERTAINTY

With the point cloud measurement and registration uncertainties propagated the total point cloud uncertainty can be computed as

$$\Sigma_{xyz_m} = \Sigma_{xyz_{ls}} + \Sigma_{xyz_{reg}} = \begin{bmatrix} \sigma_{x_{ls}}^2 & \sigma_{x_{ls},y_{ls}} & \sigma_{x_{ls},z_{ls}} \\ \sigma_{y_{ls},x_{ls}} & \sigma_{y_{ls}}^2 & \sigma_{y_{ls},z_{ls}} \\ \sigma_{z_{ls},x_{ls}} & \sigma_{z_{ls},y_{ls}} & \sigma_{z_{ls}}^2 \end{bmatrix} + \begin{bmatrix} \sigma_{x_{reg}}^2 & \sigma_{x_{reg},y_{reg}} & \sigma_{x_{reg},z_{reg}} \\ \sigma_{y_{reg},x_{reg}} & \sigma_{y_{reg}}^2 & \sigma_{y_{reg},z_{reg}} \\ \sigma_{z_{reg},x_{reg}} & \sigma_{z_{reg},y_{reg}} & \sigma_{z_{reg}}^2 \end{bmatrix}$$
$$= \begin{bmatrix} \sigma_{x_m}^2 & \sigma_{x_m,y_m} & \sigma_{x_m,z_m} \\ \sigma_{y_m,x_m} & \sigma_{y_m}^2 & \sigma_{y_m,z_m} \\ \sigma_{z_m,x_m} & \sigma_{z_m,y_m} & \sigma_{z_m}^2 \end{bmatrix} \tag{26.20}$$

where σ_{x_m}, σ_{y_m}, σ_{z_m} are the total TLS coordinate uncertainties for a registered point of the point cloud; and $\sigma_{x_m,y_m} = \sigma_{y_m,x_m}$, $\sigma_{x_m,z_m} = \sigma_{z_m,x_m}$, $\sigma_{y_m,z_m} = \sigma_{z_m,y_m}$ the covariance elements. Matrix $\Sigma_{xyz_{ls}}$ is computed from Equation (26.14) and $\Sigma_{xyz_{reg}}$ computed from Equation (26.17) or Equation (26.18) depending on the registration approach.

Using the uncertainties for each coordinate, a three-dimensional uncertainty estimate can be derived for a single point

$$\sigma_{xyz_m} = \sqrt{\sigma_{xyz_{ls}}{}^2 + \sigma_{xyz_{reg}}{}^2} \tag{26.21}$$

where σ_{xyz_m} is the three-dimensional uncertainty in the mapping frame; $\sigma_{xyz_{ls}}$ is the three-dimensional uncertainty in the laser scanner frame; and $\sigma_{xyz_{reg}}$ is the three-dimensional registration uncertainty for a point of the point cloud. The three-dimensional uncertainties are computed as

$$\sigma_{xyz_m} = \sqrt{\sigma_{x_m}^2 + \sigma_{y_m}^2 + \sigma_{z_m}^2} \tag{26.22}$$

$$\sigma_{xyz_{ls}} = \sqrt{\sigma_{x_{ls}}^2 + \sigma_{y_{ls}}^2 + \sigma_{z_{ls}}^2} \tag{26.23}$$

$$\sigma_{xyz_{reg}} = \sqrt{\sigma_{x_{reg}}^2 + \sigma_{y_{reg}}^2 + \sigma_{z_{reg}}^2} \tag{26.24}$$

Example 26.9 Using the laser scanner data and covariance matrices of Examples 26.7 and 26.8, compute the point coordinates and its uncertainty in the mapping frame.

SOLUTION First, the laser scanner coordinates in the mapping frame are computed. Equation (26.16) is used because the instrument is leveled and only one rotation about the z axis is needed. From Example 26.7, the scanner frame coordinates are (7.973 m, 148.569 m, 33.008 m). From Example 26.8, the translation parameters are the origin coordinates; therefore: $T_x = 745{,}144.450$ m, $T_y = 128{,}173.570$ m, $T_z = 395.324$ m. The rotation angle is $\theta_3 = 45°00'00''$. The coordinates of the point in the mapping frame are computed as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_m = \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} + \mathrm{R}(\theta_3) \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{ls}$$

$$= \begin{bmatrix} 745{,}144.450 \\ 128{,}173.570 \\ 395.324 \end{bmatrix} + \begin{bmatrix} \cos(45°) & -\sin(45°) & 0 \\ \sin(45°) & \cos(45°) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r\cos\alpha\cos\theta \\ r\cos\alpha\sin\theta \\ r\sin\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 745{,}144.450 \\ 128{,}173.570 \\ 395.324 \end{bmatrix} + \begin{bmatrix} 0.7071 & -0.7071 & 0 \\ 0.7071 & 0.7071 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 7.973 \\ 148.569 \\ 33.008 \end{bmatrix}$$

$$= \begin{bmatrix} 745,045.033 \\ 128{,}284.261 \\ 428.332 \end{bmatrix}$$

Next, the covariance matrix of the registered laser scanner measurements can be found using Equation (26.20) as

$$\Sigma_{xyz_m} = \Sigma_{xyz_{ls}} + \Sigma_{reg}$$

$$= \begin{bmatrix} 22.72 \times 10^{-5} & -0.96 \times 10^{-5} & -0.23 \times 10^{-5} \\ -0.96 \times 10^{-5} & 4.88 \times 10^{-5} & -4.22 \times 10^{-5} \\ -0.23 \times 10^{-5} & -4.22 \times 10^{-5} & 22.95 \times 10^{-5} \end{bmatrix}$$

$$+\begin{bmatrix} 3.22\times10^{-5} & 0.65\times10^{-5} & 0 \\ 0.65\times10^{-5} & 3.08\times10^{-5} & 0 \\ 0 & 0 & 2.50\times10^{-5} \end{bmatrix}$$

$$=\begin{bmatrix} 25.94\times10^{-5} & -0.31\times10^{-5} & -0.23\times10^{-5} \\ -0.31\times10^{-5} & 7.96\times10^{-5} & -4.22\times10^{-5} \\ -0.23\times10^{-5} & -4.22\times10^{-5} & 25.45\times10^{-5} \end{bmatrix}$$

Taking the square root of the diagonal elements of the previous matrix, the uncertainties in each direction are $\sigma_{x_m} = \sqrt{25.94\times10^{-5}} = 0.016$, $\sigma_{y_m} = \sqrt{7.96\times10^{-5}} = 0.009$, $\sigma_{z_m} = \sqrt{25.45\times10^{-5}} = 0.016$. The summary of results for the coordinates and their uncertainties are

$$x_m = 745{,}045.033\ \text{m} \pm 0.016\ \text{m}$$
$$y_m = 128{,}284.261\ \text{m} \pm 0.009\ \text{m}$$
$$z_m = 428.332\ \text{m} \pm 0.016\ \text{m}.$$

Example 26.10 Using the covariance matrices of Examples 26.7–26.9, compute the three-dimensional uncertainty in the mapping frame.

SOLUTION Using Equations (26.23) and (26.24), the three-dimensional uncertainty in the scanner frame and the three-dimensional registration uncertainty are computed as

$$\sigma_{xyz_{ls}} = \sqrt{\sigma_{x_{ls}}^2 + \sigma_{y_{ls}}^2 + \sigma_{z_{ls}}^2} = \sqrt{22.72\times10^{-5} + 4.88\times10^{-5} + 22.95\times10^{-5}}$$
$$= 0.0225$$
$$\sigma_{xyz_{reg}} = \sqrt{\sigma_{x_{reg}}^2 + \sigma_{y_{reg}}^2 + \sigma_{z_{reg}}^2} = \sqrt{3.22\times10^{-5} + 3.08\times10^{-5} + 2.50\times10^{-5}}$$
$$= 0.0094$$

By using Equation (26.21), the three-dimensional uncertainty in the mapping frame is computed as

$$\sigma_{xyz_m} = \sqrt{\sigma_{xyz_{ls}}^2 + \sigma_{xyz_{reg}}^2} = \sqrt{0.0225^2 + 0.0094^2} = 0.0244$$

A similar number can be derived by using covariance matrix Σ_{xyz_m} and Equation (26.22)

$$\sigma_{xyz_m} = \sqrt{\sigma_{x_m}^2 + \sigma_{y_m}^2 + \sigma_{z_m}^2} = \sqrt{25.94\times10^{-5} + 7.96\times10^{-5} + 25.45\times10^{-5}}$$
$$= 0.0244$$

26.8 SURVEY PLANNING



Using error propagation procedures, professionals can effectively plan high-quality laser scanning surveys. For example, professionals can use accuracy requirements for projects and decide whether the selected laser scanner instrument has requisite specifications to meet project objectives. In addition, considering the project accuracy requirements, professionals can decide on appropriate location, distribution, and quantity of scanner setups as well as set measurement thresholds (e.g., maximum range and maximum incidence angles) to meet project objectives. More information about laser scanning survey planning can be found in Jia and Lichti (2018).

Based on the project requirements and intended use of the laser scanning point cloud, the accuracy requirements may be given in three dimensions or in two dimensions (horizontal) + one dimension (vertical). The three-dimensional uncertainty was presented in Equation (26.22). The horizontal (radial) and vertical uncertainties, in the mapping frame, can be computed as

$$\sigma_{xy_m} = \sqrt{\sigma_{x_m}^2 + \sigma_{y_m}^2} \tag{26.25}$$

$$\sigma_{z_m} = \sqrt{\sigma_{z_m}^2} \tag{26.26}$$

where σ_{xy_m} the radial uncertainty; and σ_{z_m} the uncertainty in the vertical direction.

Accuracy requirements are often given at a certain confidence level (usually the 95% confidence level). Therefore, surveyors must use the linear (1D), circular (2D), and spherical (3D) error multipliers from Table 19.9 to derive the appropriate values at a certain confidence level. For instance, assuming equal uncertainty in each direction ($\sigma = \sigma_{x_m} = \sigma_{y_m} = \sigma_{z_m}$), Equation (26.22) becomes

$$\sigma_{xyz_m} = \sqrt{3\sigma^2} = \sqrt{3}\sigma = \sqrt{3}\sigma_{x_m} = \sqrt{3}\sigma_{y_m} = \sqrt{3}\sigma_{z_m} \tag{26.27}$$

Using the 95% spherical multiplier from Table 19.9, the 95% confidence level in three dimensions is

$$\sigma_{xyz_m,@95\%} = 2.7955\,\sigma = 2.7955\,\sigma_{x_m} = 2.7955\sigma_{y_m} = 2.7955\,\sigma_{z_m} \tag{26.28}$$

Rearranging Equation (26.27) as $\sigma = \frac{\sigma_{xyz_m}}{\sqrt{3}}$ and substituting it in Equation (26.28) yields

$$\sigma_{xyz_m,@95\%} = 2.7955\frac{\sigma_{xyz_m}}{\sqrt{3}} = 1.6140\sigma_{xyz_m} = 1.6140\sqrt{\sigma_{x_m}^2 + \sigma_{y_m}^2 + \sigma_{z_m}^2} \tag{26.29}$$

For 2D and 1D, the same procedure can be followed to derive the following equation at the 95% confidence level as

$$\sigma_{xy_m,@95\%} = 2.4477\frac{\sigma_{xy_m}}{\sqrt{2}} = 1.7308\,\sigma_{xy_m} = 1.7308\sqrt{\sigma_{x_m}^2 + \sigma_{y_m}^2} \tag{26.30}$$

$$\sigma_{z_m@95\%} = 1.9600\sqrt{\sigma_{z_m}^2} = 1.9600\,\sigma_{z_m} \tag{26.31}$$

Example 26.11 An engineering company has requested a TLS survey to be completed with $\sigma_{xyz_m@95\%} = 2.54$ cm (1 in). The surveyor will collect several scans of the exterior of buildings that will be combined with indoor scans to create 3D models for building information modeling (BIM) applications. At the one standard deviation level, compute the required three-dimensional accuracy in the mapping frame and the accuracy for each individual coordinate (e.g., $\sigma_{x_m} = \sigma_{y_m} = \sigma_{z_m}$).

SOLUTION Using and rearranging Equation (26.29), the three-dimensional accuracy at the one standard deviation level (σ_{xyz_m}) is

$$\sigma_{xyz_m} = \frac{\sigma_{xyz_m@95\%}}{1.6140} = \frac{2.54}{1.6140} = \pm 1.57 \text{ cm}$$

To compute the uncertainty in each coordinate, the previous number representing the 3D accuracy is divided by $\sqrt{3}$ (assuming equal uncertainty in each coordinate). Using Equation (26.27), this is

$$\sigma = \sigma_{x_m} = \sigma_{y_m} = \sigma_{z_m} = \frac{\sigma_{xyz_m}}{\sqrt{3}} = \frac{1.57}{\sqrt{3}} = 0.91 \text{ cm}$$

In the uncertainty estimates, professionals should not neglect the registration uncertainty, which is part of the total 3D error budget (σ_{xyz_m}) in addition to the measurement uncertainties based on the equipment specifications.

Example 26.12 The same surveyor has a scanner with the instrument specifications of the following table. Using the project information of the previous example (Example 26.11), compute the maximum scanning distance from the object to satisfy the project's accuracy requirements. Assume that the incidence angle will not exceed 60°, and the vertical angle will not exceed 60°. Also, assume that the registration uncertainty is ±0.005 m for each coordinate. Lastly, use a horizontal angle of 90°. Note that the horizontal angle does not affect the final 3D error, it only changes the distribution of uncertainty among the three coordinates; therefore, any horizontal angle can be used in the calculations.

Horizontal encoder uncertainty	$\pm 12''$
Vertical encoder uncertainty	$\pm 12''$
Inclination sensor uncertainty	$\pm 2.0''$
Beam divergence ($1/e^2$ definition)	0.33 mrad
Range uncertainty	1.5 mm + 10 ppm

SOLUTION In Example 26.11, it was concluded that the total 3D error budget (σ_{xyz_m}) should not exceed ± 1.57 cm. This value includes the registration uncertainty. To find the allowed measurement uncertainty, the registration uncertainty needs to be removed from σ_{xyz_m} using SLOPOV. Because the registration uncertainty is assumed to be the same in all three coordinates, the total 3D registration uncertainty is

$$\sigma_{xyz_{reg.}} = \sqrt{3(0.005)^2} = 0.009 \text{ m or } 0.9 \text{ cm}.$$

Using Equation (26.21), the maximum allowable three-dimensional measurement uncertainty is then

$$\sigma_{xyz_{ls}} = \sqrt{\sigma^2_{xyz_m} - \sigma^2_{xyz_{reg.}}} = \sqrt{1.57^2 - 0.9^2} = 1.31 \text{ cm}$$

This means that, at one standard deviation, the laser scanner coordinate uncertainty needs to be

$$\sigma_{xyz_{ls}} = \sqrt{\sigma^2_{x_{ls}} + \sigma^2_{y_{ls}} + \sigma^2_{z_{ls}}} \leq 1.31 \text{ cm}$$

Because the range is needed to estimate both the J_{ls} matrix and the range uncertainty, a trial-and-error approach can be followed, where the distance is changed iteratively until the 3D uncertainty is ≤ 1.31 cm. Using a 50.000 m distance is a reasonable starting value for typical close-range scanning. In this example, using $\theta = 90°, \alpha = 60°, I_a = 60°$, a maximum distance of 71.000 m is found. The following steps demonstrate the computations to derive the maximum distance.

The J_{ls} matrix of partial derivatives are

$$J_{ls} = \begin{bmatrix} \cos\alpha\cos\theta & -r\cos\alpha\sin\theta & -r\sin\alpha\cos\theta \\ \cos\alpha\sin\theta & r\cos\alpha\cos\theta & -r\sin\alpha\sin\theta \\ \sin\alpha & 0 & r\cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos 60°\cos 90° & -71.000\cos 60°\sin 90° & -71.000\sin 60°\cos 90° \\ \cos 60°\sin 90° & 71.000\cos 60°\cos 90° & -71.000\sin 60°\sin 90° \\ \sin 60° & 0 & 71.000\cos 60° \end{bmatrix}$$

$$= \begin{bmatrix} 0.0000 & -35.5000 & 0.0000 \\ 0.5000 & 0.0000 & -61.4878 \\ 0.8660 & 0.0000 & 35.5000 \end{bmatrix}$$

The covariance matrix of the TLS measurements is computed using Equations (26.5), (26.6), (26.9), (26.10), and (26.11), and using a maximum range of 71.00 m. The angular uncertainty due to beam divergence is estimated using Equation (26.5)

$$\sigma_{BWangle} = \frac{\gamma\rho}{4} = \frac{0.33 \times 10^{-3}\rho}{4} = \frac{68.07''}{4} = \pm 17.0''$$

To find the horizontal and vertical uncertainty, the angular error sources of the scanner encoder, beam divergence, and inclination are combined using Equations (26.9) and (26.10)

$$\sigma_\theta = \sqrt{\sigma_{\theta_{en}}^2 + \sigma_{BWangle}^2 + \sigma_{inc}^2} = \sqrt{12^2 + 17.0^2 + 2.0^2}$$
$$= \pm 20.9'' \text{ or } 10.14 \times 10^{-5} \text{ rad}$$
$$\sigma_\alpha = \sqrt{\sigma_{\alpha_{en}}^2 + \sigma_{BWangle}^2 + \sigma_{inc}^2} = \sqrt{12^2 + 17.0^2 + 2.0^2}$$
$$= \pm 20.9'' \text{ or } 10.14 \times 10^{-5} \text{ rad}$$

The range uncertainty due to the beam footprint is computed using Equation (26.6)

$$\sigma_{BWrange} = r\frac{\gamma}{4\rho}\tan I_a = 71.000\frac{17.0''}{\rho}\tan 60° = \pm 0.0101$$

For the range uncertainty, the constant and scale manufacturer values along with the beam footprint effect are combined using Equation (26.11)

$$\sigma_r = \sqrt{a^2 + (r \times b\,ppm)^2 + \sigma_{BWrange}^2} = \sqrt{0.0015^2 + \left(\frac{71.000 \times 10}{10^6}\right)^2 + 0.0101^2}$$
$$= \pm 0.0103 \text{ m}$$

The covariance matrix of the TLS measurements is created as

$$\Sigma_{TLS(noise)} = \begin{bmatrix} \sigma_r^2 & 0 & 0 \\ 0 & \sigma_\theta^2 & 0 \\ 0 & 0 & \sigma_\alpha^2 \end{bmatrix} = \begin{bmatrix} 0.0103^2 & 0 & 0 \\ 0 & (10.14 \times 10^{-5})^2 & 0 \\ 0 & 0 & (10.14 \times 10^{-5})^2 \end{bmatrix}$$

Using Equation (6.13), the covariance of the coordinates is

$$\Sigma_{xyz_{ls}} = J_{ls}\Sigma_{TLS(noise)}J_{ls}^{\mathrm{T}} = \begin{bmatrix} 1.30 \times 10^{-5} & 0 & 0 \\ 0 & 6.53 \times 10^{-5} & 2.33 \times 10^{-5} \\ 0 & 2.33 \times 10^{-5} & 9.22 \times 10^{-5} \end{bmatrix}$$

Computing the square root of the variances in the previous matrix, the uncertainty in each coordinate is derived as

$$\sigma_{x_{ls}} = \sqrt{1.30 \times 10^{-5}} = \pm 0.0036$$

$$\sigma_{y_{ls}} = \sqrt{6.53 \times 10^{-5}} = \pm 0.0081$$

$$\sigma_{z_{ls}} = \sqrt{9.22 \times 10^{-5}} = \pm 0.0096$$

Note that, unlike in the previous example, these individual coordinate uncertainties values are not maximum thresholds. Rather, they represent the estimated uncertainty angles at a horizontal angle of 90° and will differ at other horizontal angles. The individual coordinate uncertainties are combined to find the three-dimensional uncertainty at one standard deviation as

$$\sigma_{xyz_{ls}} = \sqrt{\sigma_{x_{ls}}^2 + \sigma_{y_{ls}}^2 + \sigma_{z_{ls}}^2} = \sqrt{0.0036^2 + 0.0081^2 + 0.0096^2} = 0.0131$$

Therefore, the result of 0.0131 m or 1.31 cm is equal to the 1.31 cm and will satisfy the project requirements if the maximum scanning distance from the object is less than 71 m.

Figure 26.5 shows a brute-force method using MATHCAD to determine the maximum range to achieve project specifications as discussed in Example 26.12. The line labeled "Call" shows the call to the function, which is CompRange (start, inc, a, b, γ, σ_θ, σ_α, σ_I, θ, α, I, target, maxiter) where it is starting at 10 m, incrementing by 1 m, and the remaining terms shown from Example 26.12 except for maxiter. Maxiter is a fail-safe variable to prevent the program continuing forever. For instance, it is shown as 200 iterations. Thus, if it exceeds 200 iterations, the function will return with the last value determined. The column labeled Equation lists the equation number for each computation where separate functions have been created and called from the routine. Note that the while loop will continue incrementing the range, r by *inc* for each loop until the computed value for σ exceeds the target value, which is the previously determined project specification, σ_{xyz_ls}. Once this range is found where the computed σ is greater than the target value, the loop stops. Notice, in line 14, the code increments r again even though the target value is exceeded. Thus, once out of the while loop, both r and σ must be computed. Since the stopping r was the range where the project specification was exceeded, and it was also incremented again, two increments must be subtracted from r to determine the last range where σ is less or equal to the project specification. Using this value, it recomputes σ as was previously done in the loop. Finally, in line 21, it returns the vector $[r, \sigma]^T$ where r is the maximum range and σ is the computed uncertainty for this range. Since, as previously discussed, it can also stop if the maximum iterations are exceeded, the user should check σ to see if the function

Line	Code	Equation
1	iteration $\leftarrow 0$	
2	$S_{BW} \leftarrow S_{BW.angle}(\gamma)$	(26.5)
3	$\sigma_\theta \leftarrow \sqrt{\left(\frac{S_\theta}{\rho}\right)^2 + (S_{BW})^2 + \left(\frac{S_i}{\rho}\right)^2}$	(26.9)
4	$\sigma_\alpha \leftarrow \sqrt{\left(\frac{S_\alpha}{\rho}\right)^2 + (S_{BW})^2 + \left(\frac{S_i}{\rho}\right)^2}$	(26.10)
5	$\sigma \leftarrow 0$	
6	$R \leftarrow$ start	
7	While $\sigma \leq$ target ^ iteration < maxiter	
8	iteration $\leftarrow$ iteration + 1	
9	$J_{LS} \leftarrow JJ(\theta, \alpha, r)$	(26.13)
10	$\sigma_r \leftarrow S_r(a,b,r,\gamma,I)$	(26.11)
11	$\Sigma_{TLS} \leftarrow S_{TLS}(\sigma_r, \sigma_\theta, \sigma_\alpha)$	(26.12)
12	$\Sigma_{LS} \leftarrow J \cdot \Sigma_{TLS} \cdot J_{LS}^T$	(26.14)
13	$\sigma \leftarrow \sqrt{\Sigma_{LS_{0,0}} + \Sigma_{LS_{1,1}} + \Sigma_{LS_{2,2}}}$	
14	$r \leftarrow r +$ inc	
15	$r \leftarrow r - 2\cdot$inc	
16	$J_{LS} \leftarrow JJ(\theta, \alpha, r)$	(26.13)
17	$\sigma_r \leftarrow S_r(a,b,r,\gamma,I)$	(26.11)
18	$\Sigma_{TLS} \leftarrow S_{TLS}(\sigma_r, \sigma_\theta, \sigma_\alpha)$	(26.12)
19	$\Sigma_{LS} \leftarrow J \cdot \Sigma_{TLS} \cdot J_{LS}^T$	(26.14)
20	$\sigma \leftarrow \sqrt{\Sigma_{LS_{0,0}} + \Sigma_{LS_{1,1}} + \Sigma_{LS_{2,2}}}$	
21	return $(r\ \sigma)^T$	

FIGURE 26.5 Mathcad code to compute the maximum range from scanner to achieve project specifications.

returned the proper range. This is especially true if the range is at its maximum value of "$r = start + maxiter \times inc$," which indicates the loop stopped because the maximum iterations were exceeded. It should be pointed out here that the maxiter value should be sufficiently large to allow the function to loop through all the values. For instance, if the starting value is zero and the increment is 0.5 m, then it will require 200 iterations to loop through the first 100 m. Additionally, notice that the code in lines 9 through 14 are repeated in lines 16 through 20. Thus, these operations could be rewritten to a function, which would eliminate the repetition of the lines and replace them with a call to the function.

While this function is rather brute force, it can be used very efficiently by the user with some judicious calls. For instance, the first call could use increments (inc) of 100 m to determine in which 100-m range the project specification exists. It could then be called with the proper 100-m range as the starting value with an increment of 50, which would quickly determine if the desired range was in the lower half or upper half of the previously determined 100-m range. Once determined, the start

value could be set appropriately, called again using an increment of 1 m. However, given that today's personal computers are extremely fast, this procedure is most likely unnecessary. It is only important that the maximum iterations variable is set large enough to solve the problem.

Another important factor controlling the location of scanner setups is the scanning rate/time, and the desired point density of the point cloud. The point density describes the spacing (at a certain distance) between two points collected from the same setup. Scan manufacturers typically provide an estimate for a specific range for an orthogonal surface. However, the point density will vary substantially depending on the incidence angle and surface reflectance in addition to the range. Note that point density does not always correspond to the scanning resolution and the two terms should not be confused. For example, scans can be acquired at a spacing finer than the laser footprint, which will increase point density but will generally not improve the ability to resolve finer features. The scanning rate and the point density are two important parameters in TLS survey planning, as they can control the number of setups and the associated scan time; therefore, the total survey hours required to collect a dataset. Table 26.1 shows simulated examples of scanning time for certain scanner point spacing settings. The surveyor will have to balance field time and point spacing requirements for a project. Oftentimes, a better product will result from several moderate resolution scans compared to just a few very high-resolution scans as additional scans from multiple vantage points help reduce large data occlusions in the scene. More information can be found in O'Banion (2018a) who explored the relationship between resolution, number of setups, and overall data gaps.

Professionals can create similar tables for their scanners by selecting the desired point spacings at 100 on their instrument and record the estimated scanning time given by the laser scanner instrument. Using the information in Table 26.1 and the

TABLE 26.1 Approximate Scan Duration for Different Point Spacings

Point Spacing at 100 m[a] (cm)	Scanner 1: Maximum Range of 150 m	Scanner 2: Maximum Range of 300 m
50	00:00:14	00:00:14
25	00:00:31	00:00:35
20	00:00:40	00:00:47
10	00:01:31	00:01:58
5.0	00:03:26	00:05:01
2.5	00:07:50	00:12:46
2.0	00:10:12	00:17:16
1.0	00:23:15	00:43:59

Two simulated scanners with maximum ranges of 150 and 300 m are used. Scan time is given in HH:MM:SS format.

[a]Assuming an orthogonal surface with adequate reflectance to minimize data dropouts.

least-squares fit procedures discussed in Chapter 11, two equations are estimated for the 150 m and 300 m scanners

$$150\text{ m: Time} = 1394.7 \times \text{PS}^{-1.189} \tag{26.32}$$

$$300\text{ m: Time} = 2638.6 \times \text{PS}^{-1.350} \tag{26.33}$$

where time is the scanning time in seconds and PS is the point spacing in cm. The equations provide approximate estimations and should only be used for planning. Additionally, there are other settings such as image acquisition and different scanning modes that will also affect the scan time.

As a rule of thumb, it is suggested to match the uncertainty with the point spacing in the point cloud. For example, if the project uncertainty is set to 1 cm at 100 m, then the target resolution should also be set at 1 cm in 100 m.

Example 26.13 In Example 26.12, the maximum scanning distance was found to be 71 m to maintain the project uncertainty requirements of $\sigma_{xyz_{ls}} = 1.31$ cm. Estimate the scanning time using the 150 m distance equation.

SOLUTION Matching the resolution to the uncertainty, i.e., 1.31 cm at 71 m, the scanning resolution setting and time can be found. First determine the resolution corresponding to 100 m: $1.31\left(\frac{100}{71}\right) = 1.85$ cm. Using 1.85 cm as the point spacing in Equation (26.32) results in 672 seconds or 11 minutes and 12 seconds of scanning time per setup. Professionals should also add estimated times for scanner and target setup, as well as the time that is needed to capture photographic images.

To ensure that the scanning resolution of 1.31 cm at 71 m is maintained in the previous example, the scanner setups should be made with a maximum spacing of 71 m to ensure sufficient overlap between scans. Oftentimes visibility constraints, surface incidence angles, and the availability of objects to serve as registration features between scans will warrant a much closer spacing. Additionally, any range filtering should typically be completed after registration. Figure 26.6 shows an example of TLS survey planning with considerations of the maximum scanner spacing and maximum scanning distance. The overlapping area between adjacent scans in the configuration of Figure 26.6 is approximately 39.1%. This percentage is calculated based on the area of two intersecting circles, which can be computed from (Weisstein, 2025)

$$A = 2R^2\cos^{-1}\left(\frac{d}{2R}\right) - \frac{1}{2}d\sqrt{4R^2 - d^2} \tag{26.34}$$

where A is the area of two intersecting circles; R is the area of the first circle; and d is the distance between the centers of the two circles. The maximum distance

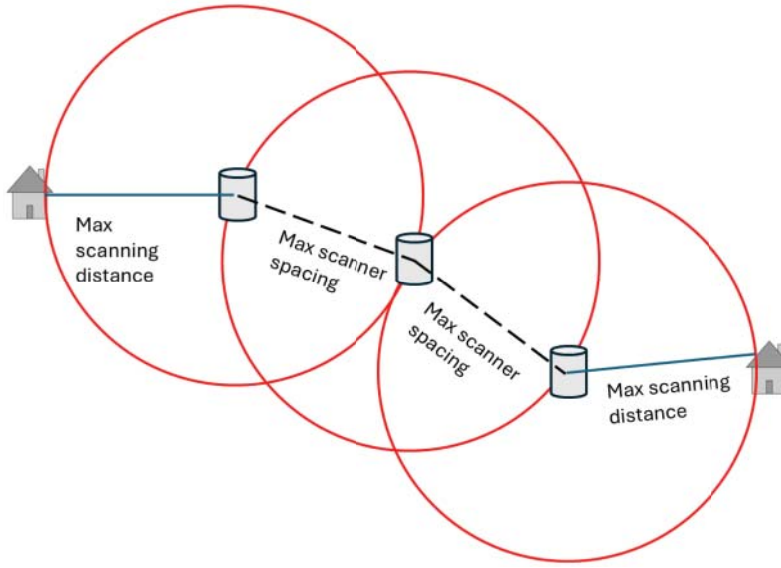


FIGURE 26.6 Planning of scanner spacing showing maximum scanner spacing and maximum scanning distance requirements.

between two scanner setups should be equal with the maximum scanning distance to maintain the desired point spacing: therefore, $d = R$. Equation (26.34) becomes

$$A = 2R^2\cos^{-1}\left(\frac{R}{2R}\right) - \frac{1}{2}R\sqrt{4R^2 - R^2} = \frac{2\pi R^2}{3} - \frac{R^2\sqrt{3}}{2} \tag{26.35}$$

Dividing the circle intersection area A by πR^2 (area of circle) provides the ratio of the area of two intersecting circles with respect to the total area of the circle. This is computed as

$$\frac{A}{\pi R^2} = \frac{2\pi R^2}{3\pi R^2} - \frac{R^2\sqrt{3}}{2\pi R^2} = \frac{2}{3} - \frac{\sqrt{3}}{2\pi} = 0.391 \tag{26.36}$$

This ratio expressed as percentage is 39.1%.

26.9 CONCLUSIONS

This chapter presented the primary measurement error sources of terrestrial laser scanning and how to perform error propagation to estimate the uncertainty within a point cloud. While performing a robust error propagation to real point cloud data can be computationally intensive given the large datasets, the processes presented

here can be readily implemented in the survey design step to select appropriate instrumentation, methods, and data collection procedures. They can also be utilized to visualize and communicate errors present in point clouds, which are highly variable. Through error propagation professionals can ensure that data are collected in such a way to satisfy project requirements and enhance quality assurance.

26.10 SOFTWARE

Solving the examples and numerical problems of this chapter by hand calculations can be extremely difficult. Instead, spreadsheets, MATHCAD, or MATLAB should be used to perform the computations. An EXCEL spreadsheet (Chapter 26.xlsx) is provided for the examples of this chapter. The EXCEL spreadsheet is demonstrating computations for a single laser scanner point. For more than one point (e.g., point cloud) using a programming language is necessary to develop an efficient computational package. The package should read an input TLS point file, create the design and covariance matrices, and perform the error propagation steps as demonstrated in this chapter. Special attention should be placed on the right size of the design and covariance matrices to avoid programming mistakes. For example, Jacobian matrix J_{ls} will have size $3n \times 3n$, and it will be a block diagonal matrix with blocks of 3×3. The covariance matrix $\Sigma_{TLS(noise)}$ will also have size $3n \times 3n$, where n is the number of points. Instead, the Jacobian matrix J_m will have size $3n \times 4$ for the four-parameter transformation and $3n \times 6$ for the six-parameter transformation. The covariance $\Sigma_{4-param.}$ will have size 4×4 for the four-parameter transformation, and 6×6 for the six-parameter transformation. The covariance matrix of the registered TLS point cloud (Σ_{xyz_m}) will also have size $3n \times 3n$.

PROBLEMS

Note: Solutions to problems marked with an asterisk are given in Appendix G.

26.1 Compute the scanner coordinates for the following measurements:

***(a)** $r = 40.136$ m, $\theta = 10°10'58''$, $\alpha = 05°45'24''$

(b) $r = 69.845$ m, $\theta = 90°22'32''$, $\alpha = 42°36'47''$

(c) $r = 154.697$ m, $\theta = 145°55'14''$, $\alpha = 33°24'54''$

(d) $r = 176.481$ m, $\theta = 256°46'19''$, $\alpha = 25°29'36''$

26.2 Compute the scanner measurements for the local scanner coordinates

***(a)** $x_{ls} = 133.658$ m, $y_{ls} = -209.745$, $z_{ls} = 26.367$

(b) $x_{ls} = 57.147$ m, $y_{ls} = -182.378$, $z_{ls} = 12.231$

(c) $x_{ls} = 163.475$ m, $y_{ls} = 167.215$, $z_{ls} = 21.324$

(d) $x_{ls} = 169.696$ m, $y_{ls} = 22.465$, $z_{ls} = -17.236$

26.3 Explain the primary angular uncertainty components in terrestrial laser scanning, including the beam divergence effect.

26.4 Explain the primary range uncertainty components in terrestrial laser scanning, including the beam divergence effect.

26.5 For the following scanner specifications and observations estimate the angular and range uncertainty. Use an angular beam divergence effect of $\pm 20''$. Use the following ranges and beam divergence effects

***(a)** Range 23.791 m and a range beam divergence effect of ± 2.0 mm.

(b) Range 458.140 m and a range beam divergence effect of ± 30.0 mm.

(c) Range 1021.342 m and a range beam divergence effect of ± 73.0 mm.

Horizontal encoder uncertainty	$\pm 10''$
Vertical encoder uncertainty	$\pm 10''$
Inclination sensor uncertainty	$\pm 2.0''$
Range uncertainty	2 mm + 20 ppm

26.6 Repeat Problem 26.5, but use an angular beam divergence effect of $\pm 10''$, and

(a) Range 23.791 m and a range beam divergence effect of ± 1.0 mm

(b) Range 458.140 m and a range beam divergence effect of ± 13.0 mm

(c) Range 1021.342 m and a range beam divergence effect of ± 30.0 mm

26.7 For a laser scanner with a wavelength of 1550 nm and a beam diameter at the front window of 5 mm (FWHM definition), calculate the beam divergence in mrad.

26.8 Suppose a scanner has a beam divergence of 0.4 mrad ($1/e^2$ definition). For an incidence angle of 45° and ranges measurements of 10.000 m, 50.000 m, 150.000 m, and 250.000 m

***(a)** Estimate the angular effect of the beam divergence

(b) Estimate the range effect of the beam footprint

(c) Create a plot to show how the range effect of the beam footprint changes with distance.

26.9 For a laser scanner with a wavelength of 1550 nm and a beam diameter at the front window of 5 mm (FWHM definition), and for incidence angles of 15°, 30°, 45°, 60°, and 75°.

(a) Create a graph showing the range uncertainty for ranges of 1 m up to 500 m. Consider the nonlinear behavior of the range uncertainty.

(b) Discuss how the range effect of the beam divergence changes with distance.

26.10 Suppose a scanner has the specifications listed in the following table. The scanner has measured a point with the following observations: $r = 258.322$ m, $\theta = 315°33'14''$, $\alpha = 19°14'00''$. The measurement took place at an incidence angle of 10°.

(a) Estimate the point coordinates in the laser scanner frame.

(b) Estimate the covariance matrix of the laser scanner measurements.

(c) Using error propagation, estimate the covariance matrix of the laser scanner coordinates.

(d) Compute the uncertainty of the laser scanner coordinates.

Horizontal encoder uncertainty	$\pm 10''$
Vertical encoder uncertainty	$\pm 10''$
Inclination sensor uncertainty	$\pm 2''$
Beam divergence ($1/e^2$ definition)	0.4 mrad
Range uncertainty	1.5 mm + 20 ppm

26.11 Suppose a scanner has the specifications listed in the following table. The scanner has measured a point with the following observations: $r = 136.361$ m, $\theta = 93°44'36''$, $\alpha = 5°49'26''$. The measurement took place at an incidence angle of 5°.

(a) Estimate the point coordinates in the laser scanner frame.

(b) Estimate the covariance matrix of the laser scanner measurements.

(c) Using error propagation, estimate the covariance matrix of the laser scanner coordinates.

(d) Compute the uncertainty of the laser scanner coordinates.

Horizontal encoder uncertainty	$\pm 12''$
Vertical encoder uncertainty	$\pm 12''$
Inclination sensor uncertainty	$\pm 3''$
Beam divergence ($1/e^2$ definition)	0.5 mrad
Range uncertainty	2 mm + 15 ppm

26.12 Repeat Problem 26.11 with $\theta = 0°00'00''$, $\theta = 45°00'00''$, $\theta = 90°00'00''$, $\theta = 135°00'00''$, $\theta = 180°00'00''$, $\theta = 225°00'00''$, $\theta = 270°00'00''$, $\theta = 315°00'00''$. Discuss how the distribution of uncertainty changes among the three coordinates when the horizontal angle changes.

26.13 Suppose a laser scanner point has the following coordinates $x_{ls} = 105.198$ m, $y_{ls} = -84.330$ m, $z_{ls} = 23.145$ m, the four-parameter transformation parameters $T_x = 745{,}159.833$, $T_y = 128{,}231.869$, $T_z = 393.705$, $\theta_3 = 30°$ and their uncertainties $\sigma_{T_x} = 0.010$, $\sigma_{T_y} = 0.010$, $\sigma_{T_z} = 0.010$, and $\sigma_{\theta_3} = 40''$. Estimate the registration uncertainty for this point.

26.14 Suppose a laser scanner point has the following coordinates $x_{ls} = -36.789$, $y_{ls} = 25.659$, $z_{ls} = 3.657$, the four-parameter transformation parameters $T_x = 745{,}593.771$, $T_y = 128{,}004.787$, $T_z = 392.427$, $\theta_3 = 224°$ and their uncertainties $\sigma_{T_x} = 0.003$, $\sigma_{T_y} = 0.003$, $\sigma_{T_z} = 0.003$, and $\sigma_{\theta_3} = 55''$. Estimate the registration uncertainty for this point.

26.15 Suppose a scanner has the specifications listed in Problem 26.10. The scanner has measured a point with the following observations: $r = 258.322$ m, $\theta = 315°33'14''$, $\alpha = 19°14'00''$. The measurement took place at an incidence angle of 10°. The 4-parameter transformation parameters $T_x = 745{,}144.450$ m, $T_y = 128{,}173.570$, $T_z = 395.324$ m, $\theta_3 = 184°$ and their uncertainties $\sigma_{T_x} = 0.005$ m, $\sigma_{T_y} = 0.005$ m, $\sigma_{T_z} = 0.005$ m, and $\sigma_{\theta_3} = 35''$. Estimate the point coordinates and their total uncertainty in the mapping frame.

(a) Estimate the point coordinates in the mapping frame.

(b) Estimate the covariance matrix of the laser scanner coordinates.

(c) Estimate the covariance matrix of the registration uncertainty.

(d) Estimate the covariance matrix of the registered laser scanner coordinates, and compute the uncertainty of the point coordinates in the mapping frame.

26.16 Use the same instrument and transformation specifications given in Problem 26.15 with the observations $r = 54.190$ m, $\theta = 25°04'54''$, $\alpha = 5°25'13''$, and an incidence angle of 45°. Assume that the instrument wavelength is 1055 nm and a beam diameter at the front window of 5 mm (FWHM definition) to estimate the beam divergence, and consider the nonlinear behavior of the range uncertainty.

26.17 A TLS survey is requested for slope stability and monitoring with accuracy specifications of $\sigma_{xyz_m,95\%}$ of 1 cm. What is the three-dimensional accuracy (one-sigma level) and expected accuracy in each coordinate?

26.18 A TLS survey for 3D modeling is requested with accuracy specifications of $\sigma_{xyz_m,@95\%}$ of 3 cm. What is the three-dimensional accuracy (one-sigma level) and expected accuracy in each coordinate?

26.19 Suppose a scanner has the specifications listed in the following table. For a project accuracy requirement of $\sigma_{xyz_m,@95\%}$ of 1 cm, compute the maximum allowable scanning distance (rounded to the nearest m) from the object to satisfy the project's accuracy requirements. Assume that the incidence angle will not exceed 45° and that vertical angle will not exceed 45°. The registration uncertainty is ±0.002 m for each coordinate.

Horizontal encoder uncertainty	$\pm 6''$
Vertical encoder uncertainty	$\pm 6''$
Inclination sensor uncertainty	$\pm 1''$
Beam divergence ($1/e^2$ definition)	0.25 mrad
Range uncertainty	1 mm + 10 ppm

26.20 Using the same instrument specifications given in Problem 26.19, compute the maximum scanning distance (rounded to the nearest m) from the object to satisfy the project's accuracy requirements of $\sigma_{xyz_m@95\%}$ equals ± 3 cm. Assume that the incidence angle will not exceed 60° and that vertical angle will not exceed 60°. The registration uncertainty is ± 0.005 m for each coordinate.

26.21 Suppose a scanner has the specifications listed in the following table. For a project accuracy requirement of $\sigma_{xyz_m@95\%}$ of ± 6 cm, compute the maximum scanning distance (rounded to the nearest m) from the object to satisfy the project's accuracy requirements. Assume that the incidence angle will not exceed 60° and that vertical angle will not exceed 60°. The registration uncertainty is ± 0.015 m for each coordinate.

Horizontal encoder uncertainty	$\pm 10''$
Vertical encoder uncertainty	$\pm 10''$
Inclination sensor uncertainty	$\pm 2''$
Beam divergence ($1/e^2$ definition)	0.35 mrad
Range uncertainty	2 mm + 10 ppm

26.22 Using the following equation, $\text{Time} = 1394.7 \times \text{PS}^{-1.189}$, that estimates the scanning time and a desired point spacing of 2.5 cm at 100 m. Estimate the total time that it would take to scan a site for the following cases:

(a) 10 scanner setups, 15 minutes for setting up the instrument and scanning targets to register the scan in the field for each setup, and 7.5 minutes for capturing images in each setup.

(b) For the same number of setups, assume that the surveyor does not capture any images and only the point cloud is collected.

(c) For the same number of setups assume that the surveyor spends only 5 minutes for scanner setup and no additional time is needed for scanning targets because a cloud-to-cloud registration approach will be used at the office. Images are collected in each setup.

(d) Discuss the differences in the results of (a), (b), and (c). In your response, consider how the scan planning process and selected parameters affect field time, office time, and project requirements/data completeness.

26.23 Repeat Problem 26.22(a) but use five scanner setups. Discuss the impact of decreasing the point spacing from 2.5 cm at 100 m to 1.25 cm at 100 m in relation to the fewer setups and data completeness (e.g., occlusions and gaps).

26.24 Repeat Problem 26.22(a) but use the following equation Time $= 2638.6 \times PS^{-1.350}$, because long range scans (up to 300 m) are needed. Discuss the impact of the lower scanning rate equation used in Problem 26.22(a), compared to the one provided in this problem.

CHAPTER 27

PLANE, THREE-DIMENSIONAL OBJECTS, AND ELEVATION MODEL ESTIMATION WITH POINT CLOUDS

27.1 INTRODUCTION

A key advantage of point clouds over conventional techniques is the abundance of thousands to millions of point samples that enable advanced modeling processes and surface estimations. Like conventional surveying technologies, model estimation from point clouds follows the fundamental principles of least squares. Such estimations have become popular in the surveying and engineering fields for applications such as three-dimensional (3D) modeling of geometric primitives for building information modeling (BIM) or topographic surface modeling to create digital elevation models (DEMs). Nevertheless, having access to high volumes of data can also create challenges as computations become more intensive and time-consuming. For simplicity in conveying fundamental concepts in this chapter, a relatively small number of points will be used in the examples to facilitate learning. Readers can explore developing their own code with the formulas provided in this chapter to support more complex processing of typical point cloud datasets. This chapter outlines the process of estimating surfaces of planes, spheres, cylinders, and polynomial models, which are fundamental building blocks for creating BIMs and DEMs.

27.2 PLANE FITTING WITH POINT CLOUDS



Estimations of planar surfaces are often developed for modeling buildings and other objects with flat planar surfaces. The general equation of a plane is

$$ax + by + cz + d = 0 \tag{27.1}$$

where a, b, c, and d are constants; and a, b, and c are the direction cosines that form the normal vector of a plane $\boldsymbol{n} = (a, b, c)$; and x, y, z are the Cartesian coordinates of the points lying on the plane. From these three values, constant d is computed as

$$d = -(ax_0 + by_0 + cz_0) \tag{27.2}$$

where (x_0, y_0, z_0) is a given point on the plane. This point is often the centroid of a point cloud data set, which serves as the new origin for the local coordinate system (Figure 27.1). To write Equation (27.1) in the form of a linear regression problem, constant c is set to 1. The value of $c = 1$ ensures that the normal vector points toward the positive side of the z-axis, while setting $c = -1$ changes the sign of the normal vector values and the normal vector points to the negative side of the z-axis. Therefore, with $c = 1$, Equation (27.1) can be simplified to

$$-ax - by - d = z \tag{27.3}$$

This form represents a linear problem that can be solved in a similar fashion to the least- squares procedures developed in Chapter 11 for fitting points to a straight line. Expanding Equation (27.3) to include observational errors, which contribute to the z-axis, the previous equation is modified as

$$-ax - by - d = z + v_z \tag{27.4}$$

In matrix representation, this becomes

$$AX = L + V \tag{27.5}$$

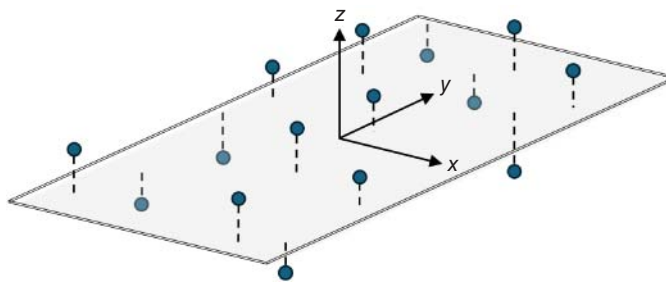


FIGURE 27.1 Example schematic of point cloud plane estimation.

where the matrices have the following form:

$$A = \begin{bmatrix} -x_1 & -y_1 & -1 \\ -x_2 & -y_2 & -1 \\ \vdots & \vdots & \vdots \\ -x_n & -y_n & -1 \end{bmatrix}, \quad X = \begin{bmatrix} a \\ b \\ d \end{bmatrix}, \quad L = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}, \quad V = \begin{bmatrix} v_{z_1} \\ v_{z_2} \\ \vdots \\ v_{z_n} \end{bmatrix} \tag{27.6}$$

Note that the plane parameters estimated in this approach are not normalized. To normalize the normal of the plane to unit length, the following normalization scale is computed:

$$k = \frac{1}{\sqrt{a^2 + b^2 + c^2}} \tag{27.7}$$

The normalized normal vector is found as

$$a' = ak \tag{27.8a}$$

$$b' = bk \tag{27.8b}$$

$$c' = ck \tag{27.8c}$$

where a', b', c' are the normalized plane parameters, and a new, normalized value for d can be computed as $d' = -(a'x_0 + b'y_0 + c'z_0)$.

In the case where the uncertainties of the z coordinates are known, then a weighted adjustment can take place. Note that in point cloud fitting problems with mapping coordinates, the centroid of the point cloud should be removed from the point cloud coordinates to avoid working with large numbers and numerical instability. The centroid coordinates are the average of each coordinate for all points. It is computed as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{cent.} = \begin{bmatrix} \frac{\sum_{i=1}^{n} x_i}{n} \\ \frac{\sum_{i=1}^{n} y_i}{n} \\ \frac{\sum_{i=1}^{n} z_i}{n} \end{bmatrix} \tag{27.9}$$

where $x_{cent.}, y_{cent.}, z_{cent.}$ are the centroid coordinates; and n is the number of points. Then the design and observation matrices of Equation (27.6) become

$$A = \begin{bmatrix} -(x_1 - x_{cent.}) & -(y_1 - y_{cent.}) & -1 \\ -(x_2 - x_{cent.}) & -(y_2 - y_{cent.}) & -1 \\ \vdots & \vdots & \vdots \\ -(x_n - x_{cent.}) & -(y_n - y_{cent.}) & -1 \end{bmatrix}, \quad L = \begin{bmatrix} z_1 - z_{cent.} \\ z_2 - z_{cent.} \\ \vdots \\ z_n - z_{cent.} \end{bmatrix} \tag{27.10}$$

In surveying, situations commonly arise with vertical planes (e.g., walls or building facades) that span in the z-axis. In such cases, the estimation is performed in a similar fashion by setting either $a = 1$ or $b = 1$ instead of $c = 1$ and re-arranging Equation (27.1) with either x or y, respectively.

Example 27.1 Determine an estimate for the plane that passes through the point cloud with the coordinates shown in the following table. Perform a weighted fit. List the estimated plane parameters, the reference standard deviation, residuals, and plane parameter uncertainty.

Number	x	y	z	σ_z	Number	x	y	z	σ_z
1	11.432	19.803	0.930	0.001	9	11.429	20.546	0.927	0.003
2	11.408	20.174	0.928	0.002	10	11.429	20.923	0.924	0.003
3	10.713	20.053	0.922	0.002	11	11.068	20.558	0.925	0.003
4	11.071	19.806	0.926	0.001	12	10.745	20.804	0.919	0.001
5	10.341	19.811	0.916	0.002	13	11.065	20.946	0.922	0.001
6	11.060	20.206	0.926	0.002	14	10.337	20.516	0.913	0.001
7	10.727	20.428	0.920	0.002	15	10.312	20.865	0.911	0.002
8	10.337	20.164	0.914	0.002	16	10.798	19.808	0.924	0.002

SOLUTION First, the centroid is estimated by computing the average of each coordinate. The centroid is computed as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{cent.} = \begin{bmatrix} \dfrac{11.432 + 11.408 + \cdots + 10.798}{16} \\ \dfrac{19.803 + 20.174 + \cdots + 19.808}{16} \\ \dfrac{0.930 + 0.928 + \cdots + 0.924}{16} \end{bmatrix} = \begin{bmatrix} 10.8920 \\ 20.3382 \\ 0.9217 \end{bmatrix}$$

Using these values, the A and L matrices are determined using Equation (27.6) as

$$A = \begin{bmatrix} -(11.432 - 10.8920) & -(19.803 - 20.3382) & -1 \\ -(11.408 - 10.8920) & -(20.174 - 20.3382) & -1 \\ \vdots & \vdots & \vdots \\ -(10.798 - 10.8920) & -(19.808 - 20.3382) & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -0.540 & 0.535 & -1 \\ -0.516 & 0.164 & -1 \\ \vdots & \vdots & \vdots \\ 0.094 & 0.530 & -1 \end{bmatrix}$$

$$L = \begin{bmatrix} 0.930 - 0.9217 \\ 0.928 - 0.9217 \\ \vdots \\ 0.924 - 0.9217 \end{bmatrix} = \begin{bmatrix} 0.0083 \\ 0.0063 \\ \vdots \\ 0.0023 \end{bmatrix}$$

Since the observations are assumed to be uncorrelated, the W matrix is computed using Equation (10.4) as

$$W = \begin{bmatrix} \frac{1}{0.001^2} & 0 & 0 & 0 \\ 0 & \frac{1}{0.002^2} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \frac{1}{0.002^2} \end{bmatrix}$$

Using these matrices, Equation (11.34*b*) yields

$$N = A^{\mathrm{T}}WA = \begin{bmatrix} 1078380 & -379494 & -31111 \\ -379494 & 1486033 & -2569 \\ -31111 & -2569 & 7333333 \end{bmatrix}$$

$$A^{\mathrm{T}}WL = \begin{bmatrix} -15448 \\ 10088 \\ 347 \end{bmatrix}$$

From Equation (11.35), the least-squares estimation for the plane parameters is

$$X = N^{-1}A^{\mathrm{T}}WL = \begin{bmatrix} -0.0131 \\ 0.0034 \\ 0.0000 \end{bmatrix}$$

The plane parameters are not normalized. Using Equations (27.7), (27.8*a*), (27.8*b*), and (27.8*c*), the parameters are normalized to unit length

$$k = \frac{1}{\sqrt{-0.0131^2 + 0.0034^2 + 1.0000^2}} = 0.9999$$

$$a' = -0.0131 \times 0.9999 = -0.0131$$

$$b' = 0.0034 \times 0.9999 = 0.0034$$

$$c' = 1.0000 \times 0.9999 = 0.9999$$

The estimated value for parameter d is zero (0.0000) since the new origin of the plane, after subtracting the centroid, is (0.0000, 0.0000, 0.0000). The d parameter in the original coordinate space of the point cloud is calculated as

$d = -(-0.0131 \times 10.8920 + 0.0034 \times 20.3382 + 0.9999 \times 0.9217) = -0.8487$. Therefore, the normalized vector of this plane is $\boldsymbol{n} = (a, b, c) = (-0.0131, 0.0034, 0.9999)$, and parameter d is -0.8487.

The residuals are

$$V^{\mathrm{T}} = [0.0006 \;\; 0.0010 \;\; -0.0017 \;\; -0.0001 \;\; 0.0003 \;\; -0.0016 \;\; \cdots \;\; -0.0017]$$

The standard deviation of unit weight is computed using Equation (12.15)

$$S_0 = \sqrt{\frac{V^{\mathrm{T}}WV}{16-3}} = 0.6936$$

and the parameter uncertainties are computed using Equation (13.24) as $\sigma_a = S_0\sqrt{q_{x_a,x_a}} = 0.0007$, $\sigma_b = S_0\sqrt{q_{x_b,x_b}} = 0.0006$, $\sigma_d = S_0\sqrt{q_{x_d,x_d}} = 0.0003$.

The problem of plane estimation can also be solved more robustly by finding the eigenvectors (a 3×3 matrix) and eigenvalues associated with the coordinates of the point cloud. In this approach, the plane direction can be found without having to specify a principal direction for the plane before estimation. In this case, the design matrix is first computed

$$A = \begin{bmatrix} x_1 - x_{cent.} & y_1 - y_{cent.} & z_1 - z_{cent.} \\ x_2 - x_{cent.} & y_2 - y_{cent.} & z_2 - z_{cent.} \\ \vdots & \vdots & \vdots \\ x_n - x_{cent.} & y_n - y_{cent.} & z_n - z_{cent.} \end{bmatrix} \tag{27.11}$$

Note that in Equation (27.11), the centroid $(x_{cent.}, y_{cent.}, z_{cent.})$ is subtracted from the coordinates to reduce their size for numerical stability. This yields a matrix of normal equations of

$$N = A^{\mathrm{T}}A \tag{27.12}$$

Equation (27.12) can also be expanded to include weights for each point as $N = A^{\mathrm{T}}WA$.

The eigenvalues and eigenvectors of matrix N can be determined by satisfying the following equation:

$$(N - \lambda \mathrm{I})X = 0 \tag{27.13}$$

where λ is the eigenvalue and X represents the eigenvector that is a solution of Equation (27.13) and therefore contains the plane orientation $\boldsymbol{n} = (a, b, c)$. A 3×3 matrix has three eigenvalues and three eigenvectors containing three elements each. The eigenvector corresponding to the smallest eigenvalue provides the plane orientation. Unfortunately, for a 3-by-3 matrix, Equation (27.13) results in a set of cubic equations, which are often difficult to solve by hand. They are also difficult to program directly. Fortunately, because eigenvectors and eigenvalues are used

in many mathematical problems in diverse domains, there are many tools available to obtain them. Thus, robust computer routines to determine the eigenvalues and associated eigenvectors of a matrix are readily available in programs such as MathCad and Matlab or as accessible libraries for programming languages such as Python and C++.

Example 27.2 Estimate the plane that passes through the point cloud in Example 27.1. Find the plane normal using the eigenvector approach.

SOLUTION To ensure numerical stability, the centroid is subtracted from the coordinates of the point cloud. The centroid for the data set in Example 27.1 is (10.892, 20.338, 0.922). The design matrix is

$$A = \begin{bmatrix} 11.432 - 10.8920 & 19.803 - 20.3382 & 0.930 - 0.9217 \\ 11.408 - 10.8920 & 20.174 - 20.3382 & 0.928 - 0.9217 \\ \vdots & \vdots & \vdots \\ 10.798 - 10.8920 & 19.808 - 20.3382 & 0.924 - 0.9217 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5400 & -0.5352 & 0.0083 \\ 0.5160 & -0.1642 & 0.0063 \\ \vdots & \vdots & \vdots \\ -0.0940 & -0.5302 & 0.0023 \end{bmatrix}$$

The weight matrix is

$$W = \begin{bmatrix} \dfrac{1}{0.001^2} & 0 & 0 & 0 \\ 0 & \dfrac{1}{0.002^2} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dfrac{1}{0.002^2} \end{bmatrix}$$

The weighted normal equations matrix is computed as

$$N = A^{\mathrm{T}}WA = \begin{bmatrix} 1078380 & -379494 & 15448 \\ -379494 & 1486033 & -10088 \\ 15448 & -10088 & 244 \end{bmatrix}$$

The eigenvalues are then determined by setting the determinant of $N - \lambda I$ equal to zero. This results in the cubic equation of

$$\lambda^3 - 2{,}564{,}656.661\lambda^2 + 1{,}458{,}776{,}535{,}992.920\lambda - 9{,}121{,}069{,}392{,}515.300 = 0$$

The creation and solution for the roots of the equation, which are the eigenvalues for the matrix, are difficult to generate by hand. Additionally, the eigenvector corresponding to the smallest eigenvalue must also be found. Thus, software is

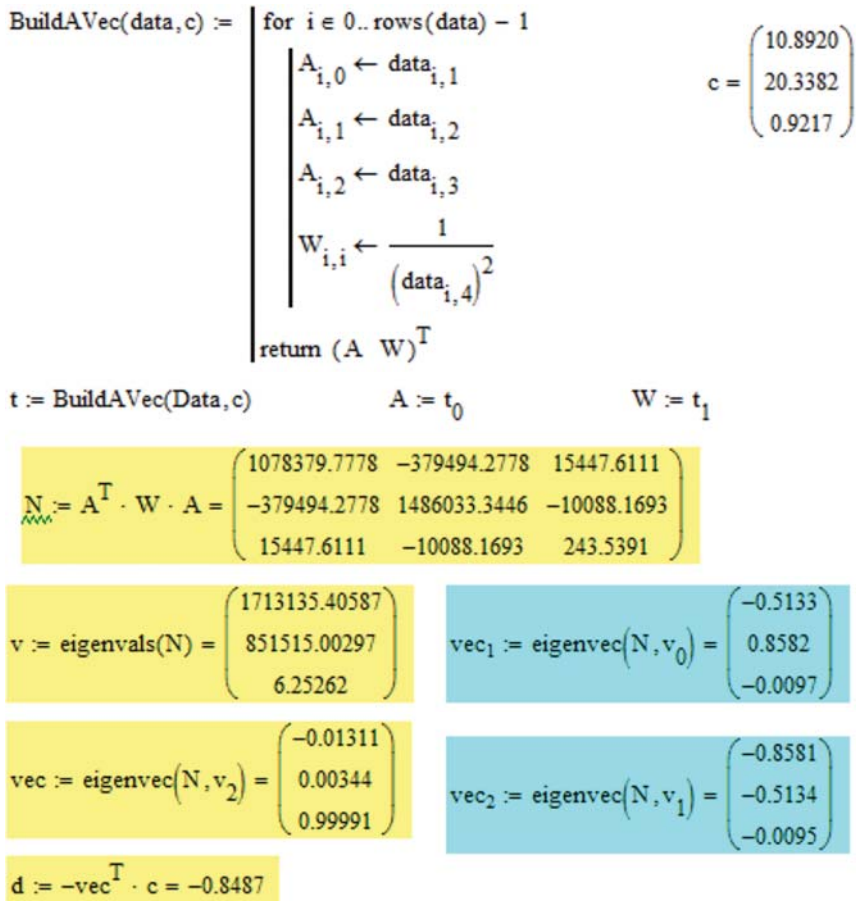


FIGURE 27.2 MathCad solution for Example 27.2.

recommended to solve these problems. Using MATHCAD, the eigenvalues, which correspond to matrix N, are $\lambda_1 = 1{,}713{,}135.406$, $\lambda_2 = 851{,}515.003$, and $\lambda_3 = 6.253$. The associated eigenvectors of each eigenvalue are

$$\boldsymbol{v}_1 = \begin{bmatrix} 0.5133 \\ -0.8582 \\ 0.0097 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} -0.8581 \\ -0.5134 \\ -0.0095 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} -0.0131 \\ 0.0034 \\ 0.9999 \end{bmatrix}$$

The plane normal is $\boldsymbol{n} = (a, b, c) = (-0.0131, 0.0034, 0.9999)$, which is identical to the solution found in Example 27.1 using the standard least-squares approach. The value for parameter d is computed as $d = -(-0.0131 \times 10.8920 + 0.0034 \times 20.3382 + 0.9999 \times 0.9217) = -0.8487$. Figure 27.2 shows the MathCad solution for this problem.

Equation (27.1) does not consider potential uncertainties in the x and y coordinates and requires constraining one direction. The general least-squares equations of Chapter 22 can be used to handle these issues. That is, Equation (27.1) must be rewritten as

$$F(x, y, z) = a\,(x + v_x) + b\,(y + v_y) + c\,(z + v_z) + d = 0 \tag{27.14}$$

In Equation (27.14), for each point x, y, z, there are associated residuals v_x, v_y, v_z. This equation is nonlinear. Its linearized form is

$$\frac{\partial F}{\partial x}v_x + \frac{\partial F}{\partial y}v_y + \frac{\partial F}{\partial z}v_z + \frac{\partial F}{\partial a}da + \frac{\partial F}{\partial b}db + \frac{\partial F}{\partial c}dc + \frac{\partial F}{\partial d}dd = a_0x + b_0y + c_0z + d_0 \tag{27.15}$$

where the partial derivatives are

$$\frac{\partial F}{\partial x} = a, \quad \frac{\partial F}{\partial y} = b, \quad \frac{\partial F}{\partial z} = c, \quad \frac{\partial F}{\partial a} = x, \quad \frac{\partial F}{\partial b} = y, \quad \frac{\partial F}{\partial c} = z, \quad \frac{\partial F}{\partial d} = 1 \tag{27.16}$$

Using Equation (22.5), the plane estimation problem can be written as

$$BV + JX = K \tag{27.17}$$

where

$$B = \begin{bmatrix} a_0 & b_0 & c_0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_0 & b_0 & c_0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & a_0 & b_0 & c_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & a_0 & b_0 & c_0 \end{bmatrix}, \quad V = \begin{bmatrix} v_{x_1} \\ v_{y_1} \\ v_{z_1} \\ \vdots \\ v_{x_n} \\ v_{y_n} \\ v_{z_n} \end{bmatrix} \tag{27.18a}$$

$$J = \begin{bmatrix} \dfrac{\partial F_1}{\partial a} & \dfrac{\partial F_1}{\partial b} & \dfrac{\partial F_1}{\partial c} & \dfrac{\partial F_1}{\partial d} \\ \dfrac{\partial F_2}{\partial a} & \dfrac{\partial F_2}{\partial b} & \dfrac{\partial F_2}{\partial c} & \dfrac{\partial F_2}{\partial d} \\ \vdots & \vdots & \vdots & \vdots \\ \dfrac{\partial F_n}{\partial a} & \dfrac{\partial F_n}{\partial b} & \dfrac{\partial F_n}{\partial c} & \dfrac{\partial F_n}{\partial d} \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_1 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ x_n & y_n & z_n & 1 \end{bmatrix} \tag{27.18b}$$

$$X = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}, \quad K = \begin{bmatrix} -(a_0\,x_1 + b_0y_1 + c_0z_1 + d_0) \\ -(a_0\,x_2 + b_0y_2 + c_0z_2 + d_0) \\ \vdots \\ -(a_0\,x_n + b_0y_n + c_0z_n + d_0) \end{bmatrix} \tag{27.18c}$$

To solve this problem, a constraint needs to be introduced for the plane parameters, and the system of equations needs to be expanded using Helmert's method discussed in Chapter 20. To normalize the vector to be of unit length, the constraint is

$$a^2 + b^2 + c^2 - 1 = 0 \tag{27.19}$$

The partial derivatives of the constraints are

$$C = \begin{bmatrix} 2a_0 & 2b_0 & 2c_0 & 0 \end{bmatrix} \tag{27.20}$$

Hence, the constraint is evaluated as

$$K_C = 0 - \left(a_0^2 + b_0^2 + c_0^2 - 1\right) \tag{27.21}$$

For this problem, initial values for the plane parameters a_0, c_0, b_0, d_0 are needed, and iterations are required to solve the nonlinear problem. All coordinates contribute as observations, and their uncertainties can be used to build the covariance matrix. Assuming that coordinates are derived from independent observations, the corresponding covariance matrix is

$$\Sigma = \begin{bmatrix} \sigma_{x_1}^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_{y_1}^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_{z_1}^2 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \sigma_{x_n}^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_{y_n}^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sigma_{z_n}^2 \end{bmatrix} \tag{27.22}$$

Using Equation (22.7), the weight matrix is then given as $W_e = (B\Sigma B^{\mathrm{T}})^{-1}$ where B, in this case, is computed from Equation (27.18*a*).

The remainder of the solution by the general least-squares method is achieved by applying Equations (22.7) through (22.14) from Chapter 22.

Example 27.3 Estimate the plane that passes through the point cloud of the following table. Perform a weighted fit using the general least-squares method. List the estimated plane parameters, the reference standard deviation, residuals, and plane parameter uncertainty.

Number	x	y	z	σ_x	σ_y	σ_z
1	11.432	19.803	4.930	0.004	0.002	0.002
2	11.100	20.350	4.926	0.006	0.006	0.008
3	10.686	20.546	4.919	0.005	0.008	0.005
4	10.426	20.786	4.914	0.002	0.001	0.001
5	11.264	20.909	4.923	0.007	0.004	0.005
6	10.750	20.961	4.919	0.004	0.004	0.002
7	11.432	20.561	4.926	0.001	0.002	0.005
8	11.154	20.001	4.927	0.008	0.008	0.009
9	11.432	20.206	4.928	0.003	0.008	0.006
10	10.713	20.053	4.922	0.003	0.003	0.006
11	11.003	20.685	4.922	0.002	0.001	0.003
12	10.335	19.811	4.916	0.002	0.003	0.001
13	10.337	20.152	4.915	0.005	0.005	0.008
14	10.261	20.483	4.911	0.006	0.006	0.004

SOLUTION Acknowledging that this plane is almost perpendicular to the z-axis, the initial values for the plane parameters are selected as $a_0 = 0.0000$, $b_0 = 0.0000$, $c_0 = 1.0000$. Using Equation (27.9), the centroid is computed as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{cent.} = \begin{bmatrix} \dfrac{11.432 + 11.100 + \cdots + 10.261}{14} \\ \dfrac{19.803 + 20.350 + \cdots + 20.483}{14} \\ \dfrac{4.930 + 4.926 + \cdots + 4.911}{14} \end{bmatrix} = \begin{bmatrix} 10.8804 \\ 20.3791 \\ 4.9213 \end{bmatrix}$$

The centroid is subtracted from the coordinates of each point. By doing so, the initial value for d can be set as 0.0000, because the new centroid of the data set is (0.0000, 0.0000, 0.0000). If the points are not reduced to their centroid, then the initial value of d can computed from $d_0 = -(ax_0 + by_0 + cz_0)$. Using Equation (27.18$b$), the Jacobian matrix for the first iteration is

$$J = \begin{bmatrix} 11.432 - 10.8804 & 19.803 - 20.3791 & 4.930 - 4.9213 & 1 \\ 11.100 - 10.8804 & 20.350 - 20.3791 & 4.926 - 4.9213 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 10.261 - 10.8804 & 20.483 - 20.3791 & 4.911 - 4.9213 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5516 & -0.5761 & 0.0087 & 1 \\ 0.2196 & -0.0291 & 0.0047 & 1 \\ \vdots & \vdots & \vdots & \vdots \\ -0.6194 & 0.1039 & -0.0103 & 1 \end{bmatrix}$$

Using Equation (27.18*c*), the K matrix for the first iteration is

$$K = \begin{bmatrix} -[0(0.5516) + 0(-0.5761) + 1(0.0087) + 0.000)] \\ -[0(0.2196) + 0(-0.0291) + 1(0.0047) + 0.000)] \\ \vdots \\ -[0(-0.6194) + 0(0.1039) + 1(-0.0103) + 0.000)] \end{bmatrix} = \begin{bmatrix} -0.0087 \\ -0.0047 \\ \vdots \\ 0.0103 \end{bmatrix}$$

Using Equation (27.20), the constraint matrix for the first iteration is

$$C = \begin{bmatrix} 0 & 0 & 2.0000 & 0 \end{bmatrix}$$

Using Equation (27.21), the K_C matrix for the first iteration is

$$K_C = [0.0000]$$

Using Equation (27.22), the covariance matrix of the observations is

$$\Sigma = \begin{bmatrix} 0.004^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.002^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.002^2 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0.006^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.006^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.004^2 \end{bmatrix}$$

The point cloud uncertainties are propagated using Equation (22.7). The following matrix is derived for the first iteration:

$$B = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that because the initial estimation of the plane parameters was (0.000, 0.000, 1.000), the B matrix has only values of 0 and 1. This matrix should be updated based on the plane parameters in each iteration, according to Equation (27.18*a*).

Using Equation (22.7), the weight matrix for the first iteration is

$$W_e = (B\Sigma B^{\mathrm{T}})^{-1} = \begin{bmatrix} 250000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 15625 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 40000 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1000000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 15625 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 62500 \end{bmatrix}$$

Using Equation (22.8), the normal equation matrices for the first iteration are

$$N = J^{\mathrm{T}} W_e J = \begin{bmatrix} 644920 & 36909 & 8214 & -880844 \\ 36909 & 687035 & -1582 & -106574 \\ 8214 & -1582 & 112 & -11110 \\ -880844 & -106574 & -11110 & 2892762 \end{bmatrix}$$

$$J^{\mathrm{T}} W_e K = \begin{bmatrix} -8214 \\ 1582 \\ -112 \\ 11110 \end{bmatrix}$$

Using Equation (20.16), the expanded system of equations for the constraint is

$$\begin{bmatrix} X \\ X_C \end{bmatrix} = \begin{bmatrix} N & C^{\mathrm{T}} \\ C & 0 \end{bmatrix}^{-1} \begin{bmatrix} J^{\mathrm{T}} W_e K \\ K_C \end{bmatrix}$$

Note that matrices J, B, W_e, K, C, and K_C must be updated in each iteration using the updated values for a, b, c, d.

The final values after three iterations are

$$X = \begin{bmatrix} -0.0129 \\ 0.0030 \\ 0.9999 \\ -0.0002 \end{bmatrix}$$

The normal vector of this plane is $\boldsymbol{n} = (a, b, c) = (-0.0129, 0.0030, 0.9999)$. The value for parameter d is computed as $d = -(-0.0129 \times 10.8804 + 0.0030 \times 20.3791 + 0.9999 \times 4.9213) = -4.8421$. Using Equation (22.10), the equivalent residuals are

$$V_e^{\mathrm{T}} = \begin{bmatrix} 0.0001 & 0.0018 & 0.0007 & -0.0002 & -0.0016 & \cdots & -0.0020 \end{bmatrix}^{\mathrm{T}}$$

From Equation (22.11), the observational residuals are

$$V = \Sigma B^{\mathrm{T}} W_e V_e = \begin{bmatrix} v_{x_1} \\ v_{y_1} \\ v_{z_1} \\ \vdots \\ v_{x_n} \\ v_{y_n} \\ v_{z_n} \end{bmatrix} = \begin{bmatrix} 0.0000 \\ 0.0000 \\ -0.0001 \\ \vdots \\ 0.0001 \\ 0.0000 \\ -0.0020 \end{bmatrix}$$

The reference standard deviation is $S_0 = 0.3121$, and the plane parameter uncertainties are $\sigma_a = S_0 \sqrt{q_{x_a x_a}} = 0.0005$, $\sigma_b = S_0 \sqrt{q_{x_b x_b}} = 0.0004$, $\sigma_c =$

$S_0\sqrt{q_{x_c,x_c}} = 0.00001$, and $\sigma_d = S_0\sqrt{q_{x_d,x_d}} = 0.0002$. The values q_{x_a,x_a}, q_{x_b,x_b}, q_{x_c,x_c}, and q_{x_d,x_d} are the diagonal elements of the expanded matrix $\begin{bmatrix} N & C^{\mathrm{T}} \\ C & 0 \end{bmatrix}^{-1}$.

27.3 SPHERE AND CYLINDER ESTIMATION WITH POINT CLOUDS



Estimation of 3D shapes is another common task that is completed with point clouds for modeling purposes. Only a portion of the shape needs to be captured if the shape is symmetric. This section covers two common 3D shapes, namely spheres and cylinders. Readers interested in fitting points to a circle are referred to Appendix C. Fitting a point cloud to a sphere is needed for registration of multiple scans where spherical targets are used. Estimation of cylinders may be in BIM processes (e.g., modeling the pipes in a building, modeling conduits in power station facilities, modeling columns in a building, and more).

27.3.1 Fitting Points to a Sphere

The general equation for a sphere is given by

$$S(h,k,p,r) = (x-h)^2 + (y-k)^2 + (z-p)^2 - r^2 = 0 \tag{27.23}$$

where x, y, z are the coordinates of the point cloud points lying on the sphere surface; h, k, p are the 3D coordinates of the sphere center; and r is the radius of the sphere. Adding the coordinate uncertainties, the sphere equation becomes

$$[(x+v_x)-h]^2 + [(y+v_y)-k]^2 + [(z+v_z)-p]^2 - r^2 = 0 \tag{27.24}$$

The linearized form is

$$\begin{aligned} &\frac{\partial S}{\partial x}v_x + \frac{\partial S}{\partial y}v_y + \frac{\partial S}{\partial z}v_z + \frac{\partial S}{\partial h}dh + \frac{\partial S}{\partial k}dk + \frac{\partial S}{\partial p}dp + \frac{\partial S}{\partial r}dr \\ &\quad = [(x+v_x)-h_0]^2 + [(y+v_y)-k_0]^2 + [(z+v_z)-p_0]^2 - r_0^2 \end{aligned} \tag{27.25}$$

where h_0, k_0, p_0, r_0 are the initial values for the sphere center and radius. The partial derivatives are

$$\begin{aligned} &\frac{\partial S}{\partial x} = 2(x-h), \quad \frac{\partial S}{\partial y} = 2(y-k), \quad \frac{\partial S}{\partial z} = 2(z-p), \quad \frac{\partial S}{\partial h} = -2(x-h), \\ &\frac{\partial S}{\partial k} = -2(y-k), \quad \frac{\partial S}{\partial p} = -2(z-p), \quad \frac{\partial S}{\partial r} = -2r \end{aligned} \tag{27.26}$$

In matrix form, the sphere estimation problem can be written as

$$BV + JX = K \tag{27.27}$$

where

$$B = \begin{bmatrix} 2(x_1 - h_0) & 2(y_1 - k_0) & 2(z_1 - p_0) & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2(x_n - h_0) & 2(y_n - k_0) & 2(z_n - p_0) \end{bmatrix} \tag{27.28a}$$

$$V = \begin{bmatrix} v_{x_1} \\ v_{y_1} \\ v_{z_1} \\ \vdots \\ v_{x_n} \\ v_{y_n} \\ v_{z_n} \end{bmatrix}; \quad X = \begin{bmatrix} \mathrm{d}h \\ \mathrm{d}k \\ \mathrm{d}p \\ \mathrm{d}r \end{bmatrix} \tag{27.28b}$$

$$J = \begin{bmatrix} -2(x_1 - h_0) & -2(y_1 - k_0) & -2(z_1 - p_0) & -2r_0 \\ -2(x_2 - h_0) & -2(y_2 - k_0) & -2(z_2 - p_0) & -2r_0 \\ \vdots & \vdots & \vdots & \vdots \\ -2(x_n - h_0) & -2(y_n - k_0) & -2(z_n - p_0) & -2r_0 \end{bmatrix} \tag{27.28c}$$

$$K = \begin{bmatrix} 0 - \left[(x_1 - h_0)^2 + (y_1 - k_0)^2 + (z_1 - p_0)^2 - {r_0}^2\right] \\ 0 - \left[(x_2 - h_0)^2 + (y_2 - k_0)^2 + (z_2 - p_0)^2 - {r_0}^2\right] \\ \vdots \\ 0 - \left[(x_n - h_0)^2 + (y_n - k_0)^2 + (z_n - p_0)^2 - {r_0}^2\right] \end{bmatrix} \tag{27.28d}$$

The sphere parameters can be computed by applying the least-squares method to Equations (27.28*a*) through (27.28*d*) and performing iterations until convergence is achieved. Assuming that coordinates are derived from independent observations, the corresponding covariance matrix for the sphere estimation is identical to Equation (27.22).

Example 27.4 Estimate the sphere center and radius that passes through the point cloud given in the following table. Perform a weighted fit using general least-squares method. List the estimated sphere parameters, the standard deviation of unit weight, residuals, and sphere parameter uncertainty.

Number	x	y	z	σ_x	σ_y	σ_z
1	50.000	100.000	4.810	0.001	0.001	0.002
2	49.838	100.000	4.879	0.002	0.001	0.002
3	49.802	100.000	5.034	0.001	0.001	0.001
4	49.907	99.998	5.181	0.003	0.002	0.001
5	49.895	99.855	4.906	0.001	0.003	0.005
6	49.885	99.843	5.067	0.001	0.003	0.002
7	50.049	99.845	4.885	0.002	0.001	0.003
8	50.064	99.815	5.033	0.001	0.002	0.001
9	50.036	99.888	5.163	0.002	0.002	0.001
10	50.168	99.944	4.910	0.003	0.003	0.003
11	50.182	99.942	5.061	0.002	0.001	0.004
12	50.115	100.039	5.154	0.003	0.001	0.004
13	50.115	100.084	4.859	0.003	0.002	0.003
14	50.161	100.116	4.999	0.003	0.001	0.003
15	49.998	100.176	4.911	0.001	0.001	0.005
16	49.999	100.189	5.069	0.002	0.003	0.005

SOLUTION The initial values for the sphere center are derived by finding the centroid of the point cloud using Equation (27.9). This is a good estimation assuming that no significant data gaps exist, and points are well distributed around the sphere. The centroid is computed as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{cent.} = \begin{bmatrix} \dfrac{50.000 + 49.838 + \cdots + 49.999}{16} \\ \dfrac{100.000 + 100.000 + \cdots + 100.189}{16} \\ \dfrac{4.810 + 4.879 + \cdots + 5.069}{16} \end{bmatrix} = \begin{bmatrix} 50.0134 \\ 99.9834 \\ 4.9951 \end{bmatrix}$$

Therefore, the initial estimation of the sphere center coordinates is $h = 50.0134$, $k = 99.9834$, $p = 4.9951$. The initial radius value can be estimated by computing the distance between the centroid and the first point of the point cloud. This is estimated as

$$r = \sqrt{(50.000 - 50.0134)^2 + (100.000 - 99.9834)^2 + (4.810 - 4.9951)^2} = 0.1863$$

Therefore, the initial radius value is $r = 0.1863$. The covariance matrix of the observations using Equation (27.22) is

$$\Sigma = \begin{bmatrix} 0.001^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.001^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.002^2 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0.002^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.003^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.005^2 \end{bmatrix}$$

The point cloud uncertainties are propagated to the sphere equation using Equation (27.28*a*), a partial listing for the B matrix for the first iteration is

$$B = \begin{bmatrix} -0.027 & 0.033 & -0.370 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.351 & 0.033 & -0.232 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -0.031 & 0.385 & -0.168 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -0.029 & 0.411 & 0.148 \end{bmatrix}$$

The elements of matrix B are computed using Equation (27.28*a*). For example, for the first row of matrix B

$$B_{1,1} = 2(x_1 - h_0) = 2(50.000 - 50.0134) = -0.027$$
$$B_{1,2} = 2(y_1 - k_0) = 2(100.000 - 99.9834) = 0.033$$
$$B_{1,3} = 2(z_1 - p_0) = 2(4.810 - 4.9951) = -0.370$$

Using Equation (22.7), the equivalent weight matrix for the first iteration is

$$W_e = (B\Sigma B^{\mathrm{T}})^{-1}$$
$$= \begin{bmatrix} 1818872 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1410963 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5379594 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1168950 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1168206 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 482597 \end{bmatrix}$$

Using Equation (27.28*c*), the J matrix of partial derivatives for the first iteration is

$$J = \begin{bmatrix} 0.027 & -0.033 & 0.370 & -0.373 \\ 0.351 & -0.033 & 0.232 & -0.373 \\ \vdots & \vdots & \vdots & \vdots \\ 0.029 & -0.411 & -0.148 & -0.373 \end{bmatrix}$$

The computation of the first row of matrix J for the first iteration is shown here as an example

$$J_{1,1} = -2(x_1 - h_0) = -2(50.000 - 50.0134) = 0.027$$
$$J_{1,2} = -2(y_1 - k_0) = -2(100.000 - 99.9834) = -0.033$$
$$J_{1,3} = -2(z_1 - p_0) = -2(4.810 - 4.9951) = 0.370$$
$$J_{1,4} = -2r = -2 \times 0.1863 = -0.373$$

Using Equation (27.28*d*), the K matrix for the first iteration is

$$K = \begin{bmatrix} 0 - [(50.000 - 50.0134)^2 + (100.000 - 99.9834)^2 + (4.810 - 4.9951)^2 - 0.1863^2] \\ 0 - [(49.838 - 50.0134)^2 + (100.000 - 99.9834)^2 + (4.879 - 4.9951)^2 - 0.1863^2] \\ \vdots \\ 0 - [(49.999 - 50.0134)^2 + (100.189 - 99.9834)^2 + (5.069 - 4.9951)^2 - 0.1863^2] \end{bmatrix}$$
$$= \begin{bmatrix} 0.0000 \\ -0.0098 \\ \vdots \\ -0.0133 \end{bmatrix}$$

The normal equation matrices for the first iteration are

$$N = J^{\mathrm{T}} W_e J = \begin{bmatrix} 1766869 & -61888 & -149858 & -694787 \\ -61888 & 1062011 & -270572 & -415220 \\ -149858 & -270572 & 1386142 & 371949 \\ -694787 & -415220 & 371949 & 3664697 \end{bmatrix}$$

$$J^{\mathrm{T}} W_e K = \begin{bmatrix} -37312 \\ 10468 \\ 13533 \\ 51260 \end{bmatrix}$$

Note that matrices J, B, W_e, and K must be updated in each iteration using the updated values for h, k, p, and r.

The final values after four iterations are

$$X = \begin{bmatrix} 49.9983 \\ 99.9991 \\ 5.0031 \\ 0.1994 \end{bmatrix}$$

Using Equation (22.1), the equivalent residuals are

$$V_e^{\mathrm{T}} = \begin{bmatrix} -0.0025 & 0.0013 & -0.0003 & 0.0002 & 0.0011 & 0.0015 & \cdots & 0.0007 \end{bmatrix}^{\mathrm{T}}$$

Using Equation (22.11), the observational residuals are

$$V = \Sigma B^{\mathrm{T}} W_e V_e = \begin{bmatrix} 0.0000 \\ 0.0000 \\ 0.0064 \\ \vdots \\ 0.0000 \\ 0.0013 \\ 0.0012 \end{bmatrix}$$

The reference standard deviation is $S_0 = 1.4033$, and the uncertainty of the sphere center and radius are $\sigma_h = S_0\sqrt{q_{x_h,x_h}} = 0.0011$, $\sigma_k = S_0\sqrt{q_{x_k,x_k}} = 0.0015$, $\sigma_p = S_0\sqrt{q_{x_p,x_p}} = 0.0013$, and $\sigma_r = S_0\sqrt{q_{x_r,x_r}} = 0.0008$. The values q_{x_h,x_h}, q_{x_k,x_k}, q_{x_p,x_p}, and q_{x_r,x_r} are the diagonal elements of N^{-1}.

27.3.2 Simplified Solution to Fitting Points to a Sphere

The sphere problem can be simplified into a linear problem to avoid iterations. First, the terms of Equation (27.23) are expanded as follows (Forbes, 1989):

$$x^2 - 2xh + h^2 + y^2 - 2yk + k^2 + z^2 - 2zp + p^2 - r^2 = 0 \tag{27.29}$$

Re-arranging Equation (27.29) to separate the observations from the unknowns

$$2xh + 2yk + 2zp - (h^2 + k^2 + p^2 - r^2) = x^2 + y^2 + z^2 \tag{27.30}$$

Recognizing that the $(h^2 + k^2 + p^2 - r^2)$ term is a constant, this is replaced by the constant term e, so that Equation (27.30) becomes

$$2xh + 2yk + 2zp - e = x^2 + y^2 + z^2 \tag{27.31}$$

Now, Equation (27.31) can be written in matrix notation as $AX = L$ with matrices

$$\begin{bmatrix} 2x_1 & 2y_1 & 2z_1 & -1 \\ 2x_2 & 2y_2 & 2z_2 & -1 \\ \vdots & \vdots & \vdots & \vdots \\ 2x_n & 2y_n & 2z_n & -1 \end{bmatrix} \begin{bmatrix} h \\ k \\ p \\ e \end{bmatrix} = \begin{bmatrix} {x_1}^2 + {y_1}^2 + {z_1}^2 \\ {x_2}^2 + {y_2}^2 + {z_2}^2 \\ \vdots \\ {x_n}^2 + {y_n}^2 + {z_n}^2 \end{bmatrix} \tag{27.32}$$

The above system of matrices can be solved via standard least-squares methods, and the radius is found as $r = \sqrt{h^2 + k^2 + p^2 - e}$. Note that the centroid should be computed using Equation (27.9) and removed from the coordinates for numerical

stability. The weight matrix is given as $W_e = (B\Sigma B^{\mathrm{T}})^{-1}$, where B in this case is computed as

$$B = \begin{bmatrix} 2x_1 & 2y_1 & 2z_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2x_2 & 2y_2 & 2z_2 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 2x_{n-1} & 2y_{n-1} & 2z_{1-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 2x_n & 2y_n & 2z_1 \end{bmatrix} \tag{27.33}$$

Example 27.5 Estimate the sphere center and radius that passes through the point cloud of Example 27.4. Perform a weighted fit and use the linearized approach. List the estimated sphere parameters, the reference standard deviation, residuals, and sphere parameter uncertainty.

SOLUTION In the calculations of this example, the centroid is estimated using Equation (27.9) and the point cloud of Example 27.4.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{cent.} = \begin{bmatrix} \dfrac{50.000 + 49.838 + \cdots + 49.999}{16} \\ \dfrac{100.000 + 100.000 + \cdots + 100.189}{16} \\ \dfrac{4.810 + 4.879 + \cdots + 5.069}{16} \end{bmatrix} = \begin{bmatrix} 50.0134 \\ 99.9834 \\ 4.9951 \end{bmatrix}$$

The centroid is subtracted from the point cloud coordinates. Using Equation (27.32), the A matrix of partial derivatives is

$$A = \begin{bmatrix} 2 \times -0.0134 & 2 \times 0.0166 & 2 \times -0.1851 & -1 \\ 2 \times -0.1754 & 2 \times 0.0166 & 2 \times -0.1161 & -1 \\ \vdots & \vdots & \vdots & \vdots \\ 2 \times -0.0144 & 2 \times 0.2056 & 2 \times 0.0739 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} -0.0268 & 0.0333 & -0.3701 & -1 \\ -0.3508 & 0.0333 & -0.2321 & -1 \\ \vdots & \vdots & \vdots & \vdots \\ -0.0288 & 0.4112 & 0.1479 & -1 \end{bmatrix}$$

and the L matrix is

$$L = \begin{bmatrix} -0.0134^2 + 0.0166^2 + 0.1851^2 \\ -0.1754^2 + 0.0166^2 + -0.1161^2 \\ \vdots \\ -0.0144^2 + 0.2056^2 + 0.0739^2 \end{bmatrix} = \begin{bmatrix} 0.0347 \\ 0.0445 \\ \vdots \\ 0.0480 \end{bmatrix}$$

The covariance matrix of the observations is

$$\Sigma = \begin{bmatrix} 0.001^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.001^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.002^2 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0.002^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.003^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.005^2 \end{bmatrix}$$

The point cloud uncertainties are propagated to the sphere equation using the B matrix in Equation (27.33), which for this problem is

$$B = \begin{bmatrix} -0.027 & 0.033 & -0.370 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.351 & 0.033 & -0.232 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -0.031 & 0.385 & -0.168 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -0.029 & 0.411 & 0.148 \end{bmatrix}$$

Using Equation (22.7), the resulting equivalent weight matrix is

$$W_e = (B\Sigma B^{\mathrm{T}})^{-1} = \begin{bmatrix} 1818872 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1410963 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5379594 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1168950 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1168206 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 482597 \end{bmatrix}$$

The normal equation matrices are

$$N = A^{\mathrm{T}} W_e A = \begin{bmatrix} 1766869 & -61888 & -149858 & 1864815 \\ -61888 & 1062011 & -270572 & 1114453 \\ -149858 & -270572 & 1386142 & -998316 \\ 1864815 & 1114453 & -998316 & 26400123 \end{bmatrix}$$

$$A^{\mathrm{T}} W_e L = \begin{bmatrix} -102028 \\ -28207 \\ 48178 \\ -1053755 \end{bmatrix}$$

Hence, the estimated values are

$$X = N^{-1} A^{\mathrm{T}} W L = \begin{bmatrix} -0.0151 \\ 0.0157 \\ 0.0080 \\ -0.0392 \end{bmatrix}$$

The centroid is then added back to the estimated values to get the sphere center in the original coordinate system, resulting in $h = 50.0134 - 0.0151 = 49.9983$, $k = 99.9834 + 0.0157 = 99.9991$, $p = 4.9951 + 0.0080 = 5.0031$. The radius is computed as

$$r = \sqrt{-0.0151^2 + 0.0157^2 + 0.0080^2 - (-0.0392)} = 0.1994$$

The equivalent residuals are

$$V_e^{\mathrm{T}} = \begin{bmatrix} 0.0025 & -0.0013 & 0.0003 & -0.0003 & -0.0011 & -0.0015 & \cdots & -0.0007 \end{bmatrix}^{\mathrm{T}}$$

Using Equation (22.11), the observational residuals are

$$V = \Sigma B^{\mathrm{T}} W_e V_e = \begin{bmatrix} -0.0001 \\ 0.0002 \\ -0.0067 \\ \vdots \\ 0.0000 \\ -0.0012 \\ -0.0012 \end{bmatrix}$$

The reference standard deviation is $S_0 = 1.4558$, and the uncertainties of the sphere center coordinates are $\sigma_h = S_0\sqrt{q_{x_h,x_h}} = 0.0011$, $\sigma_k = S_0\sqrt{q_{x_k,x_k}} = 0.0015$, and $\sigma_p = S_0\sqrt{q_{x_p,x_p}} = 0.0013$. The values q_{x_h,x_h}, q_{x_k,x_k}, and q_{x_p,x_p} are the diagonal elements of N^{-1}. To find the radius uncertainty, the sphere center uncertainty and uncertainty of parameter e need to be propagated through the radius estimation equation. The uncertainty of parameter e is $\sigma_e = S_0\sqrt{q_{x_e,x_e}} = 0.0003$. The radius uncertainty is calculated as

$$\begin{aligned} \sigma_r &= \sqrt{\left(\frac{h}{r}\right)^2\sigma_h^2 + \left(\frac{k}{r}\right)^2\sigma_k^2 + \left(\frac{p}{r}\right)^2\sigma_p^2 + \left(\frac{-e}{2r}\right)^2\sigma_e^2} \\ &= \sqrt{\begin{array}{l}\left(\frac{-0.0151}{0.1994}\right)^2 0.0011^2 + \left(\frac{0.0157}{0.1994}\right)^2 0.0015^2 + \left(\frac{0.0080}{0.1994}\right)^2 0.0013^2 \\ \quad + \left(\frac{-0.0392}{2\times 0.1994}\right)^2 0.0003^2\end{array}} \\ &= 0.0002 \end{aligned}$$

27.3.3 Fitting Points to a Cylinder

The problem of estimating a cylinder from point cloud data can be solved using different approaches. One of them is to consider the cylinder as an ellipse (or circle)

that extends along its axis up to a certain length. While this approach is suitable for some mathematical applications, it can be difficult to use with point cloud data, although several algorithms implement growing algorithms to estimate a series of ellipses (or circles) to form a cylinder. This approach is followed to reduce erroneous estimations when point clouds are incomplete and/or when points from neighboring objects are close to the cylinder points. In this chapter, the cylinder estimation is performed through the general equation of a cylinder in space (Benko et al., 2022), which is

$$C = c_4\left[x^2 + y^2 + z^2 - (c_1x + c_2y + c_3z)^2\right] + c_5x + c_6y + c_7z + c_8 = 0 \quad (27.34)$$

where the cylinder's radius is computed as

$$r = 1/|2c_4| \quad (27.35)$$

The nearest cylinder's axis point to the origin is given as

$$h = -\frac{c_5}{2c_4} \quad (27.36a)$$

$$k = -\frac{c_6}{2c_4} \quad (27.36b)$$

$$p = -\frac{c_7}{2c_4} \quad (27.36c)$$

The direction of the cylinder's axis is (c_1, c_2, c_3).
Adding the uncertainties for the point cloud, the cylinder equation becomes

$$c_4\left\{(x+v_x)^2 + (y+v_y)^2 + (z+v_z)^2 - [c_1(x+v_x) + c_2(y+v_y) + c_3(z+v_z)]^2\right\} + c_5(x+v_x) + c_6(y+v_y) + c_7(z+v_z) + c_8 = 0 \quad (27.37)$$

The linearized form is

$$\frac{\partial C}{\partial x}v_x + \frac{\partial C}{\partial y}v_y + \frac{\partial C}{\partial z}v_z + \frac{\partial C}{\partial c_1}dc_1 + \frac{\partial C}{\partial c_2}dc_2 + \frac{\partial C}{\partial c_3}dc_3 + \frac{\partial C}{\partial c_4}dc_4 + \frac{\partial C}{\partial c_5}dc_5 + \frac{\partial C}{\partial c_6}dc_6 + \frac{\partial C}{\partial c_7}dc_7 + \frac{\partial C}{\partial c_8}dc_8 = c_{40}\left\{(x+v_x)^2 + (y+v_y)^2 + (z+v_z)^2 - [c_{10}(x+v_x) + c_{20}(y+v_y) + c_{30}(z+v_z)]^2\right\} + c_{50}(x+v_x) + c_{60}(y+v_y) + c_{70}(z+v_z) + c_{80} \quad (27.38)$$

where $c_{10}, c_{20}, c_{30}, c_{40}, c_{50}, c_{60}, c_{70}, c_{80}$ are initial estimates for the cylinder parameters. Matrix B is

$$B = \begin{bmatrix} \frac{\partial C}{\partial x_1} & \frac{\partial C}{\partial y_1} & \frac{\partial C}{\partial z_1} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{\partial C}{\partial x_2} & \frac{\partial C}{\partial y_2} & \frac{\partial C}{\partial z_2} & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & \frac{\partial C}{\partial x_{n-1}} & \frac{\partial C}{\partial y_{n-1}} & \frac{\partial C}{\partial z_{n-1}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \frac{\partial C}{\partial x_n} & \frac{\partial C}{\partial y_n} & \frac{\partial C}{\partial z_n} \end{bmatrix} \tag{27.39}$$

The partial derivatives for the B matrix are

$$\frac{\partial C}{\partial x} = c_4[2x - 2c_1(c_1x + c_2y + c_3z)] + c_5$$
$$\frac{\partial C}{\partial y} = c_4[2y - 2c_2(c_1x + c_2y + c_3z)] + c_6$$
$$\frac{\partial C}{\partial z} = c_4[2z - 2c_3(c_1x + c_2y + c_3z)] + c_7$$

Matrix J becomes

$$J = \begin{bmatrix} \frac{\partial C}{\partial c_1}\Big|_1 & \frac{\partial C}{\partial c_2}\Big|_1 & \frac{\partial C}{\partial c_3}\Big|_1 & \frac{\partial C}{\partial c_4}\Big|_1 & \frac{\partial C}{\partial c_5}\Big|_1 & \frac{\partial C}{\partial c_6}\Big|_1 & \frac{\partial C}{\partial c_7}\Big|_1 & \frac{\partial C}{\partial c_8}\Big|_1 \\ \frac{\partial C}{\partial c_1}\Big|_2 & \frac{\partial C}{\partial c_2}\Big|_2 & \frac{\partial C}{\partial c_3}\Big|_2 & \frac{\partial C}{\partial c_4}\Big|_2 & \frac{\partial C}{\partial c_5}\Big|_2 & \frac{\partial C}{\partial c_6}\Big|_2 & \frac{\partial C}{\partial c_7}\Big|_2 & \frac{\partial C}{\partial c_8}\Big|_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial C}{\partial c_1}\Big|_n & \frac{\partial C}{\partial c_2}\Big|_n & \frac{\partial C}{\partial c_3}\Big|_n & \frac{\partial C}{\partial c_4}\Big|_n & \frac{\partial C}{\partial c_5}\Big|_n & \frac{\partial C}{\partial c_6}\Big|_n & \frac{\partial C}{\partial c_7}\Big|_n & \frac{\partial C}{\partial c_8}\Big|_n \end{bmatrix} \tag{27.40}$$

where for $\frac{\partial C}{\partial c_\#}\Big|_n$ the n refers to the point number to be used in the substitution. The partial derivatives for the J matrix are

$$\frac{\partial C}{\partial c_1} = -2c_4x(c_1x + c_2y + c_3z)$$
$$\frac{\partial C}{\partial c_2} = -2c_4y(c_1x + c_2y + c_3z)$$
$$\frac{\partial C}{\partial c_3} = -2c_4z(c_1x + c_2y + c_3z)$$
$$\frac{\partial C}{\partial c_4} = x^2 + y^2 + z^2 - (c_1x + c_2y + c_3z)^2$$

$$\frac{\partial C}{\partial c_5} = x$$
$$\frac{\partial C}{\partial c_6} = y$$
$$\frac{\partial C}{\partial c_7} = z$$
$$\frac{\partial C}{\partial c_8} = 1$$

The K matrix becomes

$$K = \begin{bmatrix} 0 - \left\{c_{40}\left[x_1^2 + y_1^2 + z_1^2 - (c_{10}x_1 + c_{20}y_1 + c_{30}z_1)^2\right] + c_{50}x_1 + c_{60}y_1 + c_{70}z_1 + c_{80}\right\} \\ 0 - \left\{c_{40}\left[x_2^2 + y_2^2 + z_2^2 - (c_{10}x_2 + c_{20}y_2 + c_{30}z_2)^2\right] + c_{50}x_2 + c_{60}y_2 + c_{70}z_2 + c_{80}\right\} \\ \vdots \\ 0 - \left\{c_{40}\left[x_n^2 + y_n^2 + z_n^2 - (c_{10}x_n + c_{20}y_n + c_{30}z_n)^2\right] + c_{50}x_n + c_{60}y_n + c_{70}z_n + c_{80}\right\} \end{bmatrix} \tag{27.41}$$

Following Equation (22.5), the matrix form for the cylinder estimation problem is written as

$$BV + JX = K \tag{27.42}$$

The cylinder estimation problem uses three constraints to provide robustness in the estimation. The first constraint normalizes the direction vector of the cylinder's axis to a unit vector, the second constraint places the axis at the point nearest to the origin, and the third constraint is a normalization condition for the cylinder to assist in the estimation problem. The three constraints are

$$\begin{aligned} &\text{Constraint 1: } c_1^2 + c_2^2 + c_3^2 - 1 = 0 \\ &\text{Constraint 2: } c_1c_5 + c_2c_6 + c_3c_7 = 0 \\ &\text{Constraint 3: } c_5^2 + c_6^2 + c_7^2 - 4c_4c_8 - 1 = 0 \end{aligned} \tag{27.43}$$

The system of equations needs to be expanded using Helmert's method presented in Chapter 20. The partial derivatives of the constraints are

$$C = \begin{bmatrix} 2c_1 & 2c_2 & 2c_3 & 0 & 0 & 0 & 0 & 0 \\ c_5 & c_6 & c_7 & 0 & c_1 & c_2 & c_3 & 0 \\ 0 & 0 & 0 & -4c_8 & 2c_5 & 2c_6 & 2c_7 & -4c_4 \end{bmatrix} \tag{27.44}$$

The constraints are evaluated as

$$K_C = \begin{bmatrix} 0 - \left(c_1^2 + c_2^2 + c_3^2 - 1\right) \\ 0 - (c_1c_5 + c_2c_6 + c_3c_7) \\ 0 - \left(c_5^2 + c_6^2 + c_7^2 - 4c_4c_8 - 1\right) \end{bmatrix} \tag{27.45}$$

As with the plane and sphere cases, the corresponding covariance matrix for the cylinder estimation is given by Equation (27.22).

Example 27.6 Estimate the cylinder based on the point cloud in the following table. Perform a weighted fit using the general least-squares method with constraints. List the estimated cylinder parameters, the reference standard deviation, residuals, and cylinder parameter uncertainty.

Number	x	y	z	σ_x	σ_y	σ_z
1	14.265	16.814	4.313	0.004	0.003	0.003
2	14.374	16.692	4.502	0.003	0.002	0.008
3	14.489	16.571	4.693	0.002	0.003	0.002
4	14.593	16.443	4.878	0.002	0.002	0.002
5	14.698	16.317	5.061	0.001	0.001	0.002
6	14.191	16.987	4.465	0.002	0.004	0.008
7	14.324	16.832	4.689	0.001	0.001	0.002
8	14.426	16.714	4.873	0.002	0.002	0.007
9	14.538	16.593	5.054	0.004	0.003	0.006
10	14.541	16.492	5.267	0.001	0.003	0.005
11	14.000	16.904	4.574	0.001	0.001	0.001
12	14.127	16.755	4.800	0.001	0.003	0.003
13	14.234	16.629	4.983	0.001	0.002	0.005
14	14.320	16.465	5.163	0.003	0.002	0.004
15	14.002	16.654	4.626	0.001	0.003	0.002
16	14.127	16.485	4.786	0.002	0.002	0.008
17	14.357	16.224	5.196	0.006	0.004	0.003
18	14.250	16.326	4.941	0.002	0.001	0.002
19	13.963	16.665	4.371	0.005	0.004	0.006
20	14.146	16.541	4.477	0.001	0.002	0.005
21	14.318	16.400	4.672	0.005	0.004	0.006
22	14.483	16.216	4.952	0.002	0.002	0.007

SOLUTION The first step is to find the centroid of the point cloud using Equation (27.9) and subtract it from the x, y, z coordinates. The centroid of this example is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{cent.} = \begin{bmatrix} \dfrac{14.265 + 14.374 + \cdots + 14.483}{22} \\ \dfrac{16.814 + 16.692 + \cdots + 16.216}{22} \\ \dfrac{4.313 + 4.502 + \cdots + 4.952}{22} \end{bmatrix} = \begin{bmatrix} 14.3075 \\ 16.5781 \\ 4.7880 \end{bmatrix}$$

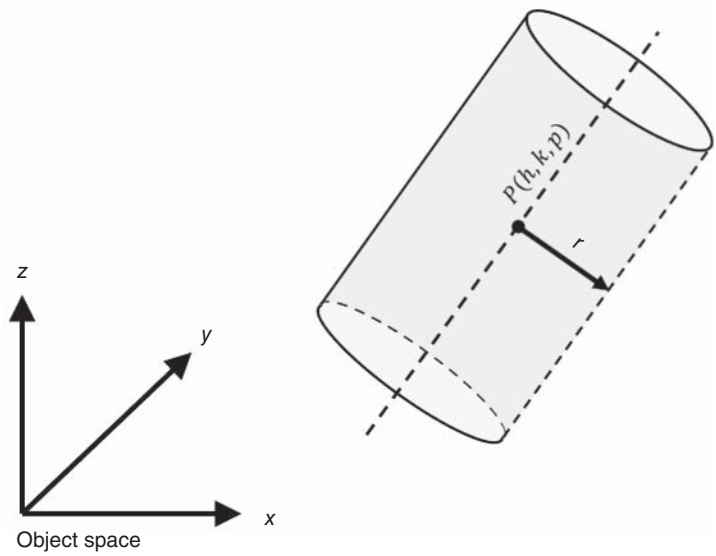


FIGURE 27.3 Cylinder geometry; $P(h, k, p)$ is a point on the cylinder's axis and r is the radius of the cylinder.

By placing the cylinder's axis at the centroid of the cylinder, as shown in Figure 27.3, the initial values for c_{50}, c_{60}, c_{70}, c_{80} simply become 0. To obtain an initial value for c_4, an estimate of the radius of the cylinder is needed. One can compute the distance between the centroid and each of the points and use the minimum distance as an estimate of the radius. The point that generates the minimum distance to the centroid is point 8 in this data set. The distance between the centroid and point 8 is 0.1993. The initial estimate for parameter c_{40} is $c_{40} = \frac{1}{2\times 0.1993} = 2.5089$.

Initial values for the axis direction can be obtained by estimating the plane parameters perpendicular to the cylinder's axis. The normal vector of that plane corresponds to the axis direction parameters c_{10}, c_{20}, c_{30}. In this example, the three closest points to the centroid are used to obtain this plane estimate. The three closest points to the centroid are points 3, 8, and 16 in this data set. To estimate the plane that fits these three points, the unweighted eigenvector plane estimation procedure outlined in Equations (27.11) through (27.13) is followed. The eigenvector approach is followed because of ease of implementation with computer languages, although any of the plane estimation approaches discussed in this chapter can be followed. Using the three closest points 8, 16, 3 and the point cloud's centroid matrix A of Equation (27.11) is formed

$$A = \begin{bmatrix} 14.489 - 14.3075 & 16.571 - 16.5781 & 4.693 - 4.7880 \\ 14.426 - 14.3075 & 16.714 - 16.5781 & 4.873 - 4.7880 \\ 14.127 - 14.3075 & 16.485 - 16.5781 & 4.786 - 4.7880 \end{bmatrix}$$

$$= \begin{bmatrix} 0.1815 & -0.0071 & -0.0950 \\ 0.1185 & 0.1359 & 0.0850 \\ -0.1815 & -0.0931 & -0.0020 \end{bmatrix}$$

The N matrix using Equation (27.12) is

$$N = A^{\mathrm{T}}A = \begin{bmatrix} 0.0796 & 0.0316 & -0.0068 \\ 0.0316 & 0.0272 & 0.0124 \\ -0.0068 & 0.0124 & 0.0163 \end{bmatrix}$$

The eigenvector corresponding to the smallest eigenvalue of value 0.000017 for matrix N is (−0.3265, 0.6799, −0.6566). Therefore, the initial values for the cylinder's axis are $c_{10} = -0.3265$, $c_{20} = 0.6799$, $c_{30} = -0.6566$. For incomplete point clouds (i.e., point clouds that partially scan the object), estimation should be derived using visualization software to plot the points so that appropriate points can be selected.

With all initial values for the cylinder's unknown parameters estimated, the matrices for the least-squares fit can be formed. Using Equation (27.22), the covariance matrix, Σ, of the point cloud coordinate observations is

$$\Sigma = \begin{bmatrix} 0.004^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.003^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.003^2 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0.002^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.002^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.007^2 \end{bmatrix}$$

A partial listing of the B matrix used to propagate the uncertainties to the equivalent weight matrix using Equation (27.39) for the first iteration is

$$B = \begin{bmatrix} 0.583 & -0.475 & -0.782 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.732 & -0.260 & -0.633 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -0.027 & -0.729 & -0.741 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0.207 & -0.414 & -0.532 \end{bmatrix}$$

The equivalent weight matrix for the first iteration using Equation (22.7) is

$$W_e = (B\Sigma B^{\mathrm{T}})^{-1} = \begin{bmatrix} 77126 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 32553 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 236157 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 50282 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 35327 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 67936 \end{bmatrix}$$

Using Equation (27.40), the J matrix for the first iteration is

$$J = \begin{bmatrix} 0.1038 & -0.5754 & 1.1587 & 0.0467 & -0.0425 & 0.2359 & -0.4750 & 1.0000 \\ -0.0812 & -0.1391 & 0.3495 & 0.0399 & 0.0665 & 0.1139 & -0.2860 & 1.0000 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0.3620 & -0.7472 & 0.3384 & 0.0198 & 0.1755 & -0.3621 & 0.1640 & 1.0000 \end{bmatrix}$$

and using Equation (27.41), the K matrix for the first iteration is

$$K = \begin{bmatrix} -0.1172 \\ -0.1000 \\ \vdots \\ -0.0496 \end{bmatrix}$$

Using Equation (27.44), the constraint matrix for the first iteration is

$$C = \begin{bmatrix} -0.6529 & 1.3598 & -1.3130 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.3265 & 0.6799 & -0.6566 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -10.0358 \end{bmatrix}$$

Using Equation (27.45), the K_C matrix for the first iteration is

$$K_C = \begin{bmatrix} 0.0000 \\ 0.0000 \\ 1.0000 \end{bmatrix}$$

Following Figure 20.2, the constrained system of equations is

$$\begin{bmatrix} X \\ X_C \end{bmatrix} = \begin{bmatrix} J^{\mathrm{T}} W_e J & C^{\mathrm{T}} \\ C & 0_{3\times 3} \end{bmatrix}^{-1} \begin{bmatrix} J^{\mathrm{T}} W_e K \\ K_C \end{bmatrix}$$

Note that $0_{3\times 3}$ is a 3-by-3 zero matrix that complements the constrained system of equations. Matrices J, B, W_e, K, C, and K_C must be updated in each iteration using the updated values for the cylinder parameters.

The final values after five iterations are

$$X = \begin{bmatrix} -0.4332 \\ 0.5014 \\ -0.7489 \\ 2.4942 \\ 0.0030 \\ 0.0368 \\ 0.0229 \\ -0.1000 \end{bmatrix}$$

Using Equation (27.35), the estimation of the radius is $r = \frac{1}{|2c_4|} = \frac{1}{|2\times 2.4942|} = 0.2005$, using Equations (27.36a) through (27.36c), the point on the cylinder's axis is $\left(-\frac{c_5}{2c_4} = -0.0006, -\frac{c_6}{2c_4} = -0.0074, -\frac{c_7}{2c_4} = -0.0046\right)$, and the direction of the cylinder's axis is ($c_1 = -0.4332$, $c_2 = -0.5014$, $c_3 = -0.7489$). Adding the centroid values, the point on the cylinder's axis in the object space coordinate system is ($h = 14.3075 - 0.0006 = 14.3069$, $k = 16.5781 - 0.0074 = 16.5708$, $p = 4.7880 - 0.0046 = 4.7834$).

The equivalent cylinder residuals are

$$V_e^{\mathrm{T}} = \begin{bmatrix} -0.0012 & -0.0015 & 0.0025 & -0.0002 & -0.0009 & 0.0012 & \cdots & 0.0009 \end{bmatrix}^{\mathrm{T}}$$

The observational residuals are

$$V = \Sigma B^{\mathrm{T}} W_e V_e = \begin{bmatrix} v_{x_1} \\ v_{y_1} \\ v_{z_1} \\ \vdots \\ v_{x_n} \\ v_{y_n} \\ v_{z_n} \end{bmatrix} = \begin{bmatrix} -0.0012 \\ 0.0000 \\ 0.0004 \\ \vdots \\ 0.0000 \\ -0.0002 \\ -0.0014 \end{bmatrix}$$

The standard deviation of unit weight is $S_0 = 1.1234$. The cylinder orientation uncertainty is $\sigma_{c_1} = S_0\sqrt{q_{x_{c_1},x_{c_1}}} = 0.0020$, $\sigma_{c_2} = S_0\sqrt{q_{x_{c_2},x_{c_2}}} = 0.0026$, and $\sigma_{c_3} = S_0\sqrt{q_{x_{c_3},x_{c_3}}} = 0.0016$. Similarly, the uncertainty for all other estimated parameters can be found. The radius uncertainty can be found through error propagation as $\sigma_r = \sqrt{\left(\frac{-1}{2c_4^2}\right)^2\sigma_{c_4}^2} = 0.0006$. The cylinder's axis point uncertainty can also be found through error propagation as $\sigma_h = \sqrt{\left(\frac{c_5}{2c_4^2}\right)^2\sigma_{c_4}^2 + \left(\frac{-1}{2c_4}\right)^2\sigma_{c_5}^2} = 0.0006$, $\sigma_k = \sqrt{\left(\frac{c_6}{2c_4^2}\right)^2\sigma_{c_4}^2 + \left(\frac{-1}{2c_4}\right)^2\sigma_{c_6}^2} = 0.0008$, and $\sigma_p = \sqrt{\left(\frac{c_7}{2c_4^2}\right)^2\sigma_{c_4}^2 + \left(\frac{-1}{2c_4}\right)^2\sigma_{c_7}^2} = 0.0006$.

27.4 RANSAC ESTIMATION

During scanning, it is possible that other objects or surfaces may interfere with the scanning process creating mixed points and outliers. The Random Sample Consensus (RANSAC) algorithm is a popular method to deal with fitting problems in the presence of outliers. RANSAC is often initially employed during the point cloud segmentation steps to remove objects that do not belong to the main object. For example, for data sets containing floor or ground surfaces, the RANSAC method can be used to segment the ground/floor plane (main fitted object) from other features/objects such as objects and people.

The RANSAC algorithm consists of the following main steps:

1. Random selection of subset data set from the original data set, given certain distance thresholds. This subset is called the inliers, meaning points that truly belong to the fitted object.
2. Fit a model to the "inlier" subset data set.
3. The entire data set is then tested against the fitted model. When a data point is not within a pre-defined error threshold, then this point is considered an outlier. Points that were initially selected as inliers are likewise reclassified as outliers if they do not meet the threshold.
4. The model is then improved by estimating the model again, this time using all the inlier points. The inlier subset of the data set is called the consensus set.
5. Steps 2 to 4 are repeated until there are no more outliers.

The convergence to a good model depends on the initial subset of data. Running the RANSAC with a different subset, and depending on the complexity of the problem, could lead to different results, and hence to different models. Therefore, caution should be placed on selecting the initial subset. An approach to deal with this shortcoming is to run the algorithm a fixed number of times, reject the models that produce small consensus sets, and keep the model with the largest consensus set. Then, subsequently, other objects or surfaces with large consensus can be fit in additional iterations until there are no more objects or surfaces with a sufficient number of points.

27.5 POLYNOMIAL AND ELEVATION ESTIMATION WITH POINT CLOUDS



Point clouds can also be used to model natural surfaces. Such surfaces can be in the form of generic polynomial> models for simple and complex surfaces.

The general equation of a 2D polynomial model is expressed as

$$f(x,y) = \sum_{i=1}^{k}\sum_{j=1}^{k} a_{ij}x^i y^j \tag{27.46}$$

where a_{ij} are polynomial parameters to be estimated through a least- squares fit to the data set $z = \{z_i = (x_i y_i)\}$, $i = 1, 2, \ldots n$, and k defines the order of the polynomial. To provide models that are statistically significant, the estimated polynomial parameters should be tested for their statistical significance using the methods developed in Chapter 18.

A common surveying data product is a DEM that is often constructed from laser scanning or photogrammetric point cloud data. Whether nonterrain objects have been removed or not, the terms Digital Terrain Model (DTM) or Digital Surface Model (DSM) are used. In this section, examples and problems assume that most nonterrain objects (e.g., cars, trees) have been removed. The main interpolation steps include: (1) creating a grid that encompasses all points where an appropriate grid step size is selected to create the grid, based on the point density of the point cloud data, and the scale of the project. (2) Neighboring points within the neighborhood of each grid point are searched and selected. (3) Next, for each grid point, a surface is fitted (using the points within the neighborhood of the grid point) and an elevation is estimated at the grid point location. Through error propagation of the estimated surface, each elevation point is accompanied by an uncertainty estimation. For the elevation estimation at each grid point, a simple bilinear interpolation can be used, although more complex surfaces can be used based on the general Equation (27.46). The bilinear interpolation is given as

$$f(x,y) = a_{00} + a_{10}x + a_{01}y + a_{11}xy = z + v_z \tag{27.47}$$

where x, y, z are the point cloud coordinates that are used to estimate the surface in each grid point of the DEM; and v_z are the residuals of the elevations z. The estimation of a bilinear surface is a linear least- squares problem.

In matrix representation, this becomes

$$AX = L + V \tag{27.48}$$

The associated matrices are given as

$$A = \begin{bmatrix} 1 & x_1 & y_1 & x_1y_1 \\ 1 & x_2 & y_2 & x_2y_2 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & y_n & x_ny_n \end{bmatrix}, \quad X = \begin{bmatrix} a_{00} \\ a_{10} \\ a_{01} \\ a_{11} \end{bmatrix}, \quad L = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}, \quad V = \begin{bmatrix} v_{z_1} \\ v_{z_2} \\ \vdots \\ v_{z_n} \end{bmatrix} \tag{27.49}$$

For numerical stability, it is best to subtract the grid point from the point cloud coordinates, similarly, to subtracting the centroid in previous examples of this

chapter. Equation (27.47) can also be modified to include the uncertainty in the x- and y-coordinates using the general least-squares method in Chapter 22.

Example 27.7 Using the data in the following table and the bilinear interpolation model, what is the elevation of a point with coordinates (10.000, 50.000) and its uncertainty?

Number	x	y	z	σ_z	Number	x	y	z	σ_z
1	9.997	49.992	20.001	0.003	8	9.982	49.995	19.997	0.001
2	10.002	50.004	19.998	0.003	9	9.987	50.016	20.000	0.001
3	10.009	50.016	20.002	0.001	10	10.004	50.007	20.000	0.001
4	10.002	50.006	19.997	0.001	11	10.013	49.988	20.000	0.002
5	10.014	49.985	20.001	0.003	12	9.984	49.987	20.002	0.003
6	9.990	50.004	20.003	0.003	13	10.008	50.003	19.998	0.001
7	10.003	49.982	19.998	0.002	14	10.005	50.016	20.001	0.003

SOLUTION The grid point coordinates (10.000, 50.000) are subtracted from all points for numerical stability. Using Equation (27.49), the least-squares estimation matrices are

$$A = \begin{bmatrix} 1.000 & 9.997 - 10.000 & 49.992 - 50.000 & (9.997 - 10.000)(49.992 - 50.000) \\ 1.000 & 10.002 - 10.000 & 50.004 - 50.000 & (10.002 - 10.000)(50.004 - 50.000) \\ \vdots & \vdots & \vdots & \vdots \\ 1.000 & 10.005 - 10.000 & 50.016 - 50.000 & (10.005 - 10.000)(50.016 - 50.000) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -0.003 & -0.008 & 2.4 \times 10^{-5} \\ 1 & 0.002 & 0.004 & 0.8 \times 10^{-5} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0.005 & 0.016 & 8.0 \times 10^{-5} \end{bmatrix}$$

$$X = \begin{bmatrix} a_{00} \\ a_{10} \\ a_{01} \\ a_{11} \end{bmatrix}, \quad L = \begin{bmatrix} 20.001 \\ 19.998 \\ \vdots \\ 20.001 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 0.003^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.003^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.001^2 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0.003^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.001^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.003^2 \end{bmatrix}$$

$$N = A^{\mathrm{T}}WA = \begin{bmatrix} 7166666.667 & -4888.889 & 34166.667 & 45.278 \\ -4888.889 & 768.056 & 45.278 & 1.547 \\ 34166.667 & 45.278 & 830.889 & -0.267 \\ 45.278 & 1.547 & -0.267 & 0.091 \end{bmatrix}$$

$$A^{\mathrm{T}}WL = \begin{bmatrix} 143327500.000 \\ -97734.833 \\ 683362.111 \\ 905.536 \end{bmatrix}$$

Thus, the estimation of the bilinear interpolation parameters is

$$X = N^{-1}A^{\mathrm{T}}WL = \begin{bmatrix} 19.9988 \\ 0.04407 \\ 0.08008 \\ -0.15100 \end{bmatrix}$$

The residuals are

$$V^{\mathrm{T}} = \begin{bmatrix} -0.0029 & 0.0012 & -0.0015 & 0.0024 & -0.0027 & -0.0043 & \cdots & -0.0007 \end{bmatrix}^{\mathrm{T}}$$

The standard deviation of unit weight is $S_0 = 1.3501$. With the bilinear model estimated, the elevation at the grid point can be interpolated. Equation (27.47) is modified to include the subtraction of the local origin (10.000, 50.000). Thus, the elevation estimation is performed using the equation

$$\begin{aligned} z = {} & 19.9988 + 0.04407(x - 10.000) + 0.08008(y - 50.000) \\ & - 0.15100(x - 10.000)(y - 50.000) \end{aligned}$$

Because the grid point was selected as the local origin, and the elevation is needed at the local origin coordinates, values $x = 10.000$ and $y = 50.000$ are used in the above equation. The elevation at this grid point is simply 19.9988. The uncertainty can be propagated to the estimated elevation using the estimated parameter uncertainty (obtained as the square root of the diagonal elements in matrix $\Sigma_{XX} = S_0^2 N^{-1}$).

$$\begin{aligned} \sigma_z = {} & \sqrt{\begin{array}{l} \sigma_{a_{00}}^2 + (x - 10.000)^2\sigma_{a_{10}}^2 + (y - 50.000)^2\sigma_{a_{01}}^2 \\ \quad + [(x - 10.000)(y - 50.000)]^2\sigma_{a_{11}}^2 \end{array}} \\ = {} & 0.0006. \end{aligned}$$

27.6 SOFTWARE

For those interested in a high-level language, the MathCad worksheet C27.XMCD has been created to demonstrate the solution of the examples in this chapter. For those who do not have MathCad, you can view an html version of the worksheet (C27.html) in your browser. In addition, Matlab .m files are provided for each example with instructions on how to load the data and run the .m files.

PROBLEMS

Note: Partial answers to problems marked with an asterisk are given in Appendix G.

27.1 Estimate the plane that passes through the following point cloud. Set $c = 1$ and perform a weighted fit. Estimate:

***(a)** the normalized plane parameters.

(b) the plane residuals.

(c) the reference standard deviation.

(d) the uncertainty of the plane parameters.

Number	x	y	z	σ_z
1	4.490	9.022	1.954	0.004
2	4.484	9.496	1.999	0.002
3	4.499	9.981	2.047	0.005
4	4.508	10.502	2.109	0.021
5	4.496	11.013	2.122	0.011
6	4.497	11.495	2.177	0.011
7	4.990	8.999	1.903	0.010
8	5.005	9.491	1.952	0.003
9	4.986	9.988	2.003	0.002
10	4.995	10.500	2.045	0.005
11	5.014	10.996	2.079	0.007
12	5.023	11.480	2.121	0.007
13	5.498	9.023	1.875	0.004
14	5.485	9.503	1.914	0.004
15	5.495	9.992	1.955	0.005
16	5.508	10.508	1.985	0.015
17	5.521	10.992	2.037	0.004
18	5.516	11.498	2.090	0.004

27.2 Estimate the plane that passes through the following point cloud. Set $c = 1$ and perform a weighted fit. Estimate:

(a) the normalized plane parameters.

(b) the plane residuals.

(c) the reference standard deviation.

(d) the uncertainty of the plane parameters.

Number	x	y	z	σ_z
1	9.503	19.020	0.824	0.004
2	9.501	19.507	0.907	0.002
3	9.504	20.001	1.005	0.005
4	9.487	20.487	1.082	0.021
5	9.495	20.986	1.177	0.011
6	9.508	21.480	1.263	0.011
7	10.000	19.020	0.830	0.010
8	10.003	19.509	0.915	0.003
9	9.990	19.997	1.004	0.002
10	10.003	20.498	1.081	0.005
11	10.007	20.988	1.173	0.007
12	9.995	21.482	1.259	0.007
13	10.494	19.017	0.826	0.004
14	10.500	19.500	0.911	0.004
15	10.499	20.009	1.000	0.005
16	10.499	20.491	1.082	0.015
17	10.508	20.987	1.167	0.004
18	10.506	21.468	1.261	0.004

27.3 Using the data in Problem 27.1, estimate the plane that passes through the point cloud. Use the eigenvector approach and perform a weighted fit.

27.4 Using the data in Problem 27.2, estimate the plane that passes through the point cloud. Use the eigenvector approach and perform a weighted fit.

27.5 Perform Problem 27.1 by doing a weighted fit using the general least-squares method with the following point uncertainties. Use initial plane values $a = 0$, $b = 0$, $c = 1$. Estimate:

(a) the plane parameters.

(b) the equivalent and observational residuals.

(c) the reference standard deviation.

(d) the uncertainty of the plane parameters.

Number	σ_x	σ_y	σ_z	Number	σ_x	σ_y	σ_z
1	0.004	0.010	0.004	10	0.009	0.007	0.005
2	0.010	0.006	0.002	11	0.007	0.007	0.007
3	0.004	0.010	0.005	12	0.012	0.010	0.007
4	0.004	0.006	0.021	13	0.007	0.010	0.004
5	0.012	0.010	0.011	14	0.010	0.007	0.004
6	0.010	0.009	0.011	15	0.005	0.008	0.005
7	0.005	0.005	0.010	16	0.007	0.009	0.015
8	0.009	0.011	0.003	17	0.010	0.005	0.004
9	0.010	0.012	0.002	18	0.007	0.004	0.004

27.6 Perform Problem 27.2 by doing a weighted fit using the general least-squares method with the following point uncertainties. Use initial plane values $a = 0$, $b = 0$, $c = 1$. Estimate:

(a) the plane parameters

(b) the equivalent and observational residuals

(c) the reference standard deviation

(d) the uncertainty of the plane parameters

Number	σ_x	σ_y	σ_z	Number	σ_x	σ_y	σ_z
1	0.003	0.004	0.004	10	0.003	0.004	0.007
2	0.003	0.002	0.006	11	0.003	0.003	0.002
3	0.004	0.002	0.005	12	0.005	0.005	0.002
4	0.010	0.006	0.004	13	0.006	0.002	0.003
5	0.005	0.003	0.003	14	0.002	0.003	0.002
6	0.008	0.003	0.002	15	0.003	0.008	0.003
7	0.005	0.005	0.003	16	0.003	0.002	0.005
8	0.003	0.002	0.003	17	0.008	0.008	0.007
9	0.010	0.002	0.004	18	0.006	0.009	0.003

27.7 Using the following data set, estimate the plane that passes through the point cloud. Perform a weighted fit and the general least-squares method. Use initial plane values $a = 0$, $b = 0.7$, $c = -0.7$. Estimate:

(a) the plane parameters.

(b) the equivalent and observational residuals.

(c) the reference standard deviation.

(d) the uncertainty of the plane parameters.

Number	x	y	z	σ_x	σ_y	σ_z
1	19.494	19.289	0.791	0.006	0.004	0.004
2	19.502	19.640	1.152	0.002	0.002	0.009
3	19.494	19.983	1.496	0.006	0.015	0.009
4	19.502	20.358	1.849	0.002	0.002	0.006
5	19.490	20.712	2.221	0.010	0.014	0.006
6	19.487	21.054	2.557	0.013	0.007	0.014
7	19.993	19.294	0.779	0.007	0.009	0.011
8	19.998	19.639	1.133	0.011	0.014	0.004
9	19.995	19.994	1.508	0.005	0.005	0.010
10	19.999	20.344	1.854	0.008	0.007	0.007
11	19.992	20.727	2.212	0.008	0.017	0.011
12	19.982	21.049	2.563	0.007	0.007	0.009
13	20.504	19.283	0.801	0.004	0.005	0.013
14	20.520	19.655	1.147	0.011	0.007	0.006
15	20.512	20.000	1.517	0.012	0.012	0.012
16	20.490	20.357	1.844	0.010	0.009	0.009
17	20.498	20.720	2.220	0.003	0.003	0.004
18	20.494	21.064	2.574	0.006	0.005	0.007

27.8 Estimate the sphere center and radius that passes through the following point cloud. Perform a weighted fit using the general least-squares method. For initial values, use the centroid of the point cloud and the distance between the centroid and the first point. Estimate:

***(a)** the sphere parameters.

(b) the equivalent and observational residuals.

(c) the reference standard deviation.

(d) the uncertainty of the sphere parameters.

Number	x	y	z	σ_x	σ_y	σ_z
1	5.001	5.001	4.749	0.002	0.002	0.003
2	4.798	5.006	4.850	0.001	0.003	0.001
3	4.761	4.988	5.080	0.005	0.004	0.009
4	4.921	5.001	5.240	0.002	0.002	0.002
5	4.872	4.819	4.889	0.007	0.008	0.009
6	4.929	4.791	5.112	0.004	0.008	0.007
7	5.074	4.785	4.883	0.004	0.005	0.007
8	5.138	4.805	5.077	0.001	0.002	0.003
9	5.211	4.932	4.889	0.005	0.007	0.006
10	5.119	4.960	5.228	0.002	0.002	0.003
11	5.242	5.075	5.033	0.005	0.003	0.008
12	5.107	5.157	4.851	0.005	0.007	0.003
13	5.087	5.235	5.038	0.008	0.007	0.006
14	4.849	5.193	4.937	0.004	0.006	0.003

27.9 Estimate the sphere center and radius that passes through the following point cloud. Perform a weighted fit using the general least-squares method. For initial values, use the centroid of the point cloud and the distance between the centroid and the first point. Estimate:

(a) the sphere parameters.

(b) the equivalent and observational residuals.

(c) the reference standard deviation.

(d) the uncertainty of the sphere parameters.

Number	x	y	z	σ_x	σ_y	σ_z
1	10.000	10.000	9.949	0.001	0.001	0.002
2	9.963	9.998	9.965	0.005	0.004	0.004
3	9.951	9.997	10.004	0.004	0.001	0.001
4	9.970	9.999	10.039	0.001	0.001	0.001
5	9.972	9.962	9.983	0.003	0.002	0.001
6	9.984	9.958	10.024	0.003	0.003	0.003
7	10.009	9.963	9.964	0.002	0.001	0.001
8	10.007	9.987	10.048	0.002	0.002	0.002
9	10.031	9.958	10.002	0.001	0.003	0.002
10	10.044	9.988	9.972	0.004	0.004	0.003
11	10.042	9.988	10.024	0.001	0.001	0.002
12	10.048	10.018	9.998	0.004	0.001	0.002
13	10.020	10.030	9.965	0.003	0.001	0.001
14	10.026	10.044	10.018	0.001	0.002	0.003
15	9.985	10.045	9.985	0.001	0.002	0.003
16	9.989	10.037	10.031	0.001	0.002	0.001

27.10 Estimate the sphere center and radius that passes through the point cloud of Problem 27.8. Perform a weighted fit and use the linear equation method. Estimate:

(a) the sphere parameters.

(b) the equivalent and observational residuals.

(c) the reference standard deviation.

(d) the uncertainty of the sphere parameters.

27.11 Estimate the sphere center and radius that passes through the point cloud of Problem 27.9. Perform a weighted fit and use the linear equation method. Estimate:

(a) the sphere parameters.

(b) the equivalent and observational residuals.

(c) the reference standard deviation.

(d) the uncertainties of the sphere parameters.

27.12 Estimate the cylinder based on the following point cloud. Perform a weighted fit using the general least-squares method with constraints. Estimate:

(a) the initial values. For the initial radius, use point 7, and for the normal estimation, use points 3, 7, and 13.

***(b)** the cylinder parameters.

(c) the equivalent and observational residuals.

(d) the reference standard deviation.

(e) the uncertainty of the cylinder parameters.

Number	x	y	z	σ_x	σ_y	σ_z
1	10.098	20.009	1.169	0.001	0.001	0.003
2	10.102	19.999	1.338	0.006	0.005	0.009
3	10.097	19.997	1.501	0.004	0.003	0.008
4	10.106	20.003	1.660	0.004	0.004	0.004
5	10.099	20.003	1.831	0.006	0.007	0.008
6	10.097	19.995	2.001	0.005	0.003	0.010
7	10.029	20.106	1.579	0.006	0.003	0.008
8	10.008	20.100	1.252	0.007	0.007	0.013
9	10.004	20.102	1.409	0.004	0.006	0.005
10	10.009	20.097	1.741	0.009	0.004	0.005
11	10.009	20.102	1.911	0.004	0.006	0.004
12	9.977	20.104	1.041	0.003	0.009	0.007
13	9.901	20.036	1.502	0.005	0.002	0.008
14	9.903	20.035	1.998	0.004	0.006	0.002
15	9.902	20.006	1.171	0.003	0.002	0.001
16	9.903	20.018	1.337	0.005	0.004	0.002
17	9.899	20.016	1.659	0.009	0.005	0.007
18	9.911	20.015	1.834	0.002	0.004	0.002
19	9.966	19.905	1.545	0.004	0.003	0.003
20	9.990	19.906	1.243	0.006	0.001	0.003
21	9.986	19.899	1.714	0.003	0.003	0.011
22	9.979	19.901	1.915	0.001	0.004	0.002

27.13 Estimate the cylinder based on the following point cloud. Perform a weighted fit using the general least-squares method with constraints. Estimate:

(a) the initial values. For the initial radius, use point 11; for the normal estimation, use points 3, 11, and 22.

(b) the cylinder parameters.

(c) the equivalent and observational residuals.

(d) the reference standard deviation.

(e) the uncertainty of the cylinder parameters.

Number	x	y	z	σ_x	σ_y	σ_z
1	10.198	10.003	10.001	0.002	0.001	0.002
2	10.203	9.753	10.005	0.002	0.002	0.005
3	10.207	9.510	10.003	0.002	0.002	0.001
4	10.199	9.259	10.008	0.004	0.005	0.004
5	10.207	9.001	10.011	0.004	0.003	0.003
6	10.103	9.877	10.171	0.003	0.004	0.003
7	10.063	9.621	10.191	0.002	0.003	0.002
8	10.060	9.375	10.192	0.004	0.004	0.008
9	10.062	9.119	10.192	0.002	0.001	0.003
10	9.878	9.994	10.146	0.005	0.004	0.005
11	9.862	9.506	10.152	0.002	0.005	0.003
12	9.837	9.751	10.121	0.003	0.001	0.003
13	9.841	9.266	10.125	0.005	0.003	0.003
14	9.843	9.002	10.122	0.008	0.009	0.008
15	9.809	9.918	9.924	0.007	0.009	0.007
16	9.809	9.587	9.927	0.004	0.004	0.002
17	9.816	9.124	9.925	0.007	0.005	0.009
18	9.875	9.370	9.852	0.003	0.003	0.006
19	9.962	9.751	9.812	0.011	0.009	0.010
20	10.016	9.998	9.800	0.003	0.003	0.002
21	10.015	9.179	9.798	0.008	0.007	0.007
22	10.063	9.507	9.815	0.001	0.001	0.001

27.14 Estimate the bilinear interpolation model and the elevation at grid point (10.150, 20.075) and its uncertainty. The points below are given in the vicinity of the grid point. Estimate:

***(a)** the interpolation parameters.

(b) the residuals.

(c) the reference standard deviation and interpolation parameter uncertainty.

(d) the interpolated elevation.

(e) the uncertainty of the interpolated elevation.

Number	x	y	z	σ_z	Number	x	y	z	σ_z
1	10.110	20.051	5.027	0.007	9	10.150	20.095	5.030	0.008
2	10.086	20.069	5.024	0.005	10	10.126	20.112	5.027	0.004
3	10.174	20.032	5.025	0.005	11	10.143	20.136	5.027	0.004
4	10.195	20.062	5.026	0.004	12	10.127	20.075	5.027	0.003
5	10.155	20.010	5.032	0.011	13	10.171	20.080	5.023	0.005
6	10.167	20.118	5.030	0.007	14	10.194	20.099	5.026	0.003
7	10.135	20.033	5.029	0.004	15	10.109	20.088	5.027	0.003
8	10.152	20.057	5.030	0.008	16	10.212	20.086	5.026	0.003

27.15 Estimate the bilinear interpolation model and the elevation at grid point (12.450, 10.015) and its uncertainty. The points below are given in the vicinity of the grid point. Estimate:

(a) the interpolation parameters.

(b) the residuals.

(c) the reference standard deviation and interpolation parameter uncertainty.

(d) the interpolated elevation.

(e) the uncertainty of the interpolated elevation.

Number	x	y	z	σ_z	Number	x	y	z	σ_z
1	12.456	10.088	0.238	0.008	9	12.427	10.191	0.210	0.004
2	12.406	10.161	0.232	0.012	10	12.471	10.195	0.206	0.008
3	12.385	10.148	0.237	0.015	11	12.451	10.128	0.228	0.002
4	12.513	10.156	0.218	0.004	12	12.489	10.173	0.215	0.006
5	12.477	10.110	0.224	0.004	13	12.470	10.151	0.222	0.006
6	12.494	10.133	0.224	0.003	14	12.446	10.213	0.203	0.016
7	12.450	10.174	0.219	0.004	15	12.425	10.147	0.233	0.005
8	12.415	10.127	0.242	0.014	16	12.432	10.106	0.235	0.003

CHAPTER 28

COMPUTER OPTIMIZATION

28.1 INTRODUCTION

Large amounts of computer time and storage requirements can be used when performing least-squares adjustments. This is due to the fact that as the problems become more complex, the matrices become larger, and the storage requirements and time consumed in doing numerical operations grow rapidly. As an example, in analyzing the storage requirements of a 25-station horizontal least-squares adjustment that has 50 distance and 50 angle observations, the coefficient matrix would have dimensions of 100 rows and 50 columns. If this adjustment were done in double precision,[1] it would require 40,000 bytes of storage for the coefficient matrix alone. The weight matrix would require an additional 80,000 bytes of storage. Also, at least two additional similarly sized intermediate matrices[2] must be formed in computing the solution. From this example, it is easy to see that large quantities of computer time and computer memory can be required in least-squares adjustments. Thus, when writing least-squares software, it is desirable to take advantage

[1]The storage requirement of a double−precision number is eight bytes.
[2]The intermediate matrices developed are A^{T} and $A^{\mathrm{T}}W$ or J^{T} and $J^{\mathrm{T}}W$ depending on whether the adjustment is linear or nonlinear.

of some storage and computing optimization techniques. In this chapter, some of these techniques are described.

28.2 STORAGE OPTIMIZATION

Many matrices used in a surveying adjustment are large but sparse. Using the example above, a single row of the coefficient matrix for a distance observation would require a 50-element row for its four nonzero elements. In fact, the entire coefficient matrix is sparsely populated by nonzero values. Similarly, the normal matrix is always symmetric, and thus nearly half its storage is used by duplicate entries. It is relatively easy to take advantage of these conditions to reduce the storage requirements.

For the normal matrix, only its upper- or lower-triangular portion of the matrix needs to be saved. In this storage scheme, the two-dimensional matrix is saved as a vector. An example of a 4×4 normal matrix is shown in Figure 28.1. Its upper- and lower-triangular components are shown separated, and their elements numbered for reference. The vector on the right of Figure 28.1 shows the storage scheme. This scheme eliminates the need to save the duplicate entries of the normal matrix, but requires some form of *relational mapping* between the original matrix indices of row i and column j and the vector's index of row i. For the upper triangular portion of the matrix, it can be shown that the vector index for any (i, j) element is computed as

$$\text{Index}(i,j) = 1/2[j(j-1)] + i \tag{28.1}$$

Using the function given in Equation (28.1), the vector index for element (2, 3) of the normal matrix would be computed as

$$\text{Index}(2,3) = 1/2[3(3-1)] + 2 = 5$$

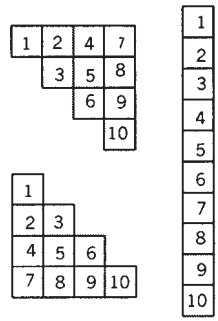


FIGURE 28.1 Structure of the normal matrix.

An equivalent mapping formula for the lower-triangular portion of the matrix is given as

$$\text{Index}(i,j) = 1/2[i(i-1)] + j \tag{28.2}$$

Equation (28.1) or (28.2) can be used to compute the storage location for each element in the upper or lower part of the normal matrix, respectively. Of course, it requires additional computational time to map the location of each element every time it is used. One method of minimizing computing time is to utilize a *mapping table*. The mapping table is a vector of integers that identifies the storage location for only the initial element of a row or column. To reduce total computational time, the indices are stored as one number less than their actual value. For the example matrix stored in Figure 28.1, it would have a relational mapping table of

$$\text{VI}^{\text{T}} = [0 \;\; 1 \;\; 3 \;\; 6 \;\; 10]$$

Although the storage location of the first element is 1, it can be seen above that it is identified as 0. Similarly, every other mapping index is one number less than its actual location in the vector, as shown in Figure 28.1. The mapping vector is found using the pseudocode shown in Table 28.1.

By using this type of mapping table, the storage location for the first element of any upper-triangular column is computed as

$$\text{Index}(i,j) = \text{VI}(j) + i \tag{28.3}$$

Using the mapping table, the storage location for the (2, 3) element is

$$\text{Index}(2,3) = \text{VI}(3) + 2 = 3 + 2 = 5$$

The equivalent formula for the lower triangular portion is

$$\text{Index}(i,j) = \text{VI}(i) + j \tag{28.4}$$

Obviously, this method of indexing an element's position requires additional storage in a mapping table, but the additional storage is offset by the decreased computational time that is created by multiplication operations in Equations (28.1) and (28.2). Although every compiler and machine are different, a set of times to compare indexing methods is shown in Table 28.2. The best times were achieved

TABLE 28.1 Creation of a Mapping Table

$\text{VI}(1) = 0$
For i going from 2 to the number of unknowns: $\text{VI}(i) = \text{VI}(i-1) + i - 1$

TABLE 28.2 Comparison of Indexing Methods

Method	Time (sec)	Extra Storage
A	4.932	0
B	3.937	2 bytes per element
C	1.638	2 bytes per element
D	1.204	Full matrix

Function A: Index = j(j − 1)/2 + i

Function B: Index = VI(j) + i;

```
Method A: Using function A
    i from 1 to 1000
        j from i to 1000
            k = Index( i,j )

Method B: Using function B
    i from 1 to 1000
        j from i to 1000
            k = Index( i,j )

Method C: Direct access to mapping table
    i from 1 to 1000
        j from i to 1000
            k = VI(j) + I

Method D: Direct matrix element access.
    i from 1 to 1000
        j from 1 to 1000
            k = A[i,j]
```

by minimizing both the number of operations and calls to the mapping function. Thus, in method *C*, the mapping table was created to be a global variable in the program, and the table was accessed directly as needed. For comparison, the time to access the matrix elements directly using the loop shown in method *D* was 1.204 seconds. Thus, method *C* uses only slightly more time than method *D* but helps avoid the additional memory that is required by the full matrix required for method *D*. Meanwhile, method *A* is slower than method *C* by a factor of 3. Since the matrix elements must be accessed repeatedly during a least-squares solution, this difference can be a significant addition to the total solution time.

The additional overhead cost of having the mapping table is minimized by declaring it as an integer array so that each element requires only two bytes of storage. In the earlier example of 25 stations, this results in 100 bytes of additional storage for the mapping table. Speed versus storage is a decision that every programmer will continually face. Unfortunately, today, many programmers disregard speed since memory is cheap. However, a well-designed program is both efficient with storage and fast.

28.3 DIRECT FORMATION OF THE NORMAL EQUATIONS

The basic weighted observation equation form is $WAX = WL + WV$. The least-squares solution for this equation is $X = (A^{\mathrm{T}}WA)^{-1}(A^{\mathrm{T}}WL) = N^{-1}C$, where $A^{\mathrm{T}}WA$ is the normal equations matrix N, and $A^{\mathrm{T}}WL$ is the constant matrix C. If the

weight matrix is diagonal, formation of the A, W, L, and $A^{\mathrm{T}}W$ matrices is unnecessary when building the normal equations, and the storage requirements for them can be eliminated. This is accomplished by forming the normal matrix directly from the observations. The tabular method in Section 11.8 showed the feasibility of this method. Notice in Table 11.2 that the contribution of each observation to the normal matrix is computed individually and, subsequently, added. This shows that there is no need to form the coefficient, constants, weight, or any intermediate matrices when deriving the normal matrix. Conceptually, this is developed as follows:

Step 1: Zero the normal and constant matrices.

Step 2: Zero a single row of the coefficient matrix.

Step 3: Based on the values of a single row of the coefficient matrix, add the proper value to the appropriate normal and constant matrix elements.

Step 4: Repeat step 3 for all observations.

This procedure works in all situations that involve a diagonal weight matrix. Procedural modifications can be developed for weight matrices with limited correlation between the observations. Computer algorithms in BASIC, FORTRAN, C, and Pascal for this method are shown in Table 28.3.

TABLE 28.3 Algorithms for Building the Normal Equations Directly from Their Observations

BASIC Language

```
For i = 1 to unknown
  ixi = VI(i) + i
  N(ixi) = N(ixi) + A(i)^2 * W(i)
  C(i) = C(i) + A(i)*W(i)*L(i)
  For j = i+1 to unknown
    N(VI(i)+j) = N(VI(i)+j)
          + A(i)*W(i)*A(j))
  Next j
Next I
```

FORTRAN Language

```
Do 100 i = 1, unknown
   ixi = VI(i) + i
   N(ixi) = N(ixi) + A(i)**2 * W(i)
   C(i) = C(i) + A(i)*W(i)*L(i)
   Do 100 j = i+1, unknown
      N(VI(i)+j) = N(VI(i)+j)
            + A(i)*W(i)*A(j)
100  Continue
```

C Language

```
for ( i=1; i<=unknown; i++) {
  ixi = VI[i]; + i;
  n[ixi] = n[ixi] + a[i]*a[i] * w[i];
  c[i] = c[i] + a[i]*w[i]*l[i];
  for ( j=i+1; j<=unknown; j++)
    n[VI[i]+j] = n[VI[i]+j]
          + a[i]*w[i]*a[j];
}// for i
```

Pascal Language

```
For i := 1 to unknown do begin
  ixi := VI[i] + i;
  N[ixi] := N[ixi] + Sqr(A[i]) * W[i];
  C[i] := C[i] + A[i]*W[i]*L[i];
  For j := i+1 to unknown do
    N[VI[i]+j] := N[VI[i]+j]
            + A[i]*W[i]*A[j]
End; { for i }
```

28.4 CHOLESKY DECOMPOSITION

Having formed only a triangular portion of the normal matrix, a Cholesky decomposition can be used to greatly reduce the time needed to find the solution. This procedure takes advantage of the fact that the normal matrix is always a positive-definite matrix.[3] Due to this property, it can be expressed as the product of a lower-triangular matrix and its transpose; that is,

$$N = LU = LL^{\mathrm{T}} = U^{\mathrm{T}}U$$

where L is a lower triangular matrix of the form

$$L = \begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{bmatrix}$$

When the normal matrix is stored as a lower-triangular matrix, it can be factored using the following pseudocode:

for $i = 1, 2, \ldots,$ number of unknowns compute

$$l_{ii} = \left(l_{ii} - \sum_{k=1}^{i-1} l_{ik}^2 \right)^{1/2} \tag{28.5}$$

for $j = i + 1, i + 2, \ldots,$ number of unknowns compute

$$l_{ji} = \frac{l_{ji} - \sum_{k=1}^{i-1} l_{ij} l_{jk}}{l_{ii}}$$

The procedures above are shown in code form in Table 28.4.

28.5 FORWARD AND BACKWARD SOLUTIONS

Being able to factor the normal matrix into a triangular matrix has the advantage that the matrix solution can be obtained without the use of an inverse and

[3] A positive-definite square matrix, A, has the property that $X^{\mathrm{T}}AX > 0$ for all nonzero vectors, X.

TABLE 28.4 Computer Algorithms for Computing Cholesky Factors of a Normal Matrix

BASIC Language

```
FOR i = 1 TO unknown
  ixi = VI(i) + i: s = 0#
   FOR k = 1 TO i - 1
       s = s + N(ix + k) ^ 2: NEXT k
   N(ixi) = SQR(N(ixi) - s)
   FOR j = i + 1 TO unknown
     s = 0#
     FOR k = 1 TO i - 1
s = s + N(VI(j) + k) * N(VI(i) + k): NEXT k
     N(VI(j) + i) = (N(VI(j) + i) - s)
     / N(ixi)
   NEXT j
NEXT i
```

FORTRAN Language

```
Do 30 i = 1,Unknown
 ixi = VI(i) + 1
 i1 = i - 1
  S = 0.0
     Do 10 k=1, i1
10   s = s + N(VI(i)+k )**2
  N(ixi) = Sqrt( N(ixi)-s )
   Do 30 j = i+1, Unknown
    s = 0.0
    Do 20 k = 1, i1
20     s = s + N( VI(j)+k )
     * N( VI(i)+k )
       N(VI[j]+i) =
 (N(VI(j)+i) -s) / N(ixi)
   30  Continue
```

C Language

```
for ( i=1; i<=unknown; i++ ) {
 ixi = VI[i] +i; s = 0.0;
 for ( k=1; k<i; k++ )
    s = s + n[VI[i]+k]*N[VI[i]+k];
 N[ixi] = sqrt( N[ixi] - s );
 for ( j=i+1; j<=unknown; j++ ) {
    s = 0;
    for ( k=1; k<i; k++)
      s = s + n[VI[j]+k] * n[VI[i]+k];
    n[jx+i] = ( n[VI[j]+i] - s ) / n[ixi];
 } // for j
} // for i
```

Pascal Language

```
For i := 1 to unknown do Begin
  ixi := VI[i]; S := 0;
  For k := 1 to Pred( i ) do
      S := S + Sqr( N[VI[i]+k] );
  N[ixi] := Sqrt( N[ixi] -S );
  For j := Succ(i) to unknown do Begin
    S := 0;
    For k := 1 to Pred(i) do
        S := S + N[VI[j]+k]*N[VI[i]+k];
    N[VI[j]+i] :=
( N[VI[j]+i] - S ) / N[ixi]
  End; { for j}
End; { for i }
```

requires less storage. The equivalent triangular matrices representing the normal equations are

$$(A^{\mathrm{T}}WA)X = NX = LUX = LL^{\mathrm{T}}X$$

$$= \begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} & \cdots & l_{n1} \\ 0 & l_{22} & l_{32} & \cdots & l_{n2} \\ 0 & 0 & l_{33} & \cdots & l_{n3} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & 0 & l_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix} = C \tag{28.6}$$

Equation (28.6) can be rewritten as $LY = C$, where $L^{\mathrm{T}}X = Y$. From this, the solution for Y can be found by taking advantage of the triangular form of L. This is known as a *forward substitution*. Steps involved in forward substitution are

Step 1: Solve for y_1 as $y_1 = c_1/l_{11}$.
Step 2: Substitute this value into row 2, and compute y_2 as $y_2 = c_2 - l_{21}y_1/l_{22}$.
Step 3: Repeat this procedure until all values for y are found using the algorithm $y_i = c_i - \sum_{k=1}^{i-1} l_{ik}y_k/l_{ii}$.

Having determined Y, the solution for the matrix system $L^TX = Y$ is computed in a manner similar to that above. However, this time, the solution starts at the lower-right corner and proceeds up the matrix L^{T}. This is called a *backward substitution*, which is done with the following steps:

Step 1: Compute x_n as $x_n = y_n/l_{nn}$.
Step 2: Solve for x_{n-1} as $x_{n-1} = y_{n-1} - l_{nn}x_n/l_{(n-1)(n-1)}$.
Step 3: Repeat this procedure until all unknowns are computed using the algorithm $x_k = y_k - \sum_{j=k+1}^{n} l_{kj}y_j/l_{kk}$.

In this process, once the original values in the constant matrix are accessed and changed, they are not needed again, and the original C matrix can be overwritten with Y and X matrices so that the entire process requires no additional storage. In fact, this method of solution also requires fewer operations than solving $X = (A^{\mathrm{T}}WA)^{-1}A^{\mathrm{T}}WL$ directly. Table 28.5 lists the computer codes for these algorithms.

28.6 USING THE CHOLESKY FACTOR TO FIND THE INVERSE OF THE NORMAL MATRIX

If necessary, the original normal matrix inverse can be found with the Cholesky factor. To derive this matrix, the inverse of the Cholesky factor is computed. The normal matrix inverse is the product of this inverse times its transpose. The Cholesky factor inverse is determined using the algorithm in Table 28.6. Code for this algorithm is shown in Table 28.7.

28.7 SPARENESS AND OPTIMIZATION OF THE NORMAL MATRIX

In the least-squares adjustment of most surveying and photogrammetry problems, it is known that certain locations in the normal matrix will contain zeros. The network

TABLE 28.5 Computer Algorithms for Forward and Backward Substitutions

BASIC Language

```
Rem Forward Substitution
For i = 1 to Unknown
  C(i) = C(i) / N(VI(i)+i)
  For j = i+1 to Unknown
    C(j) = C(j) – N(VI(i)+j) * C(i)
  Next j
Next i
Rem Backward Substitution
For i = Unknown to 1 Step –1
  For j = Unknown To i+1 Step –1
    C(i) = N(VI(i)+j) * C(j)
  Next j
  C(i) = C(i) / N(VI(i)+i)
Next i
```

FORTRAN Language

```
C Forward Substitution
  Do 100 i = 1,Unknown
    C(i) = C(i) / N(VI(i)+i)
    Do 100 j = i+1, Unknown
      C(j) = C(j) – N(VI(i)+j) * C(i)
100 Continue
C Backward Substitution
   Do 110 i = Unknown, 1,–1
     Do 120 j = Unknown, i+1, –1
120     C(i) = C(i) – N(VI(i)+j) * C(j)
     C(i) = C(i) / N(VI(i)+i)
110 Continue
```

C Language

```
//Forward Substitution
for (i=1; i<=unknown; i++){
c[i] = c[i]/n[VI[i]+i];
for (j=i+1; j<=unknown; j++){
   c[j] = c[j] – n[VI[i]+j]*c[i];
} //for j
} //for i
//backward substitution
for (i=unknown; i>=1; i––){
 for (j=unknown; j>=i+1; j––){
   c[i] = c[i] – n[VI[i]+j] * c[j];
 } //for j
c[i] = c[i]/n[VI[i]+i];
} // for i
```

Pascal Language

```
{Forward Substitution}
For i := 1 to Unknown Do Begin
  C[i] := C[i] / N[VI[i]+i];
  For j := i+1 to Unknown Do
    C[j] := C[j] – N[VI[i]+j] * C[i];
End; { for i }
{Backward Substitution}
For i := Unknown DownTo 1 do Begin
  For j := Unknown DownTo i+1 Do
     C[i] := C[i] – N[VI[i]+j] * C[j];
  C[i] := C[i] / N[VI[i]+i];
End; { for i }
```

TABLE 28.6 Pseudocode for Algorithm Computing the Inverse of a Cholesky Decomposed Matrix

```
for i going from the number of unknowns down to 1
    for k going from number of unknowns down to i + 1
              for j going from i+1 to k, sum the product N(i, j) * N(j, k)
          N(i, k) = –S / N(i, i)
   N(i, i) = 1/N(i, i)
```

shown in Figure 28.2 can be used to demonstrate this fact. In that figure, assume that distances and angles were observed for every line and arc, respectively. The following observations are made about the network's connectivity. Station 1 is connected to stations 2 and 8 by distance observations and is also connected to stations 2, 3, and 8 by angles. Notice that its connection to station 3 is due to angles turned

TABLE 28.7 Computer Algorithms to Find the Inverse of a Cholesky-Factored Matrix

BASIC Language

```
{Inverse}
For i = Unknown To 1 Step -1
   ixi = ix + i
   For k = Unknown To i+1 Step -1
      S = 0!
      For j = i+1 To k
         S = S + N(VI(i)+j) * N(VI(j)+k)
      Next j
      N(VI(i)+k) = -S / N(ixi)
   Next k
   N(ixi) = 1.0 / N(ixi)
Next i

{Inverse * Transpose of Inverse}
For j = 1 to Unknown
    For k = j to Unknown
       S = 0!
      For i = k to Unknown
        S = S + N(VI(k)+i) * N(VI(j)+i)
      Next i
      N((VI(j)+k) = S
    Next k
Next j
```

Fortran Language

```
C Inverse
 Do 10 i = Unknown, 1,-1
  ixi = VI(i) + i
  Do 11 k = Unknown, i+1,-1
   S = 0.0
   Do 12 j = i+1,k
12   S = S + N(VI(i)+j) * N(VI(j)+k)
   N(VI(i)+k) = -S / N(ixi)
11  Continue
  N(ixi) = 1.0 / N(ixi)
10 Continue

C  Inverse * Transpose of Inverse}
 Do 20 j = 1, Unknown
  Do 21 k = j, Unknown
   S = 0.0
   Do 22  i = k,Unknown
22    S = S + N(VI(k)+i) * N(VI(j)+i)
   N(ixj+k) = S
21  Continue
20 Continue
```

C Language

```
{Inverse}
for ( i=unknown; i>=1; i--) {
   ixi = ix + i;
   for( k=unknown; k>=i+1; k--) {
      S = 0.0;
      for ( j=i+1; j<=k; j++)
         S = S + N[VI[i]+j] * N[VI[j]+k];
      N[VI[i]+k] = -S / N[ixi];
   } // for k
   N[ixi] = 1.0 / N[ixi];
} // for i

{inverse * transpose of inverse}
for (j=1; j<=unknown; j++) {
    for (k=j; k<=unknown; k++) {
       S = 0.0;
      For (i=k; i<=unknown; i++)
        S = S + N[VI[k]+i] * N[VI[j]+i];
      N[VI[j]+k] = S
    } // for k
} //for j
```

Pascal Language

```
{Inverse}
For i := Unknown DownTo 1 do Begin
   ixi := VI[i] + i;
   For k := Unknown DownTo i+1 Do Begin
      S := 0.0;
      For j := i+1 To k Do
         S := S + N[VI[i]+j] * N[VI[j]+k];
      N[VI[i]+k] := -S / N[ixi];
   End; { For k }
   N[ixi] := 1.0 / N[ixi];
End; { For i }

{Inverse * Transpose of Inverse}
For j := 1 to Unknown Do Begin
    For k := j to Unknown Do Begin
       S := 0.0;
      For i := k to Unknown Do
        S := S + N[VI[k]+i] * N[VI[j]+i];
      N[VI[j]+k] := S
    End; { For k }
End; { For j }
```

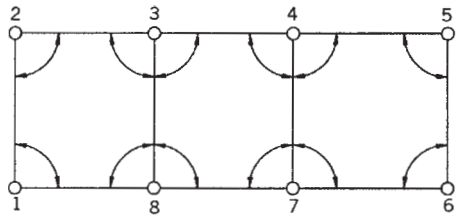


FIGURE 28.2 Horizontal network.

	1	2	3	4	5	6	7	8
1	X	X	X	0	0	0	0	X
2	X	X	X	0	0	0	0	X
3	X	X	X	X	0	0	X	X
4	0	0	X	X	X	X	X	X
5	0	0	0	X	X	X	X	0
6	0	0	0	X	X	X	X	0
7	0	0	X	X	X	X	X	X
8	X	X	X	X	0	0	X	X

FIGURE 28.3 Normal matrix.

at both 2 and 8. Since these angles directly connect stations 1 and 2, it follows that if the coordinates of 3 change, so will the coordinates of 1, and thus, with respect to station 1, the normal matrix can be expected to have nonzero elements corresponding to stations 1, 2, 3, and 8. Conversely, the positions corresponding to stations 4, 5, 6, and 7 will have zeros.

Using this analysis with station 3, because it is connected to stations 1, 2, 4, 7, and 8 by angles, zero elements can be expected in the normal matrix corresponding to stations 5 and 6. This analysis can be made for each station. The resulting symbolic normal matrix representation is shown in Figure 28.3.

A process known as *reordering the unknowns* can minimize both storage and computational time. Examine the matrix shown in Figure 28.4 that results from placing the unknowns of Figure 28.3 in the order of 6, 5, 4, 7, 3, 1, 2, and 8. This new matrix has its nonzero elements immediately adjacent to the diagonal elements, and thus the known zero elements are grouped together and appear in the leftmost columns of their respective matrix rows. By modifying the mapping table, storing the known zero elements of the original matrix in Figure 28.3 can be avoided, as is shown in the column matrix of Figure 28.4.

This storage scheme requires a mapping table that provides the storage location for the first nonzero element of each row. Since the diagonal elements are always nonzero, they can be stored in a separate column matrix without loss of efficiency. Ignoring the fact that each station has two unknowns, the mapping of the first element for each row is 1, 1, 2, 4, 7, 9, 10, 12, and 17. This mapping scheme enables finding the starting position for each row and allows for the determination of the off-diagonal length of the row. That is, row 1 starts at position 1 of the column matrix, but has a length of zero $(1 - 1)$, indicating that there are no off-diagonal elements in this row. Row 2 starts at position 1 of the column matrix and has a length of $(2 - 1) = 1$. Row 5 starts at 7 and has a length of $(9 - 7) = 2$. By optimizing for matrix sparseness and symmetry, only 24 $(16 + 8)$ elements of the original 64–element matrix need to be stored.

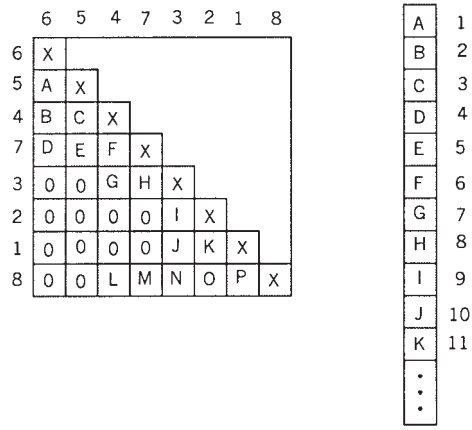


FIGURE 28.4 Reordered the normal matrix.

These storage savings, when exploited, can also result in a savings of computational time. To understand this, first examine how the Cholesky factorization procedure processes the normal matrix. For a lower triangular matrix factorization process, Figure 28.5 shows the way elements are accessed. Notice that when a particular column is modified, the elements to the left of this column are used. Also, notice that no rows above the corresponding diagonal element are used.

To see how the reorganized sparse matrix can be exploited in the factorization process, the processing steps must be understood. In Figure 28.4, the first two elements of row 5 are known zeros, and each summation loop in the factorization process should start with the third column. If the zeros in the off-diagonals are rearranged such that they occur in the leftmost columns of the rows, it is possible to avoid that portion of the operations for the rows. Notice how the rows near the lower portion of the matrix are optimized to minimize computational effort. A savings of eight multiplications is obtained for computations of column 5. In a large system, the cost savings of this optimization technique can be enormous. A complete discussion of these techniques is presented by George and Lui (1981). A comparison of the operations performed when computing column 5 in Figure 28.4 using a Cholesky decomposition of both a full and optimized solution routine is shown in Table 28.8. Note that computations for rows 6 and 7 do not exist since

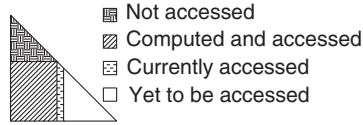


FIGURE 28.5 Computation of the Cholesky factor.

TABLE 28.8 Comparison of the Number of Operations in Computing One Column

	Full Operations	Optimized Operations
1	$s = n_{51}^2 + n_{52}^2 + \cdots + n_{54}^2$	$s = n_{53}^2 + n_{54}^2$
2	$n_{55} =$ square root of $(n_{55} - s)$	$n_{55} =$ square root of $(n_{55} - s)$
3	$j = 6$	$j = 6$
	(a) $s = n_{61}{*}n_{51} + n_{62}{*}n_{52} + \cdots + n_{64}{*}n_{54}$	no operations necessary
	(b) $n_{65} = (n_{65} - s)/n_{55}$	$j = 7$
	$j = 7$	no operations necessary
	(a) $s = n_{71}{*}n_{51} + n_{72}{*}n_{52} + \cdots + n_{74}{*}n_{54}$	$j = 8$
	(b) $n_{75} = (n_{75} - s)/n_{55}$	$s = n_{83}{*}n_{53} + n_{84}{*}n_{54}$
	$j = 8$	
	(a) $s = n_{81}{*}n_{51} + n_{82}{*}n_{52} + \cdots + n_{84}{*}n_{54}$	
	(b) $n_{85} = (n_{85} - s)/n_{55}$	

these rows had known zeros in the columns before column 5, after the reordering process. If only the more time-consuming multiplication and division operations are counted, the optimized solution requires 5 operations as compared to 19 in the nonoptimized (full) solution.

Several methods have been developed to optimize the ordering of the unknowns. Two of the more well-known reordering schemes are the *reverse Cuthill–Mckee* and the *Snay's banker's algorithm*. Both of these algorithms reorder stations based on their connectivity. The example of Figure 28.2 will be used to demonstrate the banker's algorithm. In this example, a connectivity matrix is first developed. This is simply a list of stations that are connected to a given station by either an angle or distance observation. For instance, station 1 is connected to stations 2 and 8 by distances, and additionally to station 3 by angles. The complete connectivity matrix of Figure 28.2 is listed in the first two columns of Table 28.9. In addition to the connectivity, the algorithms also need to know the number of connected points for each station. This is known as the *station's degree*.

TABLE 28.9 Connectivity Matrix

Station	Connectivity	Degree	6	5	4	7	3	1	2
1	2,3,8	3	3	3	3	3	2	–	–
2	1,3,8	3	3	3	3	3	2	1	–
3	1,2,4,7,8	5	5	5	4	3	–	–	–
4	3,5,6,7,8	5	4	3	–	–	–	–	–
5	4,6,7	3	2	–	–	–	–	–	–
6	4,5,7	3	–	–	–	–	–	–	–
7	3,4,5,6,8	5	4	3	2	–	–	–	–
8	1,2,3,4,7	5	5	5	4	3	2	1	0

To start the reordering, select the station with the lowest degree. In this example, that could be either station 1, 2, 5, or 6. For demonstration purposes, begin with station 6. This station is now removed from each station's connectivity matrix, and thus stations 4, 5, and 7 have their degrees reduced by one. This is shown in column 6. Now select the next station with the smallest degree. Since station 5 had its degree reduced to 2 when station 6 was selected, it now has the smallest degree. This process is continued, that is, the next station with the smallest degree is selected and is removed from the connectivity of the remaining stations. If two or more stations have the smallest degree, the most recently changed station with the smallest degree is chosen. In this example, this happened after choosing station 5. Station 4 was selected over stations 1 and 2 simply because it just had its degree reduced. Although station 7 qualified equally to station 4, station 4 was higher on the list and was selected for that reason alone. This happened again after selecting station 7. Station 3 was the next choice since it was the first station reduced to a degree of 3. Note that the final selection of station 8 was omitted from the list.

A similar optimized reordering can be found by starting with station 1, 2, or 5. This is left as an exercise. For a computer algorithm of this reordering, the reader should refer to the NOAA Technical Memorandum NGS 4 by Richard Snay; the computer algorithm for the *reverse Cuthill–McKee* is given in the book by George and Lui (see the Bibliography).

PROBLEMS

28.1 Explain why computer optimization techniques are necessary for doing large least-squares adjustments.

28.2 When using double precision, how much computer memory is required in forming the J, W, K, J^{T}, $J^{\mathrm{T}}W$, $J^{\mathrm{T}}WK$, and $J^{\mathrm{T}}WJ$ matrices for a 20-station adjustment having 50 total observations?

28.3 If the normal matrix of Problem 28.2 were stored in lower triangular form and computed directly from the observations, how much storage would be required?

28.4 Discuss how computational savings are created when solving the normal equations by Cholesky factorization and substitution.

28.5 Write a comparison of an operations table similar to that in Table 28.8 for column 3 of the matrix shown in Figure 28.4.

28.6 Same question as Problem 28.5 except with column 6.

28.7 Using the station reordering algorithm presented in Section 28.7, develop a connectivity matrix and reorder the stations when starting with station 1. Draw the normal matrix of the newly reordered station's which is similar to the sketch shown in Figure 28.4.

28.8 Same as Problem 28.7 except start the reordering with station 2.

28.9 Same as Problem 28.7 except start the reordering with station 5.

PROGRAMMING PROBLEM

28.10 Develop a least-squares program like those developed in Chapter 17 that builds the normal equations directly from the observations and uses a Cholesky factorization procedure to find the solution.

28.11 Develop a Mathcad® worksheet that solves Problem 28.10.

APPENDIX A

INTRODUCTION TO MATRICES

A.1 INTRODUCTION

Matrix algebra provides at least two important advantages: (1) it enables reducing complicated systems of equations to simple expressions that can be visualized and manipulated more easily, and (2) it provides a systematic, mathematical method for solving problems that is well adapted to computers. Problems are frequently encountered in surveying, geodesy, and photogrammetry, which require the solution of large systems of equations. This book deals specifically with the analysis and adjustment of redundant observations that must satisfy certain geometric conditions. This frequently results in large equation systems, which when solved according to the least-squares method yield most probable estimates for adjusted observations and unknown parameters. As will be demonstrated, matrix methods are particularly well suited for least-squares computations, and in this book, they are used for analyzing and solving these equation systems.

A.2 DEFINITION OF A MATRIX

 A matrix is a set of numbers or symbols arranged in a square or rectangular array of m rows and n columns. The arrangement is such that certain defined mathematical operations can be performed in a systematic and efficient manner. As an

example of a matrix representation, consider the following system of three linear equations involving three unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= c_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= c_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= c_3 \end{aligned} \tag{A.1}$$

In Equation (A.1), the a's are the coefficients of the unknown parameters x, and c's the constant terms. The system above can be represented in summation notation as

$$\sum_{i=1}^{3} a_{1i}x_i = c_1$$

$$\sum_{i=1}^{3} a_{2i}x_i = c_2$$

$$\sum_{i=1}^{3} a_{3i}x_i = c_3$$

It can also be represented in matrix form as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \tag{A.2}$$

In turn, Equation (A.2) can be represented in compact matrix notation as

$$AX = C \tag{A.3}$$

In Equation (A.3), capital letters (A, X, and C) are used to denote a matrix or an array of numbers or symbols. From simplified Equation (A.3), it is immediately obvious that matrix methods provide a compact shorthand notation convenient for handling large systems of equations.

A.3 SIZE OR DIMENSIONS OF A MATRIX

The size or dimension of a matrix is specified by its number of rows m and its number of columns n. Thus, the following matrix ${}_2D^3$ is a 2×3 (read 2 by 3) matrix where there are two rows and three columns (i.e., $m = 2$, $n = 3$). Notice that the

subscript indicates the number of rows, and the superscript indicates the number of columns in the notation ${}_2D^3$:

$${}_2D^3 = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 6 \\ 5 & 4 & 1 \end{bmatrix}$$

Also, the matrix E is a 3×2 matrix:

$${}_3E^2 = \begin{bmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \\ e_{31} & e_{32} \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ 4 & 3 \\ 2 & 8 \end{bmatrix}$$

Note that the position of an element in a matrix is defined by a double subscript, and that a lowercase letter is used to designate any element within a matrix. Thus, $d_{23} = 1$ is in row 2 and column 3 of the above matrix D. In general, the subscript ij indicates an element's position in a matrix where i represents the row and j the column.

A.4 TYPES OF MATRICES

Several different types of matrices exist as described below. Various symbols can be used to designate them as illustrated.

1. *Column matrix*. The number of rows can be any positive integer, but the number of columns is one. This matrix is also known as a *vector*:

$$A = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}$$

2. *Row matrix*. The number of columns can be any positive integer, but the number of rows is 1:

$$A = [6 \quad -4 \quad 2]$$

3. *Rectangular matrix*. The number of rows and columns is m and n, respectively, where m and n are any positive integers:

$$A = \begin{bmatrix} 3 & 2 & 6 \\ 5 & 4 & 1 \end{bmatrix}$$

4. *Square matrix*. The number of rows equals the number of columns:

$$A = \begin{bmatrix} 4 & 2 & -5 \\ -7 & 3 & 4 \\ 6 & -1 & 9 \end{bmatrix}$$

A square matrix, for which the determinant is zero, is termed *singular*. If the determinant is nonzero, it is termed *nonsingular* (the determinant of a matrix is discussed in Section B.3).

5. *Symmetric matrix*. The matrix is mirrored about the main diagonal going from top left to bottom right (i.e., element a_{ij} = element a_{ji}). A symmetric matrix is always a square matrix:

$$A = \begin{bmatrix} 2 & -4 & 6 \\ -4 & 7 & 3 \\ 6 & 3 & 5 \end{bmatrix}$$

6. *Diagonal matrix*. Only the elements on the main diagonal are not zero. The first element in the main diagonal is (1, 1) and its last element is (n, n). The diagonal matrix is always a square matrix:

$$A = \begin{bmatrix} 7 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

7. *Unit matrix*. A diagonal matrix with ones along the main diagonal. It is also called the identity matrix and is identified by the symbol I:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

8. *Transpose of a matrix*. It is obtained by interchanging rows and columns (i.e., element a_{ij}^{T} = element a_{ji}). Thus, the dimensions of A^{T} are the reverse of the dimensions of A. If

$$A = \begin{bmatrix} 2 & 4 & 7 \\ 5 & 3 & 1 \end{bmatrix}$$

the transpose of A, denoted A^{T} is

$$A^{\mathrm{T}} = \begin{bmatrix} 2 & 5 \\ 4 & 3 \\ 7 & 1 \end{bmatrix}$$

A.5 MATRIX EQUALITY

Two matrices are said to be equal only when they are equal element by element. Thus, the two matrices must be the same size or have the same dimensions.

$$A = \begin{bmatrix} 1 & 7 & 6 \\ 4 & 3 & 2 \end{bmatrix} = B$$

A.6 ADDITION OR SUBTRACTION OF MATRICES

Matrices can be added or subtracted, but to do so, they must have the same dimensions. If two matrices have equal dimensions, they are said to be *conformable* for addition or subtraction. In adding or subtracting matrices, elements from each unique row/column position of the two matrices are added or subtracted systematically. The sum or difference is placed in the same unique row/column location of the resulting matrix. The following example illustrates this procedure:

$$ {}_2A^3 + {}_2B^3 = \begin{bmatrix} 7 & 3 & -1 \\ 2 & -5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 5 & 6 \\ -4 & -2 & 3 \end{bmatrix} = \begin{bmatrix} 8 & 8 & 5 \\ -2 & -7 & 9 \end{bmatrix} = {}_2C^3 $$

Assuming that two matrices are conformable for addition or subtraction, the following are true:

(a) $A + B = B + A$ (commutative law)
(b) $A + (B + C) = (A + B) + C$ (associative law)

A.7 SCALAR MULTIPLICATION OF A MATRIX

Matrices can be multiplied by a scalar (i.e., a constant). Let k be any scalar quantity; then

$$kA = Ak$$

The following are examples:

$$ 4 \times \begin{bmatrix} 3 & -1 \\ 2 & 6 \\ 4 & 7 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 2 & 6 \\ 4 & 7 \\ 5 & 3 \end{bmatrix} \times 4 = \begin{bmatrix} 12 & -4 \\ 8 & 24 \\ 16 & 28 \\ 20 & 12 \end{bmatrix} $$

As illustrated in the example above, each element of the matrix A is multiplied by the scalar k to obtain the elements of C. Note that $4 \times A = A + A + A + A = A \times 4$.

A.8 MATRIX MULTIPLICATION

If matrix A is to be *postmultiplied* by matrix B (i.e., the product of AB determined, the number of columns in matrix A must equal the number of rows in matrix B). This is a basic requirement for matrix multiplication. When this condition is satisfied, the two matrices are said to be *conformable for multiplication*. The product C will

have the same number of rows as A and the same number of columns as B. Thus, the following multiplications are possible:

$$ {}_4A^2 \times {}_2B^3 = {}_4C^3 $$

$$ {}_1A^3 \times {}_3B^1 = {}_1C^1 $$

$$ {}_3A^3 \times {}_3B^1 = {}_3C^1 $$

The following multiplications are not possible since the columns in the first matrix are not equal to the number of rows in the second matrix:

$$ {}_2B^3 \times {}_4A^2 $$

$$ {}_6A^2 \times {}_6B^3 $$

To demonstrate the process of matrix multiplication, consider the following example:

$$ {}_2A^3 \times {}_3B^2 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{12} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = {}_2C^2 $$

The elements c_{ij} of matrix C are the total sums obtained by successively multiplying each element in row i of matrix A by the elements in column j of matrix B, and then summing these products. Thus, for the abovementioned example:

$$ c_{11} = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} = \sum_{i=1}^{3} a_{1i}b_{i1} $$

$$ c_{12} = a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} = \sum_{i=1}^{3} a_{1i}b_{i2} $$

$$ c_{21} = a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} = \sum_{i=1}^{3} a_{2i}b_{i1} $$

$$ c_{22} = a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} = \sum_{i=1}^{3} a_{2i}b_{i2} $$

The process above is seen more easily with a numerical example.

$$ \begin{bmatrix} 1 & 2 & 3 \\ 4 & 2 & 7 \end{bmatrix} \begin{bmatrix} 4 & 8 \\ 6 & 2 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} = \begin{bmatrix} 31 & 21 \\ 63 & 57 \end{bmatrix} $$

$$c_{11} = 1(4) + 2(6) + 3(5) = 31$$
$$c_{12} = 1(8) + 2(2) + 3(3) = 21$$
$$c_{21} = 4(4) + 2(6) + 7(5) = 63$$
$$c_{22} = 4(8) + 2(2) + 7(3) = 57$$

The student should now verify the matrix representation of Equations (A.1) and (A.2). Notice that the product of a unit matrix, I, and a conformable matrix, A (one with the same number of rows as I), equals the original matrix A. Thus,

$$_2I^2 \times {}_2A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = {}_2A^2$$

Assuming that matrices A, B, and C are conformable for multiplication and in the order indicated, then the following are true:

(c) $A(B + C) = AB + AC$(first distributive law)
(d) $(A + B)C = AC + BC$(second distributive law)
(e) $A(BC) =$ $(AB)C$(associative law)
(f) $(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$

The following cautions are also stated:

(g) AB is not generally equal to BA, and BA may not even be conformable for multiplication.
(h) If $AB = 0$, neither A nor B are necessarily $= 0$.
(i) If $AB = AC$, B does not necessarily $= C$.

Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Example of (c):

$$A(B + C) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \times \begin{bmatrix} 0 & 0 \\ 3 & 3 \end{bmatrix} = AB + AC = \begin{bmatrix} 0 & 0 \\ 3 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 6 & 6 \end{bmatrix}$$

Example of (d)[1]:

$$(A+B)C = \begin{bmatrix} 1 & 0 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 7 & 7 \end{bmatrix} = AC + BC$$
$$= \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} + \begin{bmatrix} -2 & -2 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 7 & 7 \end{bmatrix}$$

Example of (e):

$$A(BC) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -2 & -2 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 6 & 6 \end{bmatrix} = (AB)C = \begin{bmatrix} 0 & 0 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 6 & 6 \end{bmatrix}$$

Example of (f):

$$\text{Let } A = \begin{bmatrix} 2 & 6 & 4 \\ 1 & 2 & 7 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 2 \\ 9 & 0 \\ 1 & 3 \end{bmatrix}, \text{ then } AB = \begin{bmatrix} 64 & 16 \\ 28 & 23 \end{bmatrix}$$

so

$$(AB)^T = \begin{bmatrix} 64 & 28 \\ 16 & 23 \end{bmatrix} \text{ and } B^T A^T = \begin{bmatrix} 3 & 9 & 1 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 6 & 2 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 64 & 28 \\ 16 & 23 \end{bmatrix}$$

Example of (g):

$$AB = \begin{bmatrix} 0 & 0 \\ 3 & 3 \end{bmatrix} \neq \begin{bmatrix} -3 & -3 \\ 6 & 6 \end{bmatrix} = BA$$

Example of (h):

$$\text{Let } A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix}$$

Then $AB = 0$, but neither A nor B equal 0.

Example of (i):

$$\text{Let } A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \text{ and } C = \begin{bmatrix} 4 & 10 \\ -2 & -5 \end{bmatrix}$$

Then $AB = AC$, but $B \neq C$, where $AB = \begin{bmatrix} 4 & 10 \\ 2 & 5 \end{bmatrix} = AC$

[1]The multiplication symbol, ×, is generally not written in matrix equations. This convention of not using × will be followed in this book.

A.9 COMPUTER ALGORITHMS FOR MATRIX OPERATIONS

It should be apparent that addition, subtraction, and multiplication of large matrices involve many arithmetic operations. These are very tedious when done by hand but can be done quickly by a computer. In this section, general mathematical expressions are developed for performing these operations using a computer. These general mathematical expressions, when programmed for computer solution, are called *algorithms*.

A.9.1 Addition/Subtraction of Two Matrices

Consider the two matrices A and B that are shown in Figure A.1. Note that matrices A and B are *conformable* for addition. Find the sum of the two matrices and place the results in C.

Step 1: Add the first element (a_{11}) of the A matrix to the first element (b_{11}) of B, placing the result in the first element (c_{11}) of C. Repeat this process for each of the successive columns along the first row of A and B, or from $j = 1$ to $j = n$.

Step 2: Iterate step 1 for each successive row of the matrices, or for rows increasing from $i = 1$ to $i = m$. Table A.1 shows this entire operation of adding two matrices in the four computer languages of BASIC, C, FORTRAN, and Pascal.

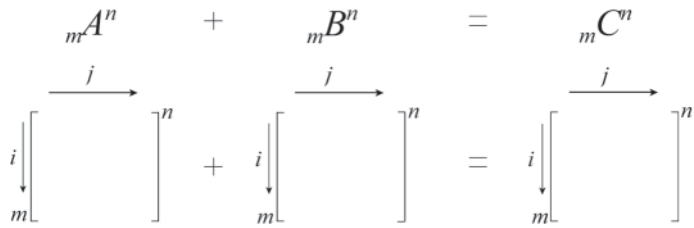


FIGURE A.1 Addition of matrices.

TABLE A.1 Addition Algorithm in BASIC, C, FORTRAN, and Pascal

BASIC Language

```
1000  For i = 1 to M
1010    For j = 1 to N
1020     C(i,j) = A(i,j) + B(i,j)
1030    Next j
1040  Next i
```

FORTRAN Language

```
      Do 100 I = 1,M
        Do 100 J = 1,N
          C(i,j) = A(i,j) + B(i,j)
100      Continue
```

C Language

```
for (i=0; i<m; i++)
  for (j=0; j<n; j++)
    C[i][j] = A[i][j] + B[i][j];
```

Pascal Language

```
For i := 1 to M do
  For j := 1 to N do
    C[i,j] := A[i,j] + B[i,j];
```

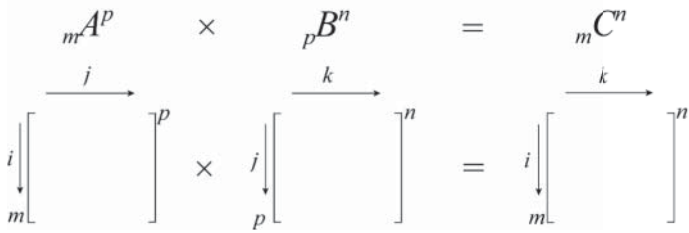


FIGURE A.2 Multiplication of matrices.

A.9.2 Matrix Multiplication

Consider the two matrices A and B that are shown in Figure A.2. Again, these matrices, A and B, are conformable for multiplication. Find the product AB and place the results in C.

Step 1: Sum the products (A row $i = 1$) × (B column $k = 1$), with the elements of A and B being increased successively from $j = 1$ to $j = p$. Place the result in c_{ik}, or c_{11}, for the first step. Mathematically, this step is represented as

$$c_{ik} = \sum_{j=1}^{p} a_{ij} b_{jk} \quad \text{with } i = 1, \text{ and } k = 1$$

Step 2: Increase k successively by 1, repeat step 1, and place the results in c_{ik}. Continue increasing k and repeating step 1 until $k = p$.

TABLE A.2 Multiplication Algorithm in BASIC, C, FORTRAN, and Pascal

BASIC Language

```
For i = 1 to M
   For k = 1 to N
      C(i,k) = 0.0
      For j = 1 to P
         C(i,k) = C(i,k) + A(i,j)*B(j,k)
      Next j :   Next k:   Next I
```

FORTRAN Language

```
      Do 100 I = 1,M
        Do 100 K = 1,N
          C(i,k) = 0.0
          Do 100 J = 1,P
            C(i,k)=C(i,k)+A(i,j)*B(j,k)
100   Continue
```

C Language

```
for (i=0; i<m; i++)
  for (k=0; k<n; k++) {
    C[i][k] = 0;
    for (j=0; j<p; j++)
      C[i][k] = C[i][k] + A[i][j]*B[j][k];
  } //for k
```

Pascal Language

```
For i := 1 to M do
  For k := 1 to N do Begin
    C[i,k] := 0.0;
    For j := 1 to P do
      C[i,k] := C[i,k] + A[i,j]*B[j,k]
  End; { for k }
```

```
Example matrices from Section A.8
A
2 2
2 1
1 2
B
2 2
-1 -1
2 2
C
1 1
1 1
```

FIGURE A.3 MATRIX data file.

Step 3: Increase i from one to two, and repeat steps 1 and 2 in entirety. Upon completion with $i = 2$, increment i by one and repeat steps 1 and 2 in entirety. Continue this process through $i = m$. This completes the matrix multiplication.

This operation is shown Table A.2 in the four languages of BASIC, C, FORTRAN, and Pascal. As a final note on multiplication, it is essential that the order in which the matrices are multiplied be specified.

A.10 USE OF THE MATRIX SOFTWARE

A software package called MATRIX is included on the companion website that is provided with this book. It includes all the matrix operations that will be necessary to study the subject of adjustment computations and solve the after-chapter problems herein that require matrix manipulation. Instructions for use of the software are given in its help file, which is part of the software.

MATRIX reads the matrices from text files and contains a text editor to create such files. The files are free format, which means that a space, comma, or tab must separate each entry. This also means that commas cannot be used as thousand separators in a single large number. Although the internal editor in MATRIX does not support tabs, a tab entry can also be used to separate entries. This feature is useful when cutting and pasting matrices from a spreadsheet. A file of matrices from Section A.8, which were used to demonstrate principles (*c*) and (*d*), is shown in Figure A.3, where the format of the files is

1. Job title.
2. Matrix name.
3. Number of rows and columns in matrix.
4. List each element in the first row.
5. Continue in the same manner for each subsequent row, by placing each row of a matrix on a new line.
6. Repeat lines 2–5 for each matrix.

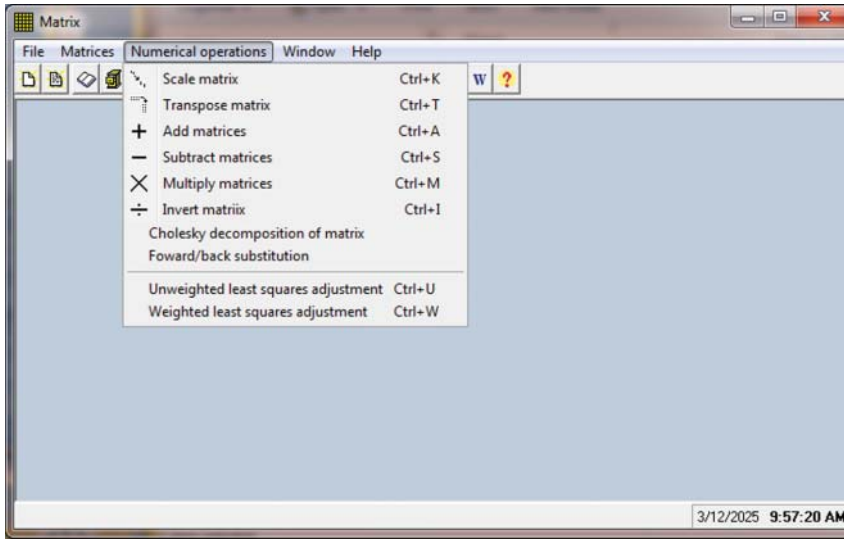


FIGURE A.4 MATRIX software.

After creating the file, it must be saved and read into the MATRIX software, shown in Figure A.4, using the *read file* option under the *matrices* menu of the software. Once read into the software, the user can perform any of the matrix operations contained in the *numerical operations* menu. When an operation is selected, a matrix selection screen will appear. Figure A.5 shows the selection screen after the multiplication option has been selected. Note that the previous addition of matrices A and B is shown as $A + B$, and the directions to the user are shown at the top of the screen, where it states select the first matrix A in the product of AB. There is a *rename* option under the *matrices* menu that allows the user to change the default name of a matrix to any name up to 20 characters long. Two options in the *numerical operations* menu allow the user to perform both a weighted and an unweighted least-squares adjustment as discussed in Chapter 11. These options create all the intermediate matrices that are necessary to perform the adjustment and compute the post-adjustment statistics as presented in Chapter 13. Additionally, the user can select a submatrix from an existing matrix or augment and stack two matrices.

Augmentation of two matrices results in placing one matrix to the right of another. For instance, if matrix A is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and B is $\begin{bmatrix} e & f \\ g & h \end{bmatrix}$, the augmentation of A and B results in the matrix $\begin{bmatrix} a & b & e & f \\ c & d & g & h \end{bmatrix}$. If B is stacked in A, the result is $\begin{bmatrix} a & b \\ c & d \\ e & f \\ g & h \end{bmatrix}$. These operations are useful in adding constraint equations using Helmert's method, as discussed in Section 20.4.

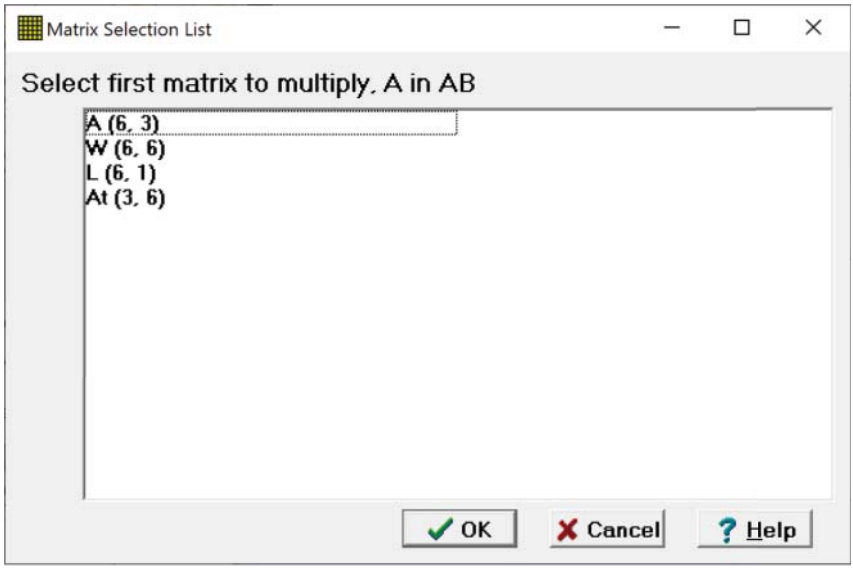


FIGURE A.5 MATRIX selection screen for multiplication of two matrices.

Users can view any result of the matrices using the *view matrix* option in the *matrices* menu. This option can also be accessed with the shortcut key *Ctrl-V*. The number of digits after the decimal point can be controlled by using the *display options* under the *matrices* menu. For example, since the additions of Section A.8 involved only integers, the user may select zero digits after the decimal point so that integers are displayed. To print the resulting matrices, the user simply needs to select the *list matrix* option in the *matrices* menu. The user will be prompted for the location and name of the resulting file, which will be created to list the selected matrices. When this option is used, the user can select multiple matrices from the matrix selection screen. These matrices will be written to the user-specified file using the display option, specified digits after the decimal point. Once saved, this file can be read into the MATRIX editor and printed. The *write file* option in the *matrices* menu allows the user to create a text file, which is readable by MATRIX, of all the matrices shown in the selection screen. The matrices will be written in scientific notation so that no accuracy is lost between the saving and reading of the resultant file.

PROBLEMS

A.1 Represent the following system of linear equations in matrix form as $AX = B$.

$$2x_1 + x_2 + 5x_3 + x_4 = 5$$

$$x_1 + x_2 - 3x_3 + 4x_4 = -1$$

$$3x_1 + 6x_2 - 2x_3 + x_4 = 8$$

$$2x_1 + 2x_2 + 2x_3 - 3x_4 = 2$$

A.2 For Problem A.1, find the product of $A^{\mathrm{T}}A$.

A.3 Do the following matrix operations

(a) $\begin{bmatrix} 1 & 2 & -1 & 0 \\ 4 & 0 & 2 & 1 \\ 2 & -5 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -4 & 1 & 2 \\ 1 & 5 & 0 & -3 \\ 2 & -2 & 3 & -1 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & -1 & 0 \\ 4 & 0 & 2 & 1 \\ 2 & -5 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & -4 & 1 & 2 \\ 1 & 5 & 0 & 3 \\ 2 & -2 & 3 & -1 \end{bmatrix}$

(c) $3\begin{bmatrix} 1 & 2 & -1 & 0 \\ 4 & 0 & 2 & 1 \\ 2 & -5 & 1 & 2 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 2 & 1 \\ 4 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 1 & 5 \\ -2 & 2 \end{bmatrix}$

A.4 Solve Problem A.2 using the MATRIX software.

A.5 Solve Problem A.3 using the MATRIX software.

A.6 Let $A = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(a) Find AB.
(b) Find B^2.
(c) Find CB.
(d) Find BA.
(e) Find B^3.
(f) Find DB.
(g) Find C^3.

A.7 Do Problem A.6 using the MATRIX software.

A.8 Multiply the following, if possible. If not possible, give reasons why the multiplication cannot be done.

(a) $\begin{bmatrix} 2 & 1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & 1 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 9 \end{bmatrix}$

(c) $\begin{bmatrix} 2 & 3 & 4 & 4 \\ 1 & 0 & -1 & 6 \\ 0 & 1 & 2 & 9 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 3 & 1 \\ 1 & 0 \\ 0 & -1 \end{bmatrix}$

(d) $\begin{bmatrix} 2 \\ 0 \end{bmatrix} \begin{bmatrix} 3 & -1 \end{bmatrix}$

A.9 Expand the following summations into their equivalent algebraic expressions.

(a) $\sum_{k=1}^{5} k$

(b) $\sum_{k=3}^{7} (k-2)$

(c) $\sum_{k=1}^{4} a_k$

(d) $\sum_{k=1}^{3} a_{2k} a_{k3}$

(e) $\sum_{i=1}^{3} \sum_{j=1}^{4} (a_{ij} + b_{ij})$

(f) $\sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{3} a_{ik} b_{kj}$

A.10 Write the algebraic expressions for the system of equations represented by the following matrix notation.

$$\begin{bmatrix} 2 & 0 \\ 1 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

A.11 Show that in general, $(AB)^{\mathrm{T}} = B^{\mathrm{T}} A^{\mathrm{T}}$.

PROGRAMMING PROBLEMS

A.12 Write a program that will read and write the elements of a matrix, compute the transpose of the matrix, and write the solution. (*Hint:* Place the reading,

writing, and transposition codes in separate subroutines/procedures/ functions to be called from the main program.)

A.13 Select one of the matrix addition codes from Table A.1 and enter it and any necessary supporting code to solve Problem A.3(a). (*Hint:* Place the code in Table A.1 in a separate subroutine/procedure/function to be called from the main program.)

A.14 Select one of the matrix multiplication programs from Table A.2, and enter this code and any necessary supporting code to solve Problem A.3(d). (*Hint:* Place the code in Table A.2 in a separate subroutine/procedure/function to be called from the main program.)

A.15 Write a program that will read and write a file of matrices, and do the matrix operations of transposition, addition, subtraction, and multiplication. Demonstrate the program by doing Problems A.2 and A.3 (a), (b), and (d). (*Hint:* Place the reading, writing, transposition, addition, subtraction, and multiplication, codes in separate subroutines/procedures/ functions to be called from the main program.)

A.16 Write a Mathcad worksheet that solves Problems A.3 and A.6.

APPENDIX B

SOLUTION OF EQUATIONS BY MATRIX METHODS

B.1 INTRODUCTION

As stated in Appendix A, an advantage offered by matrix algebra is its adaptability to computer usage. Using matrix algebra, large systems of simultaneous linear equations can be programmed for general computer solution using only a few systematic steps. For example, the simplicities of programming matrix additions and multiplications were presented in Section A.9. To solve a system of equations using matrix methods, it is first necessary to define and compute the inverse of a matrix.

B.2 INVERSE MATRIX

 If a square matrix is nonsingular (its determinant is not zero), it possesses an inverse matrix. When a system of simultaneous linear equations consisting of n equations and involving n unknowns is expressed as $AX = B$, the coefficient matrix (A) is a square matrix of dimensions $n \times n$. Consider this system of linear equations:

$$AX = B \tag{B.1}$$

The inverse of matrix A, symbolized as A^{-1}, is defined as

$$A^{-1}A = I \tag{B.2}$$

where I is the identity matrix. Premultiplying both sides of the matrix Equation (B.1) by A^{-1} gives

$$A^{-1}AX = A^{-1}B$$

Reducing yields

$$IX = A^{-1}B$$

$$X = A^{-1}B \tag{B.3}$$

Thus, the inverse is used to find the matrix of unknowns, X, or, in other words, to solve the system of equations. The following points should be emphasized regarding matrix inversions:

1. Square matrices have inverses, with the exception noted in the following.
2. When the determinant of a matrix is zero, the matrix is said to be singular and its inverse cannot be found.
3. The inversion of even a small matrix is a tedious and time-consuming operation when done by hand. However, when done by a computer, the inverse can be found quickly and easily.

B.3 THE INVERSE OF A 2 × 2 MATRIX

Several general methods are available to find a matrix inverse. Two shall be considered herein. However, before proceeding with general cases, consider the specific case of finding the inverse for a 2×2 matrix using simple elementary matrix operations. Let any 2×2 matrix be symbolized as A. Also, let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad A^{-1} = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

By applying Equation (B.2) and recalling the definition of an identity matrix I as given in Section A.4, it is possible to calculate w, x, y, and z in terms of a, b, c, and d of A^{-1}. Substituting in the appropriate values gives

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

By matrix multiplication

$$aw + by = 1 \tag{a}$$

$$ax + bz = 0 \tag{b}$$

$$cw + dy = 0 \qquad (c)$$

$$cx + dz = 1 \qquad (d)$$

The determinant of A is symbolized as $\|A\|$ and equal to $ad - bc$.

$$\left.\begin{array}{l} \text{From } (a)\ y = \dfrac{1 - aw}{b}; \quad \text{From } (b)\ \ y = -\dfrac{cw}{d} \\ \text{then } \dfrac{1 - aw}{b} = -\dfrac{cw}{d}; \quad \text{Reducing: } w = \dfrac{d}{da - bc} = \dfrac{d}{\|A\|} \end{array}\right\}$$

$$\left.\begin{array}{l} \text{From } (b)\ \ z = -\dfrac{ax}{b}; \quad \text{From } (d)\ \ z = \dfrac{1 - cx}{d} \\ \text{then } -\dfrac{ax}{b} = \dfrac{1 - cx}{d}; \quad \text{Reducing: } x = \dfrac{b}{-da + bc} = \dfrac{-b}{\|A\|} \end{array}\right\}$$

$$\left.\begin{array}{l} \text{From } (a)\ \ w = \dfrac{1 - by}{a}; \quad \text{From } (c)\ \ w = -\dfrac{dy}{c} \\ \text{then } -\dfrac{1 - by}{a} = -\dfrac{dy}{c}; \quad \text{Reducing: } y = \dfrac{c}{-ad + cb} = \dfrac{-c}{\|A\|} \end{array}\right\}$$

$$\left.\begin{array}{l} \text{From } (b)\ \ x = -\dfrac{bz}{a}; \quad \text{From } (d)\ \ x = \dfrac{1 - dz}{c} \\ \text{then } -\dfrac{bz}{a} = \dfrac{1 - dz}{c}; \quad \text{Reducing: } z = \dfrac{a}{ad - bc} = \dfrac{a}{\|A\|} \end{array}\right\}$$

Thus, for any 2×2 matrix composed of the elements $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, its inverse is simply

$$\begin{bmatrix} \dfrac{d}{\|A\|} & \dfrac{-b}{\|A\|} \\ \dfrac{-c}{\|A\|} & \dfrac{a}{\|A\|} \end{bmatrix} = \frac{1}{\|A\|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example B.1 If $A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$, find A^{-1}.

SOLUTION

$$\|A\| = 2(1) - 4(3) = -10; \quad A^{-1} = -\frac{1}{10} \begin{bmatrix} 1 & -3 \\ -4 & 2 \end{bmatrix}$$

A check on the inverse can be obtained by testing it against its definition, or $A^{-1}A = I$. Thus,

$$-\frac{1}{10} \begin{bmatrix} 1 & -3 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

B.4 INVERSES BY ADJOINTS

The inverse of A can be found using the method of adjoints with the following equation:

$$A^{-1} = \frac{\text{Adjoint of } A}{\text{Determinant of } A} = \frac{\text{Adjoint of } A}{\|A\|} \tag{B.4}$$

The *adjoint of A* is obtained by first replacing each matrix element by its signed minor or *cofactor*, and then transposing the resultant matrix. The cofactor of element a_{ij} equals $(-1)^{i+j}$ times the numerical value of the determinant for the remaining elements after row i and column j have been removed from the matrix. This procedure is illustrated in Figure B.1, where the cofactor of a_{12} is

$$(-1)^{1+2}\,(a_{21}a_{33} - a_{31}a_{23}) = a_{31}a_{23} - a_{21}a_{33}$$

Using this procedure, the inverse of the following A matrix is found:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 4 & 1 \\ 2 & 3 & 4 \end{bmatrix}$$

For this A matrix, the cofactors are calculated as follows:

$$\text{Cofactor of } a_{11} = (-1)^2\,(4 \times 4 - 1 \times 3) = 13$$

$$\text{Cofactor of } a_{21} = (-1)^3\,(3 \times 4 - 2 \times 3) = -6$$

$$\text{Cofactor of } a_{31} = (-1)^4\,(3 \times 1 - 2 \times 4) = -5$$

$$\text{Cofactor of } a_{12} = (-1)^3\,(3 \times 4 - 1 \times 2) = -10$$

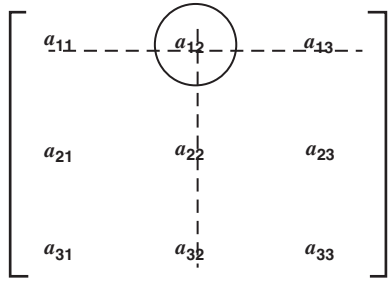


FIGURE B.1 Cofactor of the a_{12} element.

Following the abovementioned procedure, the matrix of cofactors is

$$\text{Matrix of cofactors} = \begin{bmatrix} 13 & -10 & 1 \\ -6 & 12 & -6 \\ -5 & 2 & 7 \end{bmatrix}$$

Transposing this cofactor matrix produces the following adjoint of A:

$$\text{Adjoint of } A = \begin{bmatrix} 13 & -6 & -5 \\ -10 & 12 & 2 \\ 1 & -6 & 7 \end{bmatrix}$$

The determinant of A is the sum of the products of the elements in the first row of the original matrix times their respective cofactors. Since the cofactors were already obtained in the previous step, this simplifies to

$$\|A\| = 4(13) + 3(-10) + 2(1) = 24$$

The inverse of A is now calculated as

$$A^{-1} = \frac{1}{24}\begin{bmatrix} 13 & -6 & -5 \\ -10 & 12 & 2 \\ 1 & -6 & 7 \end{bmatrix} = \begin{bmatrix} 13/24 & -1/4 & -5/24 \\ -5/12 & 1/2 & 1/12 \\ 1/24 & -1/4 & 7/24 \end{bmatrix}$$

Again, a check on the arithmetical work is obtained by using the definition of an inverse:

$$AA^{-1} = \begin{bmatrix} 13/24 & -1/4 & -5/24 \\ -5/12 & 1/2 & 1/12 \\ 1/24 & -1/4 & 7/24 \end{bmatrix}\begin{bmatrix} 4 & 3 & 2 \\ 3 & 4 & 1 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

B.5 INVERSES BY ELEMENTARY ROW TRANSFORMATIONS



A system of equations can be modified using the following three steps without changing its solution.

1. The multiplication of every element in any row by a nonzero scalar. This is simply scaling the equation. Thus, if the original equation is $2x = 4$, and this equation is scaled by three, the resultant equation $6x = 12$ still has the same solution of $x = 2$.

2. The addition (or subtraction) of the elements in any row to the elements of any other row.
3. Combinations of 1 and 2.

If elementary row transformations are successively performed on A such that A is transformed into I, and if throughout the procedure the same row transformations are also done to the same rows of the identity matrix I, the I matrix will be transformed into A^{-1}. This procedure is illustrated using the same matrix used to demonstrate the method of adjoints.

Initially, the original matrix and the identity matrix are listed side by side:

$$\begin{matrix} A & & & & I & & \end{matrix}$$
$$\left[\begin{array}{ccc|ccc} 4 & 3 & 2 & 1 & 0 & 0 \\ 3 & 4 & 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

With the following three-row transformations performed on A and I, they are transformed into matrices A_1 and I_1, respectively:

1. Multiply row 1 of matrices A and I by $1/a_{11}$ or 1/4. Place the results in row 1 of A_1 and I_1, respectively. This converts a_{11} of matrix A_1 to 1:

$$\left[\begin{array}{ccc|ccc} 1 & 3/4 & 1/2 & 1/4 & 0 & 0 \\ 3 & 4 & 1 & 0 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{array}\right]$$

2. Multiply row 1 of matrices A_1 and I_1 by a_{21} or 3. Subtract the resulting row from row 2 of matrices A and I and place the difference in row 2 of A_1 and I_1, respectively. This converts a_{21} of A_1 to zero.
3. Multiply row 1 of matrices A_1 and I_1 by a_{31} or 2. Subtract the resulting row from row 3 of matrices A and I and place the difference in row 3 of A_1 and I_1, respectively. This changes a_{31} of A_1 to zero.

After doing these operations, the transformed matrices A_1 and I_1 are

$$\begin{matrix} A_1 & & & & I_1 & & \end{matrix}$$
$$\left[\begin{array}{ccc|ccc} 1 & 3/4 & 1/2 & 1/4 & 0 & 0 \\ 0 & 7/4 & -1/2 & -3/4 & 1 & 0 \\ 0 & 3/2 & 3 & -1/2 & 0 & 1 \end{array}\right]$$

Notice that the first column of A_1 is equivalent to the first column of a 3×3 identity matrix as a result of these three-row transformations. For matrices having more than three rows, this same general procedure would be followed for each row to convert the first element in each row of A_1 to zero, except for the first row of A.

Next, the following three elementary row transformations are done on matrices A_1 and I_1 to transform them into matrices A_2 and I_2:

1. Multiply row 2 of A_1 and I_1 by $1/a_{22}$ or 4/7 and place the results in row 2 of A_2 and I_2. This converts a_{22} to 1:

$$\left[\begin{array}{ccc|ccc} 1 & 3/4 & 1/2 & 1/4 & 0 & 0 \\ 0 & 1 & -2/7 & -3/7 & 4/7 & 0 \\ 0 & 3/2 & 3 & -1/2 & 0 & 1 \end{array}\right]$$

2. Multiply row 2 of A_2 and I_2 by a_{12} or 3/4. Subtract the resulting row from row 1 of A_1 and I_1 and place the difference in row 1 of A_2 and I_2, respectively.
3. Multiply row 2 of A_2 and I_2 by a_{32}, which is 3/2. Subtract the resulting row from row 3 of A_1 and I_1 and place the difference in row 3 of A_2 and I_2, respectively.

After doing these operations, the transformed matrices A_2 and I_2 are

$$\begin{array}{ccc|ccc} A_2 & & & & I_2 & \\ \left[\begin{array}{ccc|ccc} 1 & 0 & 5/7 & 4/7 & -3/7 & 0 \\ 0 & 1 & -2/7 & -3/7 & 4/7 & 0 \\ 0 & 0 & 24/7 & 1/7 & -6/7 & 1 \end{array}\right] & & & & & \end{array}$$

Notice that after this second series of steps is completed, the second column of A_2 conforms to column two of a 3×3 identity matrix. Again, for matrices having more than three rows, this same general procedure would be followed for each row to convert the second element in each row (except the second row) of A_2 to zero.

Finally, the following three row transformations are applied to matrices A_2 and I_2 to transform them into matrices A_3 and I_3:

1. Multiply row 3 of A_2 and I_2 by $1/a_{33}$ or 7/24 and place the results in row 3 of A_3 and I_3, respectively. This converts a_{33} to 1:

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 5/7 & 4/7 & -3/7 & 0 \\ 0 & 1 & -2/7 & -3/7 & 4/7 & 0 \\ 0 & 0 & 1 & 1/24 & -1/4 & 7/24 \end{array}\right]$$

2. Multiply row 3 of A_2 and I_2 by a_{13} or 5/7. Subtract the results from row 1 of A_2 and I_2 and place the difference in row 1 of A_3 and I_3, respectively.
3. Multiply row 3 of A_2 and I_2 by a_{23} or $-2/7$. Subtract the results from row 2 of A_2 and I_2 and place the difference in row 2 of A_3 and I_3.

Following these operations, the transformed matrices A_3 and I_3 are

$$\begin{array}{ccc} A_3 & & I_3 = A^{-1} \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 12/24 & -1/4 & -5/24 \\ 0 & 1 & 0 & -5/12 & 1/2 & 1/12 \\ 0 & 0 & 1 & 1/24 & -1/4 & 7/24 \end{array}\right]$$

Notice that through these nine elementary row transformations, the original A matrix is transformed into the identity matrix, and the original identity matrix is transformed into A^{-1}, which can be verified by multiplying it by the A matrix. Also note that A^{-1} obtained by this method agrees exactly with the inverse obtained by the method of adjoints. This is because any nonsingular matrix has a unique inverse.

It should be obvious that the quantity of work involved in inverting matrices greatly increases with the matrix size, since the number of necessary row transformations is equal to the square of the number of rows or columns. Because of this, it is not considered practical to invert large matrices by hand. This work is more conveniently done with a computer. Since the procedure of elementary row transformations is systematic, it is easily programmed.

Table B.1 shows algorithms, written in BASIC, C, FORTRAN, and Pascal programming languages, for calculating the inverse of any $n \times n$ nonsingular matrix A. Students should review the code in their preferred language to gain familiarity with the computer procedures. It should be noted here that there is one additional step that good inverse routines practice, which is elementary row swapping. This is used to avoid a situation where the matrix being inversed has a zero in its diagonal element. In this case, the row is interchanged (swapped) with the following row to avoid a division by zero error. This technique is not shown in Table B.1 but can be found in computer science books on this subject.

B.6 EXAMPLE PROBLEM

Example B.2 Suppose that an EDM instrument is placed at point A in Figure B.2 and a reflector is successively placed at B, C, and D. The observed values AB, AC, and AD are shown in the figure. Calculate the unknowns X_1, X_2, and X_3 by matrix methods. The observed values are the following:

$$AB = 125.27$$

$$AC = 259.60$$

$$AD = 395.85$$

TABLE B.1 Inverse Algorithm in BASIC, C, FORTRAN, and Pascal

BASIC Language

```
REM INVERT A MATRIX
FOR k = 1 TO n
   FOR j = 1 TO n
     IF j<>k THEN A(k,j) = A(k,j)/A(k,k)
   NEXT j
   A(k,k) = 1/A(k,k)
   FOR i = 1 TO n
     IF  i<>k THEN
       FOR j=1 TO n
         IF j<>k THEN A(i,j) = A(i,j) - A(i,k)*A(k,j)
       NEXT j
       A(i,k) = -A(i,k)*A(k,k)
     END IF
  NEXT i: NEXT k
```

C Language

```
for (k=0; k<n; k++) {
  for (j=0; j<n; j++)
    if (j!=k) A[k][j] = A[k][j]/A[k][k];
  A[k][k] = 1.0/A[k][k];
  for (i=0; i<n; i++)
    if (i!=k) {
      for (j=0; j<n; j++)
        if (j!=k) A[i][j] = A[i][j] - A[i][k]*A[k][j];
      A[i][k] = -A[i][k]*A[k][k];
    } //if i<>k
 } //for k
```

FORTRAN Language

```
       Do 560 k = 1,N
         Do 520 j = 1,N
           If (j.NE.k) Then
             A(k,j) = A(k,j)/A(k,k)
520    Continue
         A(K,K) = 1.0/A(K,K)
         Do 560 i = 1,N
           If (i.EQ.k) Then GOTO 560
             Do 550 j = 1,N
               If (j.NE.k) Then
                 A(i,j) = A(i,j) - A(i,k)*A(k,j)
550        Continue
           A(i,k) = -A(i,k) * A(k,k)
560         Continue
```

Pascal Language

```
For k := 1 to N do Begin
  For j := 1 to N do
    If (j<>k) then A[k,j] := A[k,j]/A[k,k];
  A[k,k] := 1.0/A[k,k];
  For i := 1 to N do
    If (i<>k) then Begin
      For j := 1 to N do
        If (j<>k) then A[i,j] := A[i,j] - A[i,k]*A[k,j];
      A[i,k] := -A[i,k]*A[k,k];
    End; { If i<>k }
End; { for k }
```

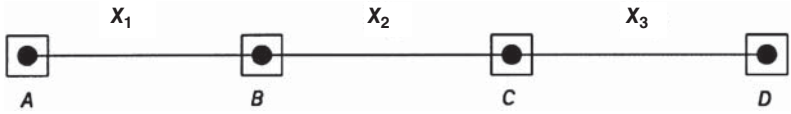


FIGURE B.2 Observation of a line.

SOLUTION Formulate the basic equations:

$$1X_1 + 0X_2 + 0X_3 = 125.27$$
$$1X_1 + 1X_2 + 0X_3 = 259.60$$
$$1X_1 + 1X_2 + 1X_3 = 395.85$$

Represented in matrix notation, these equations are $AX = L$. In this matrix equation, the individual matrices are

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad L = \begin{bmatrix} 125.27 \\ 259.60 \\ 395.85 \end{bmatrix}$$

The solution in matrix notation is $X = A^{-1}L$. Using elementary row transformations, the inverse of A is

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \end{array}\right]$$

Solving $X = A^{-1}L$, the unknowns are

$$X = A^{-1}L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 125.27 \\ 259.60 \\ 395.85 \end{bmatrix} = \begin{bmatrix} 125.27 \\ 134.33 \\ 136.25 \end{bmatrix}$$

$$X_1 = 125.27$$
$$X_2 = 134.33$$
$$X_3 = 136.25$$

B.7 EIGENVALUES AND EIGENVECTORS

When working with point clouds or computing error ellipses, it is sometimes advantageous to derive eigenvalues and their eigenvectors. For the eigenvalue, λ,

and eigenvector, v, to be an eigenvalue–eigenvector pair, they must satisfy the equation:

$$Av = \lambda v \tag{B.11}$$

where A is a square matrix. In this book, the normal equations are square positive-definite matrices.

Now, an identity matrix is introduced into Equation (B.11) so that we have equally sized matrices on both sides of the equation, and the right side of the equation is brought to the left as

$$Av - \lambda Iv = \vec{0} \tag{B.12}$$

where the identity matrix, I, has the same dimensions as A and the zero vector, $\vec{0}$, has the same number of rows as A. If v is nonzero, we can hopefully solve for λ, the eigenvalues, with the determinant of

$$\|A - \lambda I\| = 0 \tag{B.13}$$

For example, the matrix A, which is

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

has the eigenvalues 1 and 3, which pair with the eigenvectors:

$$v_1 = \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

where v_1 is the eigenvector paired with the eigenvalue 1 and v_2 is the eigenvector paired with the eigenvalue 3. Thus,

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = 1 \begin{bmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

and

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix} = 3 \begin{bmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

To show this solution, we substitute matrix A into Equation (B.13) and get:

$$\begin{Vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{Vmatrix} = 0$$

which results in the polynomial expression

$$\lambda^2 - 4\lambda + 3 = 0$$

This expression can be factored into

$$(\lambda - 1)(\lambda - 3) = 0$$

Thus, the roots for this polynomial are the eigenvalues of the A matrix, which are 1 and 3.

To solve for their corresponding eigenvectors, we substitute the roots into Equation (B.12). For example, for the eigenvalue 3, we have

$$\begin{bmatrix} 2-3 & 1 \\ 1 & 2-3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or

$$-x_1 + x_2 = 0$$

$$x_1 - x_2 = 0$$

Thus, any two-row vector where $x_1 = x_2$ will satisfy Equations (B.11) and (B.12). As shown earlier, the vector v_2 satisfies this equation. Similarly, substituting in the eigenvalue 1 results in the equation $x_1 + x_2 = 0$, or any vector where $x_1 = -x_2$ will satisfy Equation (B.11). As shown previously, v_1 satisfies this condition.

This process can be extended to an n-order matrix. However, for a three-by-three matrix, the resulting polynomial will be a cubic expression. Unfortunately, as the size of matrix A increases, the work to solve for the roots and then the eigenvectors become quite difficult and tedious to solve by hand. Thus, software is typically used to solve for eigenvalues and eigenvectors, which will be the norm in this book.

PROBLEMS

B.1 Explain when a 2×2 matrix has no inverse.

B.2 Find the inverse of A using the method of adjoints

$$A = \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

B.3 Find the inverse of A in Problem B.2 using elementary row transformations.

B.4 Solve the following system of linear equations using matrix methods.

$$x + 5y = -8$$
$$-x - 2y = -1$$

B.5 Solve the following system of linear equations using matrix methods.

$$x + y - z = -8$$
$$3x - y + z = -4$$
$$-x + 2y + 2z = 21$$

B.6 Compute the inverses of the following matrices.

$$A = \begin{bmatrix} 8 & 5 \\ 3 & 12 \end{bmatrix} \qquad B = \begin{bmatrix} 16 & 2 \\ -8 & 3 \end{bmatrix}$$

B.7 Compute the inverses of the following matrices.

$$A = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 3 & 7 \\ -1 & 0 & 4 \\ 2 & 8 & 10 \end{bmatrix}$$

B.8 Compute the inverses of the following matrices.

$$A = \begin{bmatrix} 13 & -6 & 0 \\ -6 & 18 & -6 \\ 0 & -6 & 16 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 6 \\ 2 & -3 & 4 \\ 0 & 6 & -12 \end{bmatrix}$$

B.9 Solve the following matrix system.

$$\begin{bmatrix} 13 & -6 & 0 \\ -6 & 18 & -6 \\ 0 & -6 & 16 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 740.02 \\ 612.72 \\ 1072.22 \end{bmatrix}$$

Use the MATRIX software to do each problem.

B.10 Problem B.4.

B.11 Problem B.5.

B.12 Problem B.6.

B.13 Problem B.7.

B.14 Problem B.8.

B.15 Problem B.9.

PROGRAMMING PROBLEMS

B.16 Select one of the coded matrix inverse routines from Table B.1, enter the code into a computer and use it to solve Problem B.7. (*Hint:* Place the code in Table B.1 in a separate subroutine/function/procedure to be called from the main program.)

B.17 Add a block of code to the inverse routine in the language of your choice that will inform the user when a matrix is singular.

B.18 Write a program that reads and writes a file with a nonsingular matrix; finds its inverse, and writes the results. Use this program to solve Problem B.7. (*Hint:* Place the reading, writing, and inversing code in separate subroutines/functions/procedures to be called from the main program. Provide a way to identify each matrix in the output file.)

B.19 Write a program that reads and writes a file containing a system of equations written in matrix form, solves the system using matrix operations, and writes the solution. Use this program to solve Problem B.9. (*Hint:* Place the reading, writing, and inversing code in separate subroutines/functions/procedures to be called from the main program. Provide a way to identify each matrix in the output file.)

APPENDIX C

NONLINEAR EQUATIONS AND TAYLOR'S THEOREM

C.1 INTRODUCTION

In adjustment computations, it is frequently necessary to deal with nonlinear equations. For example, some observation equations relate observed quantities to unknown parameters through the transcendental functions of sine, cosine, or tangent, while others relate them through terms raised to second- and higher-order powers. The task of solving a system of nonlinear equations is formidable. To facilitate the solution, a first-order Taylor series approximation can be used to create a set of linear equations. The equations can then be solved by matrix methods discussed in Appendix B.

C.2 TAYLOR SERIES LINEARIZATION OF NONLINEAR EQUATIONS

 Suppose that the following equation relates an observed value L to its unknown parameters x and y through nonlinear coefficients as

$$L = f(x, y) \tag{C.1}$$

By Taylor's theorem, the equation can be represented as

$$L = f(x,y) = f(x_0,y_0) + \frac{(\partial L/\partial x)_0}{1!}dx + \frac{(\partial^2 L/\partial x^2)_0}{2!}dx^2 + \cdots + \frac{(\partial^n L/\partial x^n)_0}{n!}dx^n + \frac{(\partial L/\partial y)_0}{1!}dy + \frac{(\partial^2 L/\partial y^2)_0}{2!}dy^2 + \cdots + \frac{(\partial^n L/\partial y^n)_0}{n!}dy^n + R \quad \text{(C.2)}$$

In Equation (C.2), x_0 and y_0 are approximations for x and y; $f(x_0, y_0)$ is the nonlinear function evaluated at these approximations; R is the remainder, and dx and dy are corrections to the approximations, such that

$$x = x_0 + dx$$
$$y = y_0 + dy \quad \text{(C.3)}$$

A more exact Taylor series approximation is obtained by increasing the value of n in Equation (C.2). However, as the order of each successive term increases, its significance in the overall expression decreases. If all terms containing derivatives higher than the first are dropped, the following linear expression is obtained:

$$L = f(x,y) = f(x_0,y_0) + \left(\frac{\partial L}{\partial x}\right)_0 dx + \left(\frac{\partial L}{\partial y}\right)_0 dy \quad \text{(C.4)}$$

Once the initial approximations are selected, the only unknowns in Equation (C.4) are the corrections dx and dy. Of course, by dropping the higher-order terms from the Taylor series, Equation (C.4) becomes only a good approximation for the original equation. Thus, an iterative procedure is followed in the solution that yields accurate results as dx and dy go to zero. This iterative procedure uses the following steps:

Step 1: Determine initial approximations for the unknowns. They may be obtained by guessing or from observations. However, the closer the initial approximations are to the final solution, the faster the final solution will be obtained. For some problems, initial approximations are obtained from graphical solutions or computed from available data or observations. For others, the determination of initial approximations can involve considerable computational effort.

Step 2: Substitute the initial approximations into Equation (C.4) and solve for the corrections dx and dy.

Step 3: Calculate revised values of x and y using Equations (C.3).

Step 4: Using these newly revised values for x and y, repeat steps 2 and 3.

Step 5: Continue the above procedure until the corrections dx and dy are small enough to bring x and y within tolerable accuracy. When this occurs, the solution is said to have *converged*.

C.3 NUMERICAL EXAMPLE

To clarify this procedure further, a numerical example will be solved.

Example C.1 Linearize the following pair of nonlinear equations containing the two unknowns x and y and solve for the unknown parameters x and y:

$$F: \quad x + y - 2y^2 = -4$$
$$G: \quad x^2 + y^2 = 8$$

SOLUTION Determine the partial derivative for each equation with respect to each unknown:

$$\frac{\partial F}{\partial x} = 1 \qquad \frac{\partial F}{\partial y} = 1 - 4y$$
$$\frac{\partial G}{\partial x} = 2x \qquad \frac{\partial G}{\partial y} = 2y$$

Compute an approximate solution. An estimate of $x = 1$ and $y = 1$ is used for the approximations initially.

First Iteration: Write the linearized equations in the form of Equation (C.4):

$$F: \quad dx + [1 - 4(1)]dy = -4 - (1 + 1 - 2(1)^2)$$
$$G: \quad 2(1)dx + 2(1)dy = 8 - (1^2 + 1^2)$$

From these two equations, solve for the unknowns dx and dy according to Equation (C.3):

$$dx = 1.25 \quad \text{and} \quad dy = 1.75$$

Using this solution, determine updated values for x and y:

$$x = x_0 + dx = 1 + 1.25 = 2.25$$
$$y = y_0 + dy = 1 + 1.75 = 2.75$$

Second Iteration: Continue the procedure demonstrated for the first iteration:

$$F: \quad dx + [1 - 4(2.75)]dy = -4 - (2.25 + 2.75 - 2(2.75)^2)$$
$$G: \quad 2(2.25)dx + 2(2.75)dy = 8 - (2.25^2 + 2.75^2)$$

From these two equations, $dx = -0.25$ and $dy = -0.64$, from which

$$x = x_0 + dx = 2.25 - 0.25 = 2.00$$
$$y = y_0 + dy = 2.75 - 0.64 = 2.11$$

Third Iteration: Substitute the new approximations into the equations and solve the equations for the unknown parameters:

$$F: \quad dx + [1 - 4(2.11)]dy = -4 - (2.00 + 2.11 - 2(2.11)^2)$$
$$G: \quad 2(2.00)dx + 2(2.11)dy = 8 - (2.00^2 + 2.11^2)$$

From these two equations, $dx = 0.00$ and $dy = -0.11$, from which

$$x = x_0 + dx = 2.00 + 0.00 = 2.00$$
$$y = y_0 + dy = 2.11 - 0.11 = 2.00$$

Fourth Iteration: Continue the iteration procedure.

$$F: \quad dx + [1 - 4(2.00)]dy = -4 - (2.00 + 2.00 - 2(2.00)^2)$$
$$G: \quad 2(2.00)dx + 2(2.00)dy = 8 - (2.00^2 + 2.00^2)$$

Using the two previous equations, the corrections to x and y are zero to the nearest hundredth. Thus, the solution has converged and the values of $x = 2.00$ and $y = 2.00$ are the desired unknowns. Note that the initial values for the approximations were relatively poor and four iterations were required to find the final solution. However, had better estimates been made (say, $x_0 = 2.1$ and $y_0 = 1.9$), the solution would have converged in one or two iterations and saved computational effort. Fortunately, there are accepted computational procedures to determine close approximations in most surveying problems.

C.4 USING MATRICES TO SOLVE NONLINEAR EQUATIONS



The example of Section C.3 could be solved using matrix methods. However, as in the algebraic approach, the equations must be linearized using Taylor's series. To facilitate linearization using Taylor's theorem, a *Jacobian* matrix (a matrix consisting of the partial derivatives taken with respect to the unknown variables) is formed. This is the coefficient matrix of the linearized equations. The Jacobian matrix for the example in Section C.3 is

$$J = \begin{bmatrix} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial y} \\ \dfrac{\partial G}{\partial x} & \dfrac{\partial G}{\partial y} \end{bmatrix}$$

In the preceding Jacobian matrix, the first column contains the partial derivative for each equation with respect to x, and the second column contains each the partial

derivative of each equation with respect to y. The linearized form of the equations can then be expressed in matrix notation as

$$JX = K \tag{C.5}$$

In Equation (C.5), J is the Jacobian matrix, X the matrix of unknown corrections dx and dy, and K the matrix of constants. Specifically for the example of Section C.3, these matrices are

$$J = \begin{bmatrix} 1 & 1-4y_0 \\ 2x_0 & 2y_0 \end{bmatrix} \quad X = \begin{bmatrix} dx \\ dy \end{bmatrix} \quad K = \begin{bmatrix} -4 - F(x_0, y_0) \\ 8 - G(x_0, y_0) \end{bmatrix}$$

where $F(x_0, y_0)$ and $G(x_0, y_0)$ are the equations F and G solved at the approximates of x_0 and y_0.

Beginning with a set of approximations x_0 and y_0, the J and K matrices of Equation (C.5) are formed. X is computed using the matrix methods presented in Appendix B. Having updated the unknowns according to Equations (C.3), the J and K matrices are formed again and the solution for X computed again. This procedure is iterated until convergence is achieved.

C.5 SIMPLE MATRIX EXAMPLE

Example C.2 Find the solution of the nonlinear system of equations shown below using matrix methods:

$$F: \quad x^2 + 3xy - 4y^2 = 6$$
$$G: \quad x + xy - y^2 = 3$$

SOLUTION The partial derivatives of functions F and G with respect to the unknown's, x and y, are

$$\frac{\partial F}{\partial x} = 2x + 3y \qquad \frac{\partial F}{\partial y} = 3x - 8y$$
$$\frac{\partial G}{\partial x} = 1 + y \qquad \frac{\partial G}{\partial y} = x - 2y$$

Thus, the Jacobian matrix is

$$J = \begin{bmatrix} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial y} \\ \dfrac{\partial G}{\partial x} & \dfrac{\partial G}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x_0 + 3y_0 & 3x_0 - 8y_0 \\ 1 + y_0 & x_0 - 2y_0 \end{bmatrix}$$

The system of equations to solve is

$$\begin{bmatrix} 2x_0 + 3y_0 & 3x_0 - 8y_0 \\ 1 + y_0 & x_0 - 2y_0 \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} 6 - F(x_0, y_0) \\ 3 - G(x_0, y_0) \end{bmatrix}$$

First Iteration: Using the approximate solution of $x_0 = 3$ and $y_0 = 0$ yields

$$\begin{bmatrix} 6 & 9 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} 6-9 \\ 3-3 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

The determinant for the Jacobian matrix above is $3(6) - 9(1) = 9$, and thus the matrix solution is

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 3 & -9 \\ -1 & 6 \end{bmatrix} \begin{bmatrix} -3 \\ 0 \end{bmatrix} = \begin{bmatrix} -1.0 \\ 0.3 \end{bmatrix}$$

Applying Equation (C.3), the new approximations for the unknowns used in the second iteration are

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 2.0 \\ 0.3 \end{bmatrix}$$

Second Iteration:

$$\begin{bmatrix} 4.9 & 3.6 \\ 1.3 & 1.4 \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} = \begin{bmatrix} 6-5.44 \\ 3-2.51 \end{bmatrix} = \begin{bmatrix} 0.56 \\ 0.49 \end{bmatrix}$$

From this dx and dy are found to be 0.45 and 0.77, respectively. This makes the approximation for the unknowns of x and y for the third iteration as 1.55 and 1.07, respectively. These procedures are followed until the final solution for x and y is found to be 2.00 and 1.00, respectively. Again, fewer iterations would have been required if the initial approximations had been closer to their final values.

C.6 PRACTICAL EXAMPLE

Example C.3 Assume that the x and y coordinates of three points on a circle have been observed. Their coordinates are (9.4, 5.6), (7.6, 7.2), and (3.8, 4.8), respectively. The equation for a circle with center (h, k) and radius r is $(x-h)^2 + (y-k)^2 = r^2$. Determine the coordinates of the center of the circle and its radius.

SOLUTION The equation of a circle is rewritten as $C(h, k, r) = (x-h)^2 + (y-k)^2 - r^2 = 0$. The partial derivatives with respect to the unknowns h, k, and r are

$$\frac{\partial C}{\partial h} = -2(x-h) \quad \frac{\partial C}{\partial k} = -2(y-k) \quad \frac{\partial C}{\partial r} = -2r$$

For each point observed, one equation is written, resulting in a system of three equations and three unknowns. The general linearized form of these equations expressed using matrices is

$$\begin{bmatrix} \frac{\partial C_1}{\partial h} & \frac{\partial C_1}{\partial k} & \frac{\partial C_1}{\partial r} \\ \frac{\partial C_2}{\partial h} & \frac{\partial C_2}{\partial k} & \frac{\partial C_2}{\partial r} \\ \frac{\partial C_3}{\partial h} & \frac{\partial C_2}{\partial k} & \frac{\partial C_3}{\partial r} \end{bmatrix} \begin{bmatrix} dh \\ dk \\ dr \end{bmatrix} = \begin{bmatrix} 0 - \left[(x_1 - h_0)^2 + (y_1 - k_0)^2 - r_0^2\right] \\ 0 - \left[(x_2 - h_0)^2 + (y_2 - k_0)^2 - r_0^2\right] \\ 0 - \left[(x_3 - h_0)^2 + (y_3 - k_0)^2 - r_0^2\right] \end{bmatrix} \tag{C.6}$$

After taking partial derivatives, Equation (C.6) becomes

$$\begin{bmatrix} -2(x_1 - h_0) & -2(y_1 - k_0) & -2r_0 \\ -2(x_2 - h_0) & -2(y_2 - k_0) & -2r_0 \\ -2(x_3 - h_0) & -2(y_3 - k_0) & -2r_0 \end{bmatrix} \begin{bmatrix} dh \\ dk \\ dr \end{bmatrix} = \begin{bmatrix} 0 - \left[(x_1 - h_0)^2 + (y_1 - k_0)^2 - r_0^2\right] \\ 0 - \left[(x_2 - h_0)^2 + (y_2 - k_0)^2 - r_0^2\right] \\ 0 - \left[(x_3 - h_0)^2 + (y_3 - k_0)^2 - r_0^2\right] \end{bmatrix} \tag{C.7}$$

Equations (C.7) can be simplified by multiplying each side by $-1/2$. The resulting equations are

$$\begin{bmatrix} (x_1 - h_0) & (y_1 - k_0) & r_0 \\ (x_2 - h_0) & (y_2 - k_0) & r_0 \\ (x_3 - h_0) & (y_3 - k_0) & r_0 \end{bmatrix} \begin{bmatrix} dh \\ dk \\ dr \end{bmatrix} = \begin{bmatrix} 0.5\left[(x_1 - h_0)^2 + (y_1 - k_0)^2 - r_0^2\right] \\ 0.5\left[(x_2 - h_0)^2 + (y_2 - k_0)^2 - r_0^2\right] \\ 0.5\left[(x_3 - h_0)^2 + (y_3 - k_0)^2 - r_0^2\right] \end{bmatrix} \tag{C.8}$$

Assuming approximate values for h, k, and r as 7, 4.5, and 3, respectively, Equations (C.8) are

$$\begin{bmatrix} (9.4 - 7) & (5.6 - 4.5) & 3 \\ (7.6 - 7) & (7.2 - 4.5) & 3 \\ (3.8 - 7) & (4.8 - 4.5) & 3 \end{bmatrix} \begin{bmatrix} dh \\ dk \\ dr \end{bmatrix} = \begin{bmatrix} 0.5[(9.4 - 7)^2 + (5.6 - 4.5)^2 - 3^2] \\ 0.5[(7.6 - 7)^2 + (7.2 - 4.5)^2 - 3^2] \\ 0.5[(3.8 - 7)^2 + (4.8 - 4.5)^2 - 3^2] \end{bmatrix} \tag{C.9}$$

Simplifying Equations (C.9) yields

$$\begin{bmatrix} 2.4 & 1.1 & 3 \\ 0.6 & 2.7 & 3 \\ -3.2 & 0.3 & 3 \end{bmatrix} \begin{bmatrix} dh \\ dk \\ dr \end{bmatrix} = \begin{bmatrix} -1.015 \\ -0.675 \\ 0.665 \end{bmatrix}$$

Solving this system gives results of

$$\begin{bmatrix} dh \\ dk \\ dr \end{bmatrix} = \begin{bmatrix} -0.28462 \\ -0.10769 \\ -0.07115 \end{bmatrix}$$

After applying these changes to the approximation values for the unknowns, updated values for h, k, and r of 6.7154, 4.3923, and 2.9288, respectively, are obtained. The second iteration results in corrections of 0, 0, and 0.014945. Since the correction for r is still comparatively large, the iteration process must continue. After the third iteration, suitable convergence was achieved. The final values for h, k, and r are 6.72, 4.39, and 2.94, respectively, which are within 0.00001 of a perfect solution.

Sometimes, more than one method is available for solving a problem. For example, in the preceding problem an alternative linear form of the equation of a circle can be used. That equation is $x^2 + y^2 + 2dx + 2ey + f = 0$, where the center of the circle is at $(-d, -e)$ and the circle's radius is found as $\sqrt{d^2 + e^2 - f}$. Note that the equation is linear in terms of its unknowns (d, e, f), and thus iterations are not necessary in solving for the unknowns. Writing a rearranged form of this linear equation for each of three observed sets of (x, y) coordinates yields

$$\begin{aligned} 2dx_1 + 2ey_1 + f &= -\left(x_1^2 + y_1^2\right) \\ 2dx_2 + 2ey_2 + f &= -\left(x_2^2 + y_2^2\right) \\ 2dx_3 + 2ey_3 + f &= -\left(x_3^2 + y_3^2\right) \end{aligned} \tag{C.10}$$

Equations (C.10) can, in turn, be represented in matrix notation as

$$\begin{bmatrix} 2x_1 & 2y_1 & 1 \\ 2x_2 & 2y_2 & 1 \\ 2x_3 & 2y_3 & 1 \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} -\left(x_1^2 + y_1^2\right) \\ -\left(x_2^2 + y_2^2\right) \\ -\left(x_3^2 + y_3^2\right) \end{bmatrix}$$

Solving this matrix system, the center of the circle is again found to be (6.72, 4.39) and its radius is determined to be 2.94, without any iterations.

C.7 CONCLUDING REMARKS

In this appendix, the Taylor series has been applied to solve the unknowns in nonlinear equations. Many equations in surveying, geodesy, and photogrammetry are nonlinear. In surveying, examples include the distance and angle formulas, which are nonlinear in terms of station coordinates. The Taylor series is used to linearize these equations and find least-squares solutions. Thus, when performing least-squares adjustments of plane observations, the techniques presented in this appendix must be used in the solutions.

The MATRIX software can be used in conjunction with a spreadsheet to solve these nonlinear problems. As an example, matrices for Example C.3 are formulated using a spreadsheet as shown in Figure C.1. The "final" values for h, k, and r are formed by summing the columns to their left. These values are used to formulate the J and K matrices. Thus, as the updates to the approximate values for

Example C.3

Point	x	y	Parameters	0	1	2	3	Final
1	9.4	5.6	h	7	–0.28	–7E-17		6.7154
2	7.6	7.2	k	4.5	–0.11	5E-16		4.3923
3	3.8	4.8	r	3	–0.07	0.015		2.9438

Example C.3

J

3 3

2.685 1.208 2.944

0.885 2.808 2.944

–2.915 0.408 2.944

K

3 1

0.000

0.000

0.000

Copy shaded area to MATRIX or solve in spreadsheet

Be sure to use a paste special/values.

Spreadsheet solution X

–7E-17

–6E-16

–4E-05

FIGURE C.1 Spreadsheet for Example C.3.

the unknowns are entered into the appropriate cells, the matrices are automatically updated. These updated matrices can then be cut and pasted directly into the MATRIX editor, saved, and the problem solved. Of course, for small problems such as this, the matrix routines available in most spreadsheets can be used also. This was done in Figure C.1. However, as the problem grows, the MATRIX software can easily handle their solution. These procedures are demonstrated in the spreadsheet *Example C-3.xls* on the companion website for this book.

PROBLEMS

C.1 Solve for the unknowns x and y in the following nonlinear equations using Taylor's theorem. (Use $x_0 = 5$ and $y_0 = 5$ for initial approximations.)

$$x^2y - 3x^2 = 75$$
$$x^2 - y = 19$$

C.2 Solve for the unknown values of x, y, and z in the following three nonlinear equations using the Taylor series. (Use $x_0 = y_0 = z_0 = 2$ for initial approximations.)

$$x^2 - y^2 + 2xy + z = 4$$
$$-x + y + z = 4$$
$$-2x^2 - y + z^3 = 23$$

C.3 Use the MATRIX software to solve Problem C.1.

C.4 Use the MATRIX software to solve Problem C.2.

C.5 Find center and radius of a circle using the equation $(x-h)^2+(y-k)^2=r^2$ given the coordinates of points A, B, and C, on the circle. Follow the procedures discussed in Section C.5. Use initial approximations of $h_0=5$, $k_0=4$, and $r_0=2$ for the first iteration:

$$A\text{: } (7.2, 5.2) \quad B\text{: } (4.0, 6.4) \quad C\text{: } (4.0, 2.4)$$

C.6 Repeat Problem C.5 using the linear equation: $x^2+y^2+2dx+2ey+f=0$.

C.7 Repeat Problem C.5 using the points A: (0.50, −0.70), B: (1.00, 0.00), C: (0.70, 0.70).
Use initial approximations of $h_0=0$, $k_0=0$, and $r_0=1$.

C.8 Repeat Problem C.7 using the linear equation: $x^2+y^2+2dx+2ey+f=0$.

C.9 Use the ADJUST software to solve Problem C.5.

C.10 Use the ADJUST software to solve Problem C.7.

C.11 The distance formula between two stations i and j is

$$D_{ij}=\sqrt{(x_j-x_i)^2+(y_j-y_i)^2}.$$

Write the linearized form for this equation in terms of the variables x_i, y_i, x_j, and y_j.

C.12 The azimuth formula between two stations i and j is

$$\alpha_{ij}=\tan^{-1}\left(\frac{x_j-x_i}{y_j-y_i}\right)$$

Write the linearized form for this equation in terms of the variables x_i, y_i, x_j, and y_j.

C.13 The formula for an angle $\angle jik$ is $\alpha_{ik}-\alpha_{ij}$, where α is defined in Problem C.12. Write the linearized form of this equation in terms of the variables x_i, y_i, x_j, y_j, x_k, and y_k.

PROGRAMMING PROBLEMS

C.14 Create a programmed package that solves Problem C.6.

C.15 Create a programmed package that solves Problem C.5.

APPENDIX D

THE NORMAL ERROR DISTRIBUTION CURVE AND OTHER STATISTICAL TABLES

D.1 DEVELOPMENT FOR NORMAL DISTRIBUTION CURVE EQUATION

In Section 2.4, the histogram and frequency polygon were presented as methods for graphically portraying random error distributions. If a large number of these distributions were examined for sets, observations in surveying, it would be found that they conform to normal (or Gaussian) distributions. The general laws governing normal distributions are stated as follows:

1. Positive and negative errors occur with equal probability and equal frequency.
2. Small errors are more common than large errors.
3. Large errors seldom occur, and there is a limit to the size of the greatest random error that will occur in any set of observations.

A curve that conforms to these laws is plotted with the size of the error on the abscissa and probability of occurrence on the ordinate (Figure 3.3). This curve is

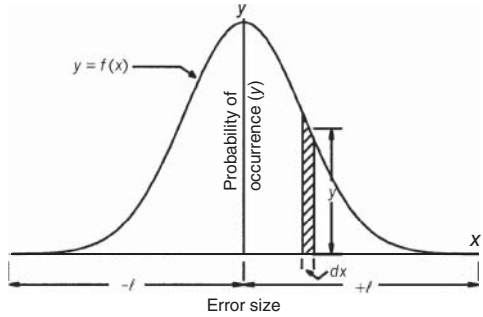


FIGURE D.1 The normal distribution curve.

repeated in Figure D.1 and is called the *normal distribution curve*, the *normal curve of error*, or simply the *probability curve*. A smooth curve of this same shape would be obtained if, for a very large group of observations, a histogram is plotted with an infinitesimally small class interval. In this section, the equation for this curve is developed.

Assume that the normal distribution curve is continuous and that the probability of an error occurring between x and $x + dx$ is given by the function $y = f(x)$. Further assume that this is the equation for the probability curve. The form of $f(x)$ will now be determined. Since, as explained in Chapter 3, probabilities are equivalent to areas under the probability curve, the probabilities of errors occurring within the ranges of (x_1 and $x_1 + dx_1$), (x_2 and $x_2 + dx_2$), etc., are $f(x_1)dx_1, f(x_2)dx_2, \ldots, f(x_n)dx_n$. The total area under the probability curve represents the total probability or simply the integer one. Then, for a finite number of possible errors:

$$f(x_1)dx_1 + f(x_2)dx_2 + \cdots + f(x_n)dx_n = 1 \tag{D.1}$$

If the total range of errors $x_1, x_2, \ldots, x_n$ is between ± 1, then, considering an infinite number of errors that makes the curve continuous, the area under the curve can be set equal to

$$1 = \int_{-1}^{1} f(x)dx$$

But because the area under the curve from $+1$ to $+\infty$ and from -1 to $-\infty$ is essentially zero, the integration limits are extended to $\pm\infty$, as

$$1 = \int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} ydx \tag{D.2}$$

Now suppose that quantity M has been observed, and that it is equal to some function of n unknown parameters $z_1, z_2, \ldots, z_n$ such that $M = f(z_1, z_2, \ldots, z_n)$. Also, let $x_1, x_2, \ldots, x_m$ be the errors of m observations $M_1, M_2, \ldots, M_m$, and let

$f(x_1)\,dx_1, f(x_2)\,dx_2, \ldots, f(x_m)\,dx_m$ be the probabilities of errors falling within the ranges of (x_1 and dx_1), (x_2 and dx_2), etc. By Equation (3.1), the probability P of the simultaneous occurrence of all of these errors is equal to the product of the individual probabilities, thus

$$P = [f(x_1)\,dx_1]\,[f(x_2)\,dx_2]\,\ldots\,[f(x_m)\,dx_m]$$

Then by logs:

$$\log P = \log f(x_1) + \log f(x_2) + \cdots + \log f(x_m) + \log dx_1 + \log dx_2 + \cdots + \log dx_m \tag{D.3}$$

The most probable values of the errors will occur when P is maximized or when the log of P is maximized. To maximize a function, it is differentiated with respect to each unknown parameter z, and the results are set equal to zero. After logarithmic differentiation of Equation (D.3), the following n equations result: (Note that dx's are constants independent of the z's and therefore their differentials with respect to the z's are zero.)

$$\begin{aligned}
\frac{1}{P}\frac{\partial P}{\partial z_1} &= \frac{1}{f(x_1)}\frac{df(x_1)}{dx_1}\frac{dx_1}{dz_1} + \frac{1}{f(x_2)}\frac{df(x_2)}{dx_2}\frac{dx_2}{dz_1} + \cdots + \frac{1}{f(x_m)}\frac{df(x_m)}{dx_m}\frac{dx_m}{dz_1} = 0\\
\frac{1}{P}\frac{\partial P}{\partial z_2} &= \frac{1}{f(x_1)}\frac{df(x_1)}{dx_1}\frac{dx_1}{dz_2} + \frac{1}{f(x_2)}\frac{df(x_2)}{dx_2}\frac{dx_2}{dz_2} + \cdots + \frac{1}{f(x_m)}\frac{df(x_m)}{dx_m}\frac{dx_m}{dz_2} = 0\\
&\vdots\\
\frac{1}{P}\frac{\partial P}{\partial z_n} &= \frac{1}{f(x_1)}\frac{df(x_1)}{dx_1}\frac{dx_1}{dz_n} + \frac{1}{f(x_2)}\frac{df(x_2)}{dx_2}\frac{dx_2}{dz_n} + \cdots + \frac{1}{f(x_m)}\frac{df(x_m)}{dx_m}\frac{dx_m}{dz_n} = 0
\end{aligned} \tag{D.4}$$

Now let

$$f'(x) = \frac{df(x)}{dx} \tag{D.5}$$

Substituting Equation (D.5) into (D.4) yields

$$\begin{aligned}
\frac{f'(x_1)dx_1}{f(x_1)dz_1} + \frac{f'(x_2)dx_2}{f(x_2)dz_1} + \cdots + \frac{f'(x_m)dx_m}{f(x_m)dz_1} &= 0\\
\frac{f'(x_1)dx_1}{f(x_1)dz_2} + \frac{f'(x_2)dx_2}{f(x_2)dz_2} + \cdots + \frac{f'(x_m)dx_m}{f(x_m)dz_2} &= 0\\
&\vdots\\
\frac{f'(x_1)dx_1}{f(x_1)dz_n} + \frac{f'(x_2)dx_2}{f(x_2)dz_n} + \cdots + \frac{f'(x_m)dx_m}{f(x_m)dz_n} &= 0
\end{aligned} \tag{D.6}$$

Thus, far $f(x)$ and $f'(x)$ are general, regardless of the number of unknown parameters. Now consider the special case where there is only one unknown z and M_1, M_2, … , M_m are m observed values of z. If z^* is the true value of the quantity, the errors associated with the observations are

$$x_1 = z^* - M_1, x_2 = z^* - M_2, \ \ldots, x_m = z^* - M_m \tag{D.7}$$

Differentiating Equation (D.7) with respect to z gives

$$1 = \frac{dx_1}{dz} = \frac{dx_2}{dz} = \cdots = \frac{dx_m}{dz} \tag{D.8}$$

Then, for this special case, substituting Equations (D.7) and (D.8) into Equations (D.6), they reduce to a single equation:

$$\frac{f'(z^* - M_1)}{f(z^* - M_1)} + \frac{f'(z^* - M_2)}{f(z^* - M_2)} + \cdots + \frac{f'(z^* - M_m)}{f(z^* - M_m)} = 0 \tag{D.9}$$

Equation (D.9) for this special case in consideration is also general for any value of m and for any observed values $M_1, M_2, \ldots, M_m$. Thus, let the values of M be

$$M_2 = M_3 = \cdots = M_m = M_1 - mN$$

where N is chosen for convenience as $N = (M_1 - M_2)/m$.

The arithmetic mean is the most probable value for this case of a single quantity having been observed several times; therefore, z^* the most probable value in this case is

$$\begin{aligned} z^* &= \frac{M_1 + M_2 + \cdots + M_n}{m} \\ &= \frac{M_1 + (m-1)(M_1 - mN)}{m} \\ &= M_1 - mN + N \\ &= M_1 - N(m-1) \\ z^* - M_1 &= -N(m-1) = N(1-m) \end{aligned} \tag{D.10}$$

Recall that $N = (M_1 - M_2)/m$, from which $M_1 = mN + M_2$. Substituting into Equation (D.10) yields

$$z^* - (mN + M_2) = N(1-m)$$

$$z^* - M_2 = N$$

Similarly, since $N = (M_1 - M_3)/m = (M_1 - M_4)/m$ and so on:

$$z^* - M_3 = N$$

$$z^* - M_4 = N$$

$$\vdots$$

Substituting these expressions into Equation (D.9) yields

$$\frac{f'[n(1-m)]}{f[N(1-M)]} + \frac{(m-1)f'(N)}{f(N)} = 0 \tag{D.11}$$

Rearranging yields

$$\frac{f[N(1-m)]}{Nf[N(1-m)](1-m)} = \frac{f(N)}{f(N)N} = \text{constant}$$

because N in this case is a constant. Thus,

$$\frac{f'(x)}{xf(x)} = \text{constant} = K \tag{D.12}$$

Substituting Equation (D.5) into (D.12), yields

$$f'(x) = xf(x)K = \frac{df(x)}{dx}$$

From which $\frac{df(x)}{dx} = xf(x)K$. Integrating gives

$$\log_e f(x) = \frac{1}{2}Kx^2 + C_1$$

$$f(x) = e^{C_1}e^{Kx^2/2}$$

But letting

$$e^{C_1} = C$$

Then

$$f(x) = Ce^{(Kx^2)/2} \tag{D.13}$$

In Equation (D.13), since $f(x)$ decreases as x increases, and thus the exponent must be negative. Arbitrarily letting

$$h = \sqrt{\frac{K}{2}} \tag{D.14}$$

and incorporating the negative into Equation (D.13), there results

$$f(x) = Ce^{-h^2x^2} \tag{D.15}$$

To find the value of the constant C, substitute Equation (D.15) into (D.2)

$$\int_{-\infty}^{\infty} Ce^{-h^2x^2}dx = 1$$

Also, arbitrarily set $t = hx$, then $dt = hdx$ and $dx = dt/h$, from which, after changing variables, we obtain

$$\frac{C}{h} = \int_{-\infty}^{\infty} e^{-t^2}dt = 1$$

The value of the definite integral is $\sqrt{\pi}$ from which[1]

$$\frac{C}{h\sqrt{\pi}} = 1$$

$$C = \frac{h}{\sqrt{\pi}} \tag{D.16}$$

Substituting Equation (D.16) into (D.15) gives

$$f(x) = \frac{h}{\sqrt{\pi}}e^{-h^2x^2} \tag{D.17}$$

Note that from Equation (D.14) that $h = \sqrt{K/2}$. For the normal distribution, $K = 1/\sigma^2$. Substituting this into Equation (D.17) yields

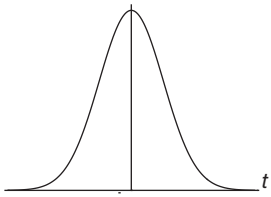
$$f(x) = \frac{1}{\sqrt{2\sigma^2\pi}}e^{-[1/2\sigma^2]x^2} = \frac{1}{\sigma\sqrt{2\pi}}e^{-x^2/2\sigma^2} \tag{D.18}$$

where the terms are as defined in Equation (3.2).

This is the general equation for the probability curve, having been derived in this instance from the consideration of a special case. In Table D.1, values for areas under the standard normal distribution function from negative infinity to t are tabulated.

[1]The technique of integrating this nonelementary function is beyond the scope of this book but can be found in advanced references.

TABLE D.1 Percentage Points for the Standard Normal Distribution Function

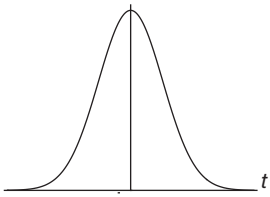


$$N_z(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

	0	1	2	3	4	5	6	7	8	9
−3.2	0.00069	0.00066	0.00064	0.00062	0.00060	0.00058	0.00056	0.00054	0.00052	0.00050
−3.1	0.00097	0.00094	0.00090	0.00087	0.00084	0.00082	0.00079	0.00076	0.00074	0.00071
−3.0	0.00135	0.00131	0.00126	0.00122	0.00118	0.00114	0.00111	0.00107	0.00104	0.00100
−2.9	0.00187	0.00181	0.00175	0.00169	0.00164	0.00159	0.00154	0.00149	0.00144	0.00139
−2.8	0.00256	0.00248	0.00240	0.00233	0.00226	0.00219	0.00212	0.00205	0.00199	0.00193
−2.7	0.00347	0.00336	0.00326	0.00317	0.00307	0.00298	0.00289	0.00280	0.00272	0.00264
−2.6	0.00466	0.00453	0.00440	0.00427	0.00415	0.00402	0.00391	0.00379	0.00368	0.00357
−2.5	0.00621	0.00604	0.00587	0.00570	0.00554	0.00539	0.00523	0.00508	0.00494	0.00480
−2.4	0.00820	0.00798	0.00776	0.00755	0.00734	0.00714	0.00695	0.00676	0.00657	0.00639
−2.3	0.01072	0.01044	0.01017	0.00990	0.00964	0.00939	0.00914	0.00889	0.00866	0.00842
−2.2	0.01390	0.01355	0.01321	0.01287	0.01255	0.01222	0.01191	0.01160	0.01130	0.01101
−2.1	0.01786	0.01743	0.01700	0.01659	0.01618	0.01578	0.01539	0.01500	0.01463	0.01426
−2.0	0.02275	0.02222	0.02169	0.02118	0.02068	0.02018	0.01970	0.01923	0.01876	0.01831
−1.9	0.02872	0.02807	0.02743	0.02680	0.02619	0.02559	0.02500	0.02442	0.02385	0.02330
−1.8	0.03593	0.03515	0.03438	0.03362	0.03288	0.03216	0.03144	0.03074	0.03005	0.02938
−1.7	0.04457	0.04363	0.04272	0.04182	0.04093	0.04006	0.03920	0.03836	0.03754	0.03673
−1.6	0.05480	0.05370	0.05262	0.05155	0.05050	0.04947	0.04846	0.04746	0.04648	0.04551
−1.5	0.06681	0.06552	0.06426	0.06301	0.06178	0.06057	0.05938	0.05821	0.05705	0.05592
−1.4	0.08076	0.07927	0.07780	0.07636	0.07493	0.07353	0.07215	0.07078	0.06944	0.06811
−1.3	0.09680	0.09510	0.09342	0.09176	0.09012	0.08851	0.08691	0.08534	0.08379	0.08226
−1.2	0.11507	0.11314	0.11123	0.10935	0.10749	0.10565	0.10383	0.10204	0.10027	0.09853
−1.1	0.13567	0.13350	0.13136	0.12924	0.12714	0.12507	0.12302	0.12100	0.11900	0.11702
−1.0	0.15866	0.15625	0.15386	0.15151	0.14917	0.14686	0.14457	0.14231	0.14007	0.13786
−0.9	0.18406	0.18141	0.17879	0.17619	0.17361	0.17106	0.16853	0.16602	0.16354	0.16109
−0.8	0.21186	0.20897	0.20611	0.20327	0.20045	0.19766	0.19489	0.19215	0.18943	0.18673
−0.7	0.24196	0.23885	0.23576	0.23270	0.22965	0.22663	0.22363	0.22065	0.21770	0.21476
−0.6	0.27425	0.27093	0.26763	0.26435	0.26109	0.25785	0.25463	0.25143	0.24825	0.24510
−0.5	0.30854	0.30503	0.30153	0.29806	0.29460	0.29116	0.28774	0.28434	0.28096	0.27760
−0.4	0.34458	0.34090	0.33724	0.33360	0.32997	0.32636	0.32276	0.31918	0.31561	0.31207
−0.3	0.38209	0.37828	0.37448	0.37070	0.36693	0.36317	0.35942	0.35569	0.35197	0.34827
−0.2	0.42074	0.41683	0.41294	0.40905	0.40517	0.40129	0.39743	0.39358	0.38974	0.38591
−0.1	0.46017	0.45620	0.45224	0.44828	0.44433	0.44038	0.43644	0.43251	0.42858	0.42465
0.0	0.50000	0.49601	0.49202	0.48803	0.48405	0.48006	0.47608	0.47210	0.46812	0.46414

(*continues*)

TABLE D.1 (*Continued*)



$$N_z(t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

t	0	1	2	3	4	5	6	7	8	9
0.0	0.50000	0.50399	0.50798	0.51197	0.51595	0.51994	0.52392	0.52790	0.53188	0.53586
0.1	0.53983	0.54380	0.54776	0.55172	0.55567	0.55962	0.56356	0.56749	0.57142	0.57535
0.2	0.57926	0.58317	0.58706	0.59095	0.59483	0.59871	0.60257	0.60642	0.61026	0.61409
0.3	0.61791	0.62172	0.62552	0.62930	0.63307	0.63683	0.64058	0.64431	0.64803	0.65173
0.4	0.65542	0.65910	0.66276	0.66640	0.67003	0.67364	0.67724	0.68082	0.68439	0.68793
0.5	0.69146	0.69497	0.69847	0.70194	0.70540	0.70884	0.71226	0.71566	0.71904	0.72240
0.6	0.72575	0.72907	0.73237	0.73565	0.73891	0.74215	0.74537	0.74857	0.75175	0.75490
0.7	0.75804	0.76115	0.76424	0.76730	0.77035	0.77337	0.77637	0.77935	0.78230	0.78524
0.8	0.78814	0.79103	0.79389	0.79673	0.79955	0.80234	0.80511	0.80785	0.81057	0.81327
0.9	0.81594	0.81859	0.82121	0.82381	0.82639	0.82894	0.83147	0.83398	0.83646	0.83891
1.0	0.84134	0.84375	0.84614	0.84849	0.85083	0.85314	0.85543	0.85769	0.85993	0.86214
1.1	0.86433	0.86650	0.86864	0.87076	0.87286	0.87493	0.87698	0.87900	0.88100	0.88298
1.2	0.88493	0.88686	0.88877	0.89065	0.89251	0.89435	0.89617	0.89796	0.89973	0.90147
1.3	0.90320	0.90490	0.90658	0.90824	0.90988	0.91149	0.91309	0.91466	0.91621	0.91774
1.4	0.91924	0.92073	0.92220	0.92364	0.92507	0.92647	0.92785	0.92922	0.93056	0.93189
1.5	0.93319	0.93448	0.93574	0.93699	0.93822	0.93943	0.94062	0.94179	0.94295	0.94408
1.6	0.94520	0.94630	0.94738	0.94845	0.94950	0.95053	0.95154	0.95254	0.95352	0.95449
1.7	0.95543	0.95637	0.95728	0.95818	0.95907	0.95994	0.96080	0.96164	0.96246	0.96327
1.8	0.96407	0.96485	0.96562	0.96638	0.96712	0.96784	0.96856	0.96926	0.96995	0.97062
1.9	0.97128	0.97193	0.97257	0.97320	0.97381	0.97441	0.97500	0.97558	0.97615	0.97670
2.0	0.97725	0.97778	0.97831	0.97882	0.97932	0.97982	0.98030	0.98077	0.98124	0.98169
2.1	0.98214	0.98257	0.98300	0.98341	0.98382	0.98422	0.98461	0.98500	0.98537	0.98574
2.2	0.98610	0.98645	0.98679	0.98713	0.98745	0.98778	0.98809	0.98840	0.98870	0.98899
2.3	0.98928	0.98956	0.98983	0.99010	0.99036	0.99061	0.99086	0.99111	0.99134	0.99158
2.4	0.99180	0.99202	0.99224	0.99245	0.99266	0.99286	0.99305	0.99324	0.99343	0.99361
2.5	0.99379	0.99396	0.99413	0.99430	0.99446	0.99461	0.99477	0.99492	0.99506	0.99520
2.6	0.99534	0.99547	0.99560	0.99573	0.99585	0.99598	0.99609	0.99621	0.99632	0.99643
2.7	0.99653	0.99664	0.99674	0.99683	0.99693	0.99702	0.99711	0.99720	0.99728	0.99736
2.8	0.99744	0.99752	0.99760	0.99767	0.99774	0.99781	0.99788	0.99795	0.99801	0.99807
2.9	0.99813	0.99819	0.99825	0.99831	0.99836	0.99841	0.99846	0.99851	0.99856	0.99861
3.0	0.99865	0.99869	0.99874	0.99878	0.99882	0.99886	0.99889	0.99893	0.99896	0.99900
3.1	0.99903	0.99906	0.99910	0.99913	0.99916	0.99918	0.99921	0.99924	0.99926	0.99929
3.2	0.99931	0.99934	0.99936	0.99938	0.99940	0.99942	0.99944	0.99946	0.99948	0.99950

D.2 OTHER STATISTICAL TABLES

On the remaining pages of this chapter are three often-used statistical tables. Application of these tables and their interpretation are discussed in detail in Chapter 4. The equations used to generate each of these tables are also presented.

D.2.1 The χ^2 Distribution

Chi-squared is a density function for the distribution of sample variances computed from sets with selected degrees of freedom for a population. The use of this distribution to construct confidence intervals for the population variance and to perform hypothesis testing involving the population variance is discussed in detail in Chapter 4. The χ^2 distribution is illustrated Figure D.2.

The χ^2 distribution critical values given in Table D.2 were generated using the following function (Critical χ^2 values for both tails of the distribution were derived with a program using numerical integration routines similar to those used in STATS.):

$$F(x) = \int_0^{\chi^2} \frac{1}{2^{\upsilon/2}\Gamma(\upsilon/2)} u^{(\upsilon-2)/2} e^{-u/2} du$$

where υ is the degrees of freedom, and Γ is known as the gamma function, which is defined as

$$\Gamma(\upsilon) = \int_0^{\infty} u^{\upsilon-1} e^{-u} du$$

It is computed as $\Gamma(\upsilon) = (\upsilon - 1)! = (\upsilon - 1)(\upsilon - 2)(\upsilon - 3) \cdots (3)(2)(1)$.

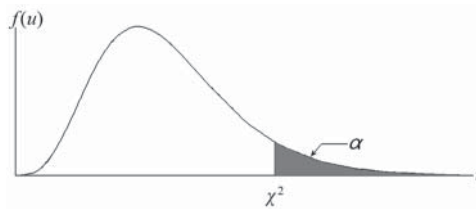


FIGURE D.2 χ^2 distribution.

TABLE D.2 Critical Values for the χ^2 Distribution

ν \ α	0.999	0.995	0.990	0.975	0.950	0.900	0.500	0.100	0.050	0.025	0.010	0.005	0.001
1	1.6*E*−6	3.9*E*−5	1.6*E*−4	0.001	0.004	0.016	0.455	2.706	3.841	5.024	6.635	7.879	10.83
2	0.0020	0.0100	0.0201	0.051	0.103	0.211	1.386	4.605	5.991	7.378	9.210	10.60	13.82
3	0.0243	0.0717	0.115	0.216	0.352	0.584	2.366	6.251	7.815	9.348	11.34	12.84	16.27
4	0.0908	0.207	0.297	0.484	0.711	1.064	3.357	7.779	9.488	11.14	13.28	14.86	18.47
5	0.210	0.412	0.554	0.831	1.145	1.610	4.351	9.236	11.07	12.83	15.09	16.75	20.52
6	0.381	0.676	0.872	1.237	1.635	2.204	5.348	10.64	12.59	14.45	16.81	18.55	22.46
7	0.598	0.989	1.239	1.690	2.167	2.833	6.346	12.02	14.07	16.01	18.48	20.28	24.32
8	0.857	1.344	1.646	2.180	2.733	3.490	7.344	13.36	15.51	17.53	20.09	21.95	26.12
9	1.152	1.735	2.088	2.700	3.325	4.168	8.343	14.68	16.92	19.02	21.67	23.59	27.88
10	1.479	2.156	2.558	3.247	3.940	4.865	9.342	15.99	18.31	20.48	23.21	25.19	29.59
11	1.834	2.603	3.053	3.816	4.575	5.578	10.34	17.28	19.68	21.92	24.72	26.76	31.26
12	2.214	3.074	3.571	4.404	5.226	6.304	11.34	18.55	21.03	23.34	26.22	28.30	32.91
13	2.617	3.565	4.107	5.009	5.892	7.042	12.34	19.81	22.36	24.74	27.69	29.82	34.53
14	3.041	4.075	4.660	5.629	6.571	7.790	13.34	21.06	23.68	26.12	29.14	31.32	36.12
15	3.483	4.601	5.229	6.262	7.261	8.547	14.34	22.31	25.00	27.49	30.58	32.80	37.70
16	3.942	5.142	5.812	6.908	7.962	9.312	15.34	23.54	26.30	28.85	32.00	34.27	39.25
17	4.416	5.697	6.408	7.564	8.672	10.09	16.34	24.77	27.59	30.19	33.41	35.72	40.79
18	4.905	6.265	7.015	8.231	9.390	10.86	17.34	25.99	28.87	31.53	34.81	37.16	42.31
19	5.407	6.844	7.633	8.907	10.12	11.65	18.34	27.20	30.14	32.85	36.19	38.58	43.82
20	5.921	7.434	8.260	9.591	10.85	12.44	19.34	28.41	31.41	34.17	37.57	40.00	45.31
21	6.447	8.034	8.897	10.28	11.59	13.24	20.34	29.62	32.67	35.48	38.93	41.40	46.80
22	6.983	8.643	9.542	10.98	12.34	14.04	21.34	30.81	33.92	36.78	40.29	42.80	48.27
23	7.529	9.260	10.20	11.69	13.09	14.85	22.34	32.01	35.17	38.08	41.64	44.18	49.73
24	8.085	9.886	10.86	12.40	13.85	15.66	23.34	33.20	36.42	39.36	42.98	45.56	51.18
25	8.649	10.52	11.52	13.12	14.61	16.47	24.34	34.38	37.65	40.65	44.31	46.93	52.62
26	9.222	11.16	12.20	13.84	15.38	17.29	25.34	35.56	38.89	41.92	45.64	48.29	54.05
27	9.803	11.81	12.88	14.57	16.15	18.11	26.34	36.74	40.11	43.19	46.96	49.64	55.48
28	10.39	12.46	13.56	15.31	16.93	18.94	27.34	37.92	41.34	44.46	48.28	50.99	56.89
29	10.99	13.12	14.26	16.05	17.71	19.77	28.34	39.09	42.56	45.72	49.59	52.34	58.30
30	11.59	13.79	14.95	16.79	18.49	20.60	29.34	40.26	43.77	46.98	50.89	53.67	59.70
35	14.69	17.19	18.51	20.57	22.47	24.80	34.34	46.06	49.80	53.20	57.34	60.27	66.62
40	17.92	20.71	22.16	24.43	26.51	29.05	39.34	51.81	55.76	59.34	63.69	66.77	73.40
50	24.67	27.99	29.71	32.36	34.76	37.69	49.33	63.17	67.50	71.42	76.15	79.49	86.66
60	31.74	35.53	37.48	40.48	43.19	46.46	59.33	74.40	79.08	83.30	88.38	91.95	99.61
120	77.76	83.85	86.92	91.57	95.70	100.62	119.33	140.23	146.57	152.21	158.95	163.65	173.62

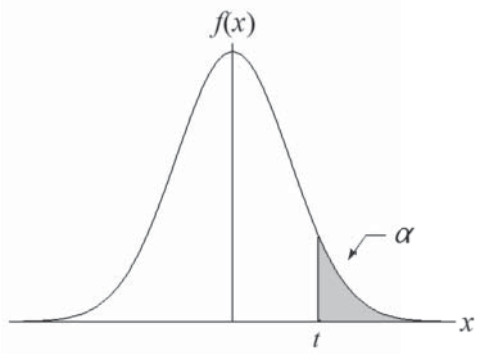


FIGURE D.3 *t* distribution.

D.2.2 *t* Distribution

The *t* distribution function, shown in Figure D.3, is used to derive confidence intervals for the population mean when the sample set is small. It is also used in hypothesis testing to check the validity of a sample mean against a population mean. The uses for this distribution are discussed in greater detail in Chapter 4.

The *t* distribution tables were generated using the following function (Critical *t* values for the upper tail of the distribution were derived with a program using numerical integration routines similar to those available in STATS.):

$$\alpha = \int \frac{\Gamma(\upsilon+1)/2}{\sqrt{\upsilon\pi}\Gamma(\upsilon/2)}\left(1+\frac{x^2}{\upsilon}\right)^{-(\upsilon+1)/2} dx$$

where Γ is the gamma function as defined in Section D.2.1, and υ is the degrees of freedom in the function. In Table D.3, critical values of *t* are listed that are required to achieve the percentage points listed in the top row. The distribution is symmetrical, and thus

$$F(-t) = 1 - F(t)$$

TABLE D.3 Critical Values for the *t* Distribution

$\nu \downarrow$ $\alpha \rightarrow$	0.400	0.350	0.300	0.250	0.200	0.150	0.100	0.050	0.025	0.010	0.005	0.001	0.0005
1	0.325	0.510	0.727	1.000	1.376	1.963	3.078	6.314	12.706	31.82	63.66	318.3	636.6
2	0.289	0.445	0.617	0.816	1.061	1.386	1.886	2.920	4.303	6.965	9.925	22.33	31.60
3	0.277	0.424	0.584	0.765	0.978	1.250	1.638	2.353	3.182	4.541	5.841	10.21	12.92
4	0.271	0.414	0.569	0.741	0.941	1.190	1.533	2.132	2.776	3.747	4.604	7.173	8.610
5	0.267	0.408	0.559	0.727	0.920	1.156	1.476	2.015	2.571	3.365	4.032	5.893	6.869
6	0.265	0.404	0.553	0.718	0.906	1.134	1.440	1.943	2.447	3.143	3.707	5.208	5.959
7	0.263	0.402	0.549	0.711	0.896	1.119	1.415	1.895	2.365	2.998	3.499	4.785	5.408
8	0.262	0.399	0.546	0.706	0.889	1.108	1.397	1.860	2.306	2.896	3.355	4.501	5.041
9	0.261	0.398	0.543	0.703	0.883	1.100	1.383	1.833	2.262	2.821	3.250	4.297	4.781
10	0.260	0.397	0.542	0.700	0.879	1.093	1.372	1.812	2.228	2.764	3.169	4.144	4.587
11	0.260	0.396	0.540	0.697	0.876	1.088	1.363	1.796	2.201	2.718	3.106	4.025	4.437
12	0.259	0.395	0.539	0.695	0.873	1.083	1.356	1.782	2.179	2.681	3.055	3.930	4.318
13	0.259	0.394	0.538	0.694	0.870	1.079	1.350	1.771	2.160	2.650	3.012	3.852	4.221
14	0.258	0.393	0.537	0.692	0.868	1.076	1.345	1.761	2.145	2.624	2.977	3.787	4.140
15	0.258	0.393	0.536	0.691	0.866	1.074	1.341	1.753	2.131	2.602	2.947	3.733	4.073
16	0.258	0.392	0.535	0.690	0.865	1.071	1.337	1.746	2.120	2.583	2.921	3.686	4.015
17	0.257	0.392	0.534	0.689	0.863	1.069	1.333	1.740	2.110	2.567	2.898	3.646	3.965
18	0.257	0.392	0.534	0.688	0.862	1.067	1.330	1.734	2.101	2.552	2.878	3.610	3.922
19	0.257	0.391	0.533	0.688	0.861	1.066	1.328	1.729	2.093	2.539	2.861	3.579	3.883
20	0.257	0.391	0.533	0.687	0.860	1.064	1.325	1.725	2.086	2.528	2.845	3.552	3.850
21	0.257	0.391	0.532	0.686	0.859	1.063	1.323	1.721	2.080	2.518	2.831	3.527	3.819
22	0.256	0.390	0.532	0.686	0.858	1.061	1.321	1.717	2.074	2.508	2.819	3.505	3.792
23	0.256	0.390	0.532	0.685	0.858	1.060	1.319	1.714	2.069	2.500	2.807	3.485	3.768
24	0.256	0.390	0.531	0.685	0.857	1.059	1.318	1.711	2.064	2.492	2.797	3.467	3.745
25	0.256	0.390	0.531	0.684	0.856	1.058	1.316	1.708	2.060	2.485	2.787	3.450	3.725
26	0.256	0.390	0.531	0.684	0.856	1.058	1.315	1.706	2.056	2.479	2.779	3.435	3.707
27	0.256	0.389	0.531	0.684	0.855	1.057	1.314	1.703	2.052	2.473	2.771	3.421	3.690
28	0.256	0.389	0.530	0.683	0.855	1.056	1.313	1.701	2.048	2.467	2.763	3.408	3.674
29	0.256	0.389	0.530	0.683	0.854	1.055	1.311	1.699	2.045	2.462	2.756	3.396	3.659
30	0.256	0.389	0.530	0.683	0.854	1.055	1.310	1.697	2.042	2.457	2.750	3.385	3.646
35	0.255	0.388	0.529	0.682	0.852	1.052	1.306	1.690	2.030	2.438	2.724	3.340	3.591
40	0.255	0.388	0.529	0.681	0.851	1.050	1.303	1.684	2.021	2.423	2.704	3.307	3.551
60	0.254	0.387	0.527	0.679	0.848	1.045	1.296	1.671	2.000	2.390	2.660	3.232	3.460
120	0.254	0.386	0.526	0.677	0.845	1.041	1.289	1.658	1.980	2.358	2.617	3.160	3.373
∞	0.253	0.385	0.525	0.675	0.842	1.037	1.282	1.645	1.960	2.326	2.576	3.291	3.300

D.2.3 *F* Distribution (Also Known as *Fisher Distribution*)

This F distribution function, shown in Figure D.4, is used to derive confidence intervals for the ratio of two population variances. It is also used in hypothesis testing for this same ratio and to determine the appropriate multiplier for determining the sizes of error ellipses at various probabilities. The uses for this distribution are discussed in Chapter 4.

Critical F values for the upper tail of the distribution were derived with a program using numerical integration routines similar to those used in STATS. The tables were generated using the following function:

$$\alpha = \int_F^{\infty} \frac{\Gamma(\upsilon_1 + \upsilon_2)/2}{\Gamma(\upsilon_1/2)\Gamma(\upsilon_2/2)} \left(\frac{\upsilon_1}{\upsilon_2}\right)^{\upsilon_1/2} \frac{x^{(\upsilon_1-2)/2}}{1 + (\upsilon_1/\upsilon_2)x^{(\upsilon_1+\upsilon_2)/2}} dx$$

where Γ is the gamma function as defined in Section D.2.1, υ_1 is the numerator degrees of freedom, and υ_2 is the denominator degrees of freedom.

For critical values in the lower tail of the distribution, the following relationship can be used in conjunction with the tabular values given in the following pages (Table D.4):

$$F_{\alpha,\upsilon_1,\upsilon_2} = \frac{1}{F_{1-\alpha,\upsilon_2,\upsilon_1}}$$

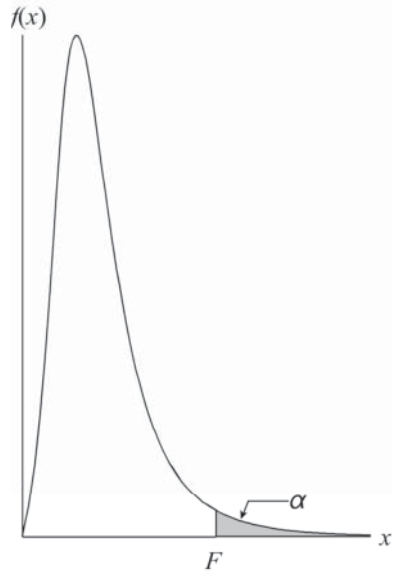


FIGURE D.4 F distribution.

TABLE D.4 Critical Values for the F Distribution

$\alpha = 0.20$

$v_2 \downarrow$ \ $v_1 \rightarrow$	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120
1	9.47	12.00	13.06	13.64	14.01	14.26	14.44	14.58	14.68	14.77	14.90	15.04	15.17	15.24	15.31	15.37	15.44	15.51
2	3.56	4.00	4.16	4.24	4.28	4.32	4.34	4.36	4.37	4.38	4.40	4.42	4.43	4.44	4.45	4.46	4.46	4.47
3	2.68	2.89	2.94	2.96	2.97	2.97	2.97	2.98	2.98	2.98	2.98	2.98	2.98	2.98	2.98	2.98	2.98	2.98
4	2.35	2.47	2.48	2.48	2.48	2.47	2.47	2.47	2.46	2.46	2.46	2.45	2.44	2.44	2.44	2.44	2.43	2.43
5	2.18	2.26	2.25	2.24	2.23	2.22	2.21	2.20	2.20	2.19	2.18	2.18	2.17	2.16	2.16	2.15	2.15	2.14
6	2.07	2.13	2.11	2.09	2.08	2.06	2.05	2.04	2.03	2.03	2.02	2.01	2.00	1.99	1.98	1.98	1.97	1.96
7	2.00	2.04	2.02	1.99	1.97	1.96	1.94	1.93	1.93	1.92	1.91	1.89	1.88	1.87	1.86	1.86	1.85	1.84
8	1.95	1.98	1.95	1.92	1.90	1.88	1.87	1.86	1.85	1.84	1.83	1.81	1.80	1.79	1.78	1.77	1.76	1.75
9	1.91	1.93	1.90	1.87	1.85	1.83	1.81	1.80	1.79	1.78	1.76	1.75	1.73	1.72	1.71	1.70	1.69	1.68
10	1.88	1.90	1.86	1.83	1.80	1.78	1.77	1.75	1.74	1.73	1.72	1.70	1.68	1.67	1.66	1.65	1.64	1.63
11	1.86	1.87	1.83	1.80	1.77	1.75	1.73	1.72	1.70	1.69	1.68	1.66	1.64	1.63	1.62	1.61	1.60	1.59
12	1.84	1.85	1.80	1.77	1.74	1.72	1.70	1.69	1.67	1.66	1.65	1.63	1.61	1.60	1.59	1.58	1.56	1.55
13	1.82	1.83	1.78	1.75	1.72	1.69	1.68	1.66	1.65	1.64	1.62	1.60	1.58	1.57	1.56	1.55	1.53	1.52
14	1.81	1.81	1.76	1.73	1.70	1.67	1.65	1.64	1.63	1.62	1.60	1.58	1.56	1.55	1.53	1.52	1.51	1.49
15	1.80	1.80	1.75	1.71	1.68	1.66	1.64	1.62	1.61	1.60	1.58	1.56	1.54	1.53	1.51	1.50	1.49	1.47
16	1.79	1.78	1.74	1.70	1.67	1.64	1.62	1.61	1.59	1.58	1.56	1.54	1.52	1.51	1.49	1.48	1.47	1.45
17	1.78	1.77	1.72	1.68	1.65	1.63	1.61	1.59	1.58	1.57	1.55	1.53	1.50	1.49	1.48	1.46	1.45	1.43
18	1.77	1.76	1.71	1.67	1.64	1.62	1.60	1.58	1.56	1.55	1.53	1.51	1.49	1.48	1.46	1.45	1.43	1.42
19	1.76	1.75	1.70	1.66	1.63	1.61	1.58	1.57	1.55	1.54	1.52	1.50	1.48	1.46	1.45	1.44	1.42	1.40
20	1.76	1.75	1.70	1.65	1.62	1.60	1.58	1.56	1.54	1.53	1.51	1.49	1.47	1.45	1.44	1.42	1.41	1.39
21	1.75	1.74	1.69	1.65	1.61	1.59	1.57	1.55	1.53	1.52	1.50	1.48	1.46	1.44	1.43	1.41	1.40	1.38
22	1.75	1.73	1.68	1.64	1.61	1.58	1.56	1.54	1.53	1.51	1.49	1.47	1.45	1.43	1.42	1.40	1.39	1.37
23	1.74	1.73	1.68	1.63	1.60	1.57	1.55	1.53	1.52	1.51	1.49	1.46	1.44	1.42	1.41	1.39	1.38	1.36
24	1.74	1.72	1.67	1.63	1.59	1.57	1.55	1.53	1.51	1.50	1.48	1.46	1.43	1.42	1.40	1.39	1.37	1.35
25	1.73	1.72	1.66	1.62	1.59	1.56	1.54	1.52	1.51	1.49	1.47	1.45	1.42	1.41	1.39	1.38	1.36	1.34
26	1.73	1.71	1.66	1.62	1.58	1.56	1.53	1.52	1.50	1.49	1.47	1.44	1.42	1.40	1.39	1.37	1.35	1.33
27	1.73	1.71	1.66	1.61	1.58	1.55	1.53	1.51	1.49	1.48	1.46	1.44	1.41	1.40	1.38	1.36	1.35	1.33
28	1.72	1.71	1.65	1.61	1.57	1.55	1.52	1.51	1.49	1.48	1.46	1.43	1.41	1.39	1.37	1.36	1.34	1.32
29	1.72	1.70	1.65	1.60	1.57	1.54	1.52	1.50	1.49	1.47	1.45	1.43	1.40	1.39	1.37	1.35	1.33	1.31
30	1.72	1.70	1.64	1.60	1.57	1.54	1.52	1.50	1.48	1.47	1.45	1.42	1.39	1.38	1.36	1.35	1.33	1.31
50	1.69	1.66	1.60	1.56	1.52	1.49	1.47	1.45	1.43	1.42	1.39	1.37	1.34	1.32	1.30	1.28	1.26	1.24
60	1.68	1.65	1.59	1.55	1.51	1.48	1.46	1.44	1.42	1.41	1.38	1.35	1.32	1.31	1.29	1.27	1.24	1.22
80	1.67	1.64	1.58	1.53	1.50	1.47	1.44	1.42	1.41	1.39	1.37	1.34	1.31	1.29	1.27	1.25	1.22	1.19
120	1.66	1.63	1.57	1.52	1.48	1.45	1.43	1.41	1.39	1.37	1.35	1.32	1.29	1.27	1.25	1.23	1.20	1.17

$\alpha = 0.10$

$v_2 \downarrow$ \ $v_1 \rightarrow$	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120
1	39.85	49.49	53.59	55.83	57.23	58.20	58.90	59.43	59.85	60.19	60.70	61.21	61.73	61.99	62.26	62.52	62.79	63.05
2	8.53	9.00	9.16	9.24	9.29	9.33	9.35	9.37	9.38	9.39	9.41	9.42	9.44	9.45	9.46	9.47	9.47	9.48
3	5.54	5.46	5.39	5.34	5.31	5.28	5.27	5.25	5.24	5.23	5.22	5.20	5.18	5.18	5.17	5.16	5.15	5.14
4	4.54	4.32	4.19	4.11	4.05	4.01	3.98	3.95	3.94	3.92	3.90	3.87	3.84	3.83	3.82	3.80	3.79	3.78
5	4.06	3.78	3.62	3.52	3.45	3.40	3.37	3.34	3.32	3.30	3.27	3.24	3.21	3.19	3.17	3.16	3.14	3.12
6	3.78	3.46	3.29	3.18	3.11	3.05	3.01	2.98	2.96	2.94	2.90	2.87	2.84	2.82	2.80	2.78	2.76	2.74
7	3.59	3.26	3.07	2.96	2.88	2.83	2.78	2.75	2.72	2.70	2.67	2.63	2.59	2.58	2.56	2.54	2.51	2.49
8	3.46	3.11	2.92	2.81	2.73	2.67	2.62	2.59	2.56	2.54	2.50	2.46	2.42	2.40	2.38	2.36	2.34	2.32
9	3.36	3.01	2.81	2.69	2.61	2.55	2.51	2.47	2.44	2.42	2.38	2.34	2.30	2.28	2.25	2.23	2.21	2.18
10	3.28	2.92	2.73	2.61	2.52	2.46	2.41	2.38	2.35	2.32	2.28	2.24	2.20	2.18	2.16	2.13	2.11	2.08
11	3.23	2.86	2.66	2.54	2.45	2.39	2.34	2.30	2.27	2.25	2.21	2.17	2.12	2.10	2.08	2.05	2.03	2.00
12	3.18	2.81	2.61	2.48	2.39	2.33	2.28	2.24	2.21	2.19	2.15	2.10	2.06	2.04	2.01	1.99	1.96	1.93
13	3.14	2.76	2.56	2.43	2.35	2.28	2.23	2.20	2.16	2.14	2.10	2.05	2.01	1.98	1.96	1.93	1.90	1.88
14	3.10	2.73	2.52	2.39	2.31	2.24	2.19	2.15	2.12	2.10	2.05	2.01	1.96	1.94	1.91	1.89	1.86	1.83
15	3.07	2.70	2.49	2.36	2.27	2.21	2.16	2.12	2.09	2.06	2.02	1.97	1.92	1.90	1.87	1.85	1.82	1.79
16	3.05	2.67	2.46	2.33	2.24	2.18	2.13	2.09	2.06	2.03	1.99	1.94	1.89	1.87	1.84	1.81	1.78	1.75
17	3.03	2.64	2.44	2.31	2.22	2.15	2.10	2.06	2.03	2.00	1.96	1.91	1.86	1.84	1.81	1.78	1.75	1.72
18	3.01	2.62	2.42	2.29	2.20	2.13	2.08	2.04	2.00	1.98	1.93	1.89	1.84	1.81	1.78	1.75	1.72	1.69
19	2.99	2.61	2.40	2.27	2.18	2.11	2.06	2.02	1.98	1.96	1.91	1.86	1.81	1.79	1.76	1.73	1.70	1.67
20	2.97	2.59	2.38	2.25	2.16	2.09	2.04	2.00	1.96	1.94	1.89	1.84	1.79	1.77	1.74	1.71	1.68	1.64
21	2.96	2.57	2.36	2.23	2.14	2.08	2.02	1.98	1.95	1.92	1.87	1.83	1.78	1.75	1.72	1.69	1.66	1.62
22	2.95	2.56	2.35	2.22	2.13	2.06	2.01	1.97	1.93	1.90	1.86	1.81	1.76	1.73	1.70	1.67	1.64	1.60
23	2.94	2.55	2.34	2.21	2.11	2.05	1.99	1.95	1.92	1.89	1.84	1.80	1.74	1.72	1.69	1.66	1.62	1.59
24	2.93	2.54	2.33	2.19	2.10	2.04	1.98	1.94	1.91	1.88	1.83	1.78	1.73	1.70	1.67	1.64	1.61	1.57
25	2.92	2.53	2.32	2.18	2.09	2.02	1.97	1.93	1.89	1.87	1.82	1.77	1.72	1.69	1.66	1.63	1.59	1.56
26	2.91	2.52	2.31	2.17	2.08	2.01	1.96	1.92	1.88	1.86	1.81	1.76	1.71	1.68	1.65	1.61	1.58	1.54
27	2.90	2.51	2.30	2.17	2.07	2.00	1.95	1.91	1.87	1.85	1.80	1.75	1.70	1.67	1.64	1.60	1.57	1.53
28	2.89	2.50	2.29	2.16	2.06	2.00	1.94	1.90	1.87	1.84	1.79	1.74	1.69	1.66	1.63	1.59	1.56	1.52
29	2.89	2.50	2.28	2.15	2.06	1.99	1.93	1.89	1.86	1.83	1.78	1.73	1.68	1.65	1.62	1.58	1.55	1.51
30	2.88	2.49	2.28	2.14	2.05	1.98	1.93	1.88	1.85	1.82	1.77	1.72	1.67	1.64	1.61	1.57	1.54	1.50
50	2.81	2.41	2.20	2.06	1.97	1.90	1.84	1.80	1.76	1.73	1.68	1.63	1.57	1.54	1.50	1.46	1.42	1.38
60	2.79	2.39	2.18	2.04	1.95	1.87	1.82	1.77	1.74	1.71	1.66	1.60	1.54	1.51	1.48	1.44	1.40	1.35
80	2.77	2.37	2.15	2.02	1.92	1.85	1.79	1.75	1.71	1.68	1.63	1.57	1.51	1.48	1.44	1.40	1.36	1.31
120	2.75	2.35	2.13	1.99	1.90	1.82	1.77	1.72	1.68	1.65	1.60	1.55	1.48	1.45	1.41	1.37	1.32	1.26

(*continues*)

TABLE D.4 (*Continued*)

$\alpha = 0.05$

$v_2 \downarrow$ \ $v_1 \rightarrow$	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	243.9	245.9	248	249	250	251	252	253.2
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77
26	4.22	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80	1.75
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79	1.73
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75	1.70
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.68
50	4.03	3.18	2.79	2.56	2.40	2.29	2.20	2.13	2.07	2.03	1.95	1.87	1.78	1.74	1.69	1.63	1.58	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47
80	3.96	3.11	2.72	2.49	2.33	2.21	2.13	2.06	2.00	1.95	1.88	1.79	1.70	1.65	1.60	1.54	1.48	1.41
120	3.92	3.07	2.68	2.45	2.29	2.18	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55	1.50	1.43	1.35

$\alpha = 0.025$

$v_2 \downarrow$ \ $v_1 \rightarrow$	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120
1	647.8	799.5	864.2	899.6	921.8	937.1	948.2	956.7	963.3	968.6	976.7	984.9	993.1	997.2	1001	1006	1010	1014
2	38.51	39.00	39.17	39.25	39.30	39.33	39.36	39.37	39.39	39.40	39.41	39.43	39.45	39.46	39.46	39.47	39.48	39.48
3	17.44	16.04	15.44	15.10	14.88	14.73	14.62	14.54	14.47	14.42	14.34	14.25	14.17	14.12	14.08	14.04	13.99	13.95
4	12.22	10.65	9.98	9.60	9.36	9.20	9.07	8.98	8.90	8.84	8.75	8.66	8.56	8.51	8.46	8.41	8.36	8.31
5	10.01	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68	6.62	6.52	6.43	6.33	6.28	6.23	6.18	6.12	6.07
6	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52	5.46	5.37	5.27	5.17	5.12	5.07	5.01	4.96	4.90
7	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82	4.76	4.67	4.57	4.47	4.41	4.36	4.31	4.25	4.20
8	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36	4.30	4.20	4.10	4.00	3.95	3.89	3.84	3.78	3.73
9	7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	4.03	3.96	3.87	3.77	3.67	3.61	3.56	3.51	3.45	3.39
10	6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.78	3.72	3.62	3.52	3.42	3.37	3.31	3.26	3.20	3.14
11	6.72	5.26	4.63	4.28	4.04	3.88	3.76	3.66	3.59	3.53	3.43	3.33	3.23	3.17	3.12	3.06	3.00	2.94
12	6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44	3.37	3.28	3.18	3.07	3.02	2.96	2.91	2.85	2.79
13	6.41	4.97	4.35	4.00	3.77	3.60	3.48	3.39	3.31	3.25	3.15	3.05	2.95	2.89	2.84	2.78	2.72	2.66
14	6.30	4.86	4.24	3.89	3.66	3.50	3.38	3.29	3.21	3.15	3.05	2.95	2.84	2.79	2.73	2.67	2.61	2.55
15	6.20	4.76	4.15	3.80	3.58	3.41	3.29	3.20	3.12	3.06	2.96	2.86	2.76	2.70	2.64	2.59	2.52	2.46
16	6.11	4.69	4.08	3.73	3.50	3.34	3.22	3.12	3.05	2.99	2.89	2.79	2.68	2.63	2.57	2.51	2.45	2.38
17	6.04	4.62	4.01	3.66	3.44	3.28	3.16	3.06	2.98	2.92	2.82	2.72	2.62	2.56	2.50	2.44	2.38	2.32
18	5.98	4.56	3.95	3.61	3.38	3.22	3.10	3.01	2.93	2.87	2.77	2.67	2.56	2.50	2.44	2.38	2.32	2.26
19	5.92	4.51	3.90	3.56	3.33	3.17	3.05	2.96	2.88	2.82	2.72	2.62	2.51	2.45	2.39	2.33	2.27	2.20
20	5.87	4.46	3.86	3.51	3.29	3.13	3.01	2.91	2.84	2.77	2.68	2.57	2.46	2.41	2.35	2.29	2.22	2.16
21	5.83	4.42	3.82	3.48	3.25	3.09	2.97	2.87	2.80	2.73	2.64	2.53	2.42	2.37	2.31	2.25	2.18	2.11
22	5.79	4.38	3.78	3.44	3.22	3.05	2.93	2.84	2.76	2.70	2.60	2.50	2.39	2.33	2.27	2.21	2.14	2.08
23	5.75	4.35	3.75	3.41	3.18	3.02	2.90	2.81	2.73	2.67	2.57	2.47	2.36	2.30	2.24	2.18	2.11	2.04
24	5.72	4.32	3.72	3.38	3.15	2.99	2.87	2.78	2.70	2.64	2.54	2.44	2.33	2.27	2.21	2.15	2.08	2.01
25	5.69	4.29	3.69	3.35	3.13	2.97	2.85	2.75	2.68	2.61	2.51	2.41	2.30	2.24	2.18	2.12	2.05	1.98
26	5.66	4.27	3.67	3.33	3.10	2.94	2.82	2.73	2.65	2.59	2.49	2.39	2.28	2.22	2.16	2.09	2.03	1.95
27	5.63	4.24	3.65	3.31	3.08	2.92	2.80	2.71	2.63	2.57	2.47	2.36	2.25	2.19	2.13	2.07	2.00	1.93
28	5.61	4.22	3.63	3.29	3.06	2.90	2.78	2.69	2.61	2.55	2.45	2.34	2.23	2.17	2.11	2.05	1.98	1.91
29	5.59	4.20	3.61	3.27	3.04	2.88	2.76	2.67	2.59	2.53	2.43	2.32	2.21	2.15	2.09	2.03	1.96	1.89
30	5.57	4.18	3.59	3.25	3.03	2.87	2.75	2.65	2.57	2.51	2.41	2.31	2.20	2.14	2.07	2.01	1.94	1.87
50	5.34	3.97	3.39	3.05	2.83	2.67	2.55	2.46	2.38	2.32	2.22	2.11	1.99	1.93	1.87	1.80	1.72	1.64
60	5.29	3.93	3.34	3.01	2.79	2.63	2.51	2.41	2.33	2.27	2.17	2.06	1.94	1.88	1.82	1.74	1.67	1.58
80	5.22	3.86	3.28	2.95	2.73	2.57	2.45	2.35	2.28	2.21	2.11	2.00	1.88	1.82	1.75	1.68	1.60	1.51
120	5.15	3.80	3.23	2.89	2.67	2.52	2.39	2.30	2.22	2.16	2.05	1.94	1.82	1.76	1.69	1.61	1.53	1.43

(*continues*)

TABLE D.4 (*Continued*)

$\alpha = 0.01$

$v_2 \downarrow$ \ $v_1 \rightarrow$	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120
1	4052	5000	5403	5625	5764	5859	5928	5982	6022	6056	6106	6157	6209	6235	6261	6287	6313	6339
2	98.5	99.0	99.2	99.2	99.3	99.3	99.4	99.4	99.4	99.4	99.4	99.4	99.4	99.5	99.5	99.5	99.5	99.5
3	34.1	30.8	29.5	28.7	28.2	27.9	27.7	27.5	27.3	27.2	27.1	26.9	26.7	26.6	26.5	26.4	26.3	26.2
4	21.2	18.0	16.7	16.0	15.5	15.2	15.0	14.8	14.7	14.6	14.4	14.2	14.0	13.9	13.8	13.7	13.7	13.6
5	16.25	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16	10.05	9.89	9.72	9.55	9.47	9.38	9.29	9.20	9.11
6	13.74	10.92	9.78	9.15	8.75	8.47	8.26	8.10	7.98	7.87	7.72	7.56	7.40	7.31	7.23	7.14	7.06	6.97
7	12.24	9.55	8.45	7.85	7.46	7.19	6.99	6.84	6.72	6.62	6.47	6.31	6.16	6.07	5.99	5.91	5.82	5.74
8	11.26	8.65	7.59	7.01	6.63	6.37	6.18	6.03	5.91	5.81	5.67	5.52	5.36	5.28	5.20	5.12	5.03	4.95
9	10.56	8.02	6.99	6.42	6.06	5.80	5.61	5.47	5.35	5.26	5.11	4.96	4.81	4.73	4.65	4.57	4.48	4.40
10	10.04	7.56	6.55	5.99	5.64	5.39	5.20	5.06	4.94	4.85	4.71	4.56	4.41	4.33	4.25	4.17	4.08	4.00
11	9.64	7.21	6.22	5.67	5.32	5.07	4.89	4.74	4.63	4.54	4.40	4.25	4.10	4.02	3.94	3.86	3.78	3.69
12	9.33	6.93	5.95	5.41	5.06	4.82	4.64	4.50	4.39	4.30	4.16	4.01	3.86	3.78	3.70	3.62	3.54	3.45
13	9.07	6.70	5.74	5.21	4.86	4.62	4.44	4.30	4.19	4.10	3.96	3.82	3.66	3.59	3.51	3.43	3.34	3.25
14	8.86	6.51	5.56	5.04	4.69	4.46	4.28	4.14	4.03	3.94	3.80	3.66	3.51	3.43	3.35	3.27	3.18	3.09
15	8.68	6.36	5.42	4.89	4.56	4.32	4.14	4.00	3.89	3.80	3.67	3.52	3.37	3.29	3.21	3.13	3.05	2.96
16	8.53	6.23	5.29	4.77	4.44	4.20	4.03	3.89	3.78	3.69	3.55	3.41	3.26	3.18	3.10	3.02	2.93	2.84
17	8.40	6.11	5.18	4.67	4.34	4.10	3.93	3.79	3.68	3.59	3.46	3.31	3.16	3.08	3.00	2.92	2.83	2.75
18	8.28	6.01	5.09	4.58	4.25	4.01	3.84	3.71	3.60	3.51	3.37	3.23	3.08	3.00	2.92	2.84	2.75	2.66
19	8.18	5.93	5.01	4.50	4.17	3.94	3.77	3.63	3.52	3.43	3.30	3.15	3.00	2.92	2.84	2.76	2.67	2.58
20	8.09	5.85	4.94	4.43	4.10	3.87	3.70	3.56	3.46	3.37	3.23	3.09	2.94	2.86	2.78	2.69	2.61	2.52
21	8.01	5.78	4.87	4.37	4.04	3.81	3.64	3.51	3.40	3.31	3.17	3.03	2.88	2.80	2.72	2.64	2.55	2.46
22	7.94	5.72	4.82	4.31	3.99	3.76	3.59	3.45	3.35	3.26	3.12	2.98	2.83	2.75	2.67	2.58	2.50	2.40
23	7.88	5.66	4.76	4.26	3.94	3.71	3.54	3.41	3.30	3.21	3.07	2.93	2.78	2.70	2.62	2.54	2.45	2.35
24	7.82	5.61	4.72	4.22	3.90	3.67	3.50	3.36	3.26	3.17	3.03	2.89	2.74	2.66	2.58	2.49	2.40	2.31
25	7.77	5.57	4.68	4.18	3.85	3.63	3.46	3.32	3.22	3.13	2.99	2.85	2.70	2.62	2.54	2.45	2.36	2.27
26	7.72	5.53	4.64	4.14	3.82	3.59	3.42	3.29	3.18	3.09	2.96	2.81	2.66	2.58	2.50	2.42	2.33	2.23
27	7.67	5.49	4.60	4.11	3.78	3.56	3.39	3.26	3.15	3.06	2.93	2.78	2.63	2.55	2.47	2.38	2.29	2.20
28	7.63	5.45	4.57	4.07	3.75	3.53	3.36	3.23	3.12	3.03	2.90	2.75	2.60	2.52	2.44	2.35	2.26	2.17
29	7.60	5.42	4.54	4.04	3.73	3.50	3.33	3.20	3.09	3.00	2.87	2.73	2.57	2.49	2.41	2.33	2.23	2.14
30	7.56	5.39	4.51	4.02	3.70	3.47	3.30	3.17	3.07	2.98	2.84	2.70	2.55	2.47	2.39	2.30	2.21	2.11
50	7.17	5.06	4.20	3.72	3.41	3.19	3.02	2.89	2.78	2.70	2.56	2.42	2.27	2.18	2.10	2.01	1.91	1.80
60	7.08	4.98	4.13	3.65	3.34	3.12	2.95	2.82	2.72	2.63	2.50	2.35	2.20	2.12	2.03	1.94	1.84	1.73
80	6.96	4.88	4.04	3.56	3.26	3.04	2.87	2.74	2.64	2.55	2.42	2.27	2.12	2.03	1.94	1.85	1.75	1.63
120	6.85	4.79	3.95	3.48	3.17	2.96	2.79	2.66	2.56	2.47	2.34	2.19	2.03	1.95	1.86	1.76	1.66	1.53

$\alpha = 0.005$

$v_2 \downarrow$ \ $v_1 \rightarrow$	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120
1	16,211	20,000	21,615	22,500	23,056	23,437	23,715	23,925	24,091	24,224	24,426	24,630	24,836	24,940	25,044	25,148	25,253	253,591
2	198.5	199.0	199.2	199.2	199.3	199.4	199.4	199.4	199.4	199.4	199.4	199.4	199.4	199.5	199.5	199.5	199.5	199.5
3	55.55	49.80	47.47	46.19	45.39	44.84	44.43	44.13	43.88	43.69	43.39	43.06	42.78	42.62	42.47	42.31	42.15	41.99
4	31.33	26.28	24.26	23.15	22.46	21.97	21.62	21.35	21.14	20.97	20.70	20.44	20.17	20.03	19.89	19.75	19.61	19.47
5	22.77	18.31	16.53	15.55	14.94	14.51	14.20	13.96	13.77	13.62	13.38	13.15	12.90	12.78	12.66	12.53	12.40	12.27
6	18.62	14.54	12.91	12.03	11.46	11.07	10.79	10.57	10.39	10.25	10.03	9.81	9.59	9.47	9.36	9.24	9.12	9.00
7	16.23	12.40	10.88	10.05	9.52	9.16	8.89	8.68	8.51	8.38	8.18	7.97	7.75	7.64	7.53	7.42	7.31	7.19
8	14.68	11.04	9.60	8.80	8.30	7.95	7.69	7.50	7.34	7.21	7.01	6.81	6.61	6.50	6.40	6.29	6.18	6.06
9	13.61	10.10	8.72	7.96	7.47	7.13	6.88	6.69	6.54	6.42	6.23	6.03	5.83	5.73	5.62	5.52	5.41	5.30
10	12.82	9.43	8.08	7.34	6.87	6.54	6.30	6.12	5.97	5.85	5.66	5.47	5.27	5.17	5.07	4.97	4.86	4.75
11	12.22	8.91	7.60	6.88	6.42	6.10	5.86	5.68	5.54	5.42	5.24	5.05	4.86	4.76	4.65	4.55	4.44	4.34
12	11.75	8.51	7.23	6.52	6.07	5.76	5.52	5.35	5.20	5.09	4.91	4.72	4.53	4.43	4.33	4.23	4.12	4.01
13	11.37	8.19	6.93	6.23	5.79	5.48	5.25	5.08	4.94	4.82	4.64	4.46	4.27	4.17	4.07	3.97	3.87	3.76
14	11.06	7.92	6.68	6.00	5.56	5.26	5.03	4.86	4.72	4.60	4.43	4.25	4.06	3.96	3.86	3.76	3.66	3.55
15	10.79	7.70	6.48	5.80	5.37	5.07	4.85	4.67	4.54	4.42	4.25	4.07	3.88	3.79	3.69	3.58	3.48	3.37
16	10.57	7.51	6.30	5.64	5.21	4.91	4.69	4.52	4.38	4.27	4.10	3.92	3.73	3.64	3.54	3.44	3.33	3.22
17	10.38	7.35	6.16	5.50	5.07	4.78	4.56	4.39	4.25	4.14	3.97	3.79	3.61	3.51	3.41	3.31	3.21	3.10
18	10.21	7.21	6.03	5.37	4.96	4.66	4.44	4.28	4.14	4.03	3.86	3.68	3.50	3.40	3.30	3.20	3.10	2.99
19	10.07	7.09	5.92	5.27	4.85	4.56	4.34	4.18	4.04	3.93	3.76	3.59	3.40	3.31	3.21	3.11	3.00	2.89
20	9.94	6.99	5.82	5.17	4.76	4.47	4.26	4.09	3.96	3.85	3.68	3.50	3.32	3.22	3.12	3.02	2.92	2.81
21	9.83	6.89	5.73	5.09	4.68	4.39	4.18	4.01	3.88	3.77	3.60	3.43	3.24	3.15	3.05	2.95	2.84	2.73
22	9.72	6.81	5.65	5.02	4.61	4.32	4.11	3.94	3.81	3.70	3.54	3.36	3.18	3.08	2.98	2.88	2.77	2.66
23	9.63	6.73	5.58	4.95	4.54	4.26	4.05	3.88	3.75	3.64	3.47	3.30	3.12	3.02	2.92	2.82	2.71	2.60
24	9.55	6.66	5.52	4.89	4.49	4.20	3.99	3.83	3.69	3.59	3.42	3.25	3.06	2.97	2.87	2.77	2.66	2.55
25	9.47	6.60	5.46	4.83	4.43	4.15	3.94	3.78	3.64	3.54	3.37	3.20	3.01	2.92	2.82	2.72	2.61	2.50
26	9.40	6.54	5.41	4.79	4.38	4.10	3.89	3.73	3.60	3.49	3.33	3.15	2.97	2.87	2.77	2.67	2.56	2.45
27	9.34	6.49	5.36	4.74	4.34	4.06	3.85	3.69	3.56	3.45	3.28	3.11	2.93	2.83	2.73	2.63	2.52	2.41
28	9.28	6.44	5.32	4.70	4.30	4.02	3.81	3.65	3.52	3.41	3.25	3.07	2.89	2.79	2.69	2.59	2.48	2.37
29	9.23	6.39	5.28	4.66	4.26	3.98	3.77	3.61	3.48	3.38	3.21	3.04	2.86	2.76	2.66	2.56	2.45	2.33
30	9.18	6.35	5.24	4.62	4.23	3.95	3.74	3.58	3.45	3.34	3.18	3.01	2.82	2.73	2.63	2.52	2.42	2.30
50	8.62	5.90	4.83	4.23	3.85	3.58	3.38	3.22	3.09	2.99	2.82	2.65	2.47	2.37	2.27	2.16	2.05	1.93
60	8.49	5.79	4.73	4.14	3.76	3.49	3.29	3.13	3.01	2.90	2.74	2.57	2.39	2.29	2.19	2.08	1.96	1.83
80	8.33	5.66	4.61	4.03	3.65	3.39	3.19	3.03	2.91	2.80	2.64	2.47	2.29	2.19	2.08	1.97	1.85	1.72
120	8.18	5.54	4.50	3.92	3.55	3.28	3.09	2.93	2.81	2.71	2.54	2.37	2.19	2.09	1.98	1.87	1.75	1.61

(*continues*)

TABLE D.4 (*Continued*)

$\alpha = 0.001$

$v_2 \downarrow$ \ $v_1 \rightarrow$	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120
1	405,269	500,004	540,387	562,506	576,412	585,943	592,881	598,151	602,292	605,625	610,676	615,772	620,913	623,504	626,107	628,720	631,345	633,980
2	998.5	999.0	999.1	999.2	999.3	999.3	999.4	999.4	999.4	999.4	999.4	999.4	999.5	999.5	999.5	999.5	999.5	999.5
3	167.0	148.5	141.1	137.1	134.6	132.9	131.6	130.6	129.9	129.3	128.3	127.4	126.4	125.9	125.5	125.0	124.5	124.0
4	74.14	61.25	56.18	53.44	51.71	50.53	49.66	49.00	48.47	48.05	47.41	46.76	46.10	45.77	45.43	45.09	44.75	44.40
5	47.18	37.12	33.20	31.09	29.75	28.83	28.16	27.65	27.24	26.92	26.42	25.91	25.39	25.13	24.87	24.60	24.33	24.06
6	35.51	27.00	23.70	21.92	20.80	20.03	19.46	19.03	18.69	18.41	17.99	17.56	17.12	16.90	16.67	16.44	16.21	15.98
7	29.25	21.69	18.77	17.20	16.21	15.52	15.02	14.63	14.33	14.08	13.71	13.32	12.93	12.73	12.53	12.33	12.12	11.91
8	25.41	18.49	15.83	14.39	13.48	12.86	12.40	12.05	11.77	11.54	11.19	10.84	10.48	10.30	10.11	9.92	9.73	9.53
9	22.86	16.39	13.90	12.56	11.71	11.13	10.70	10.37	10.11	9.89	9.57	9.24	8.90	8.72	8.55	8.37	8.19	8.00
10	21.04	14.91	12.55	11.28	10.48	9.93	9.52	9.20	8.96	8.75	8.45	8.13	7.80	7.64	7.47	7.30	7.12	6.94
11	19.69	13.81	11.56	10.35	9.58	9.05	8.66	8.35	8.12	7.92	7.63	7.32	7.01	6.85	6.68	6.52	6.35	6.18
12	18.64	12.97	10.80	9.63	8.89	8.38	8.00	7.71	7.48	7.29	7.00	6.71	6.40	6.25	6.09	5.93	5.76	5.59
13	17.82	12.31	10.21	9.07	8.35	7.86	7.49	7.21	6.98	6.80	6.52	6.23	5.93	5.78	5.63	5.47	5.30	5.14
14	17.14	11.78	9.73	8.62	7.92	7.44	7.08	6.80	6.58	6.40	6.13	5.85	5.56	5.41	5.25	5.10	4.94	4.77
15	16.59	11.34	9.34	8.25	7.57	7.09	6.74	6.47	6.26	6.08	5.81	5.54	5.25	5.10	4.95	4.80	4.64	4.47
16	16.12	10.97	9.01	7.94	7.27	6.80	6.46	6.19	5.98	5.81	5.55	5.27	4.99	4.85	4.70	4.54	4.39	4.23
17	15.72	10.66	8.73	7.68	7.02	6.56	6.22	5.96	5.75	5.58	5.32	5.05	4.78	4.63	4.48	4.33	4.18	4.02
18	15.38	10.39	8.49	7.46	6.81	6.35	6.02	5.76	5.56	5.39	5.13	4.87	4.59	4.45	4.30	4.15	4.00	3.84
19	15.08	10.16	8.28	7.27	6.62	6.18	5.85	5.59	5.39	5.22	4.97	4.70	4.43	4.29	4.14	3.99	3.84	3.68
20	14.82	9.95	8.10	7.10	6.46	6.02	5.69	5.44	5.24	5.08	4.82	4.56	4.29	4.15	4.00	3.86	3.70	3.54
21	14.59	9.77	7.94	6.95	6.32	5.88	5.56	5.31	5.11	4.95	4.70	4.44	4.17	4.03	3.88	3.74	3.58	3.42
22	14.38	9.61	7.80	6.81	6.19	5.76	5.44	5.19	4.99	4.83	4.58	4.33	4.06	3.92	3.78	3.63	3.48	3.32
23	14.20	9.47	7.67	6.70	6.08	5.65	5.33	5.09	4.89	4.73	4.48	4.23	3.96	3.82	3.68	3.53	3.38	3.22
24	14.03	9.34	7.55	6.59	5.98	5.55	5.23	4.99	4.80	4.64	4.39	4.14	3.87	3.74	3.59	3.45	3.29	3.14
25	13.88	9.22	7.45	6.49	5.89	5.46	5.15	4.91	4.71	4.56	4.31	4.06	3.79	3.66	3.52	3.37	3.22	3.06
26	13.74	9.12	7.36	6.41	5.80	5.38	5.07	4.83	4.64	4.48	4.24	3.99	3.72	3.59	3.44	3.30	3.15	2.99
27	13.61	9.02	7.27	6.33	5.73	5.31	5.00	4.76	4.57	4.41	4.17	3.92	3.66	3.52	3.38	3.23	3.08	2.92
28	13.50	8.93	7.19	6.25	5.66	5.24	4.93	4.69	4.50	4.35	4.11	3.86	3.60	3.46	3.32	3.18	3.02	2.86
29	13.39	8.85	7.12	6.19	5.59	5.18	4.87	4.64	4.45	4.29	4.05	3.80	3.54	3.41	3.27	3.12	2.97	2.81
30	13.29	8.77	7.05	6.12	5.53	5.12	4.82	4.58	4.39	4.24	4.00	3.75	3.49	3.36	3.22	3.07	2.92	2.76
40	12.61	8.25	6.59	5.70	5.13	4.73	4.44	4.21	4.02	3.87	3.64	3.40	3.14	3.01	2.87	2.73	2.57	2.41
50	12.22	7.96	6.34	5.46	4.90	4.51	4.22	4.00	3.82	3.67	3.44	3.20	2.95	2.82	2.68	2.53	2.38	2.21
60	11.97	7.77	6.17	5.31	4.76	4.37	4.09	3.86	3.69	3.54	3.32	3.08	2.83	2.69	2.55	2.41	2.25	2.08
120	11.38	7.32	5.78	4.95	4.42	4.04	3.77	3.55	3.38	3.24	3.02	2.78	2.53	2.40	2.26	2.11	1.95	1.77

APPENDIX E

CONFIDENCE INTERVALS FOR THE MEAN

Table E.1 represents 1000 95% confidence intervals constructed from sample sets selected from a population with a mean (μ) of 25.4 and a variance (σ^2) of 1.69. These data are discussed in Chapter 4. (*Note:* Intervals with asterisks fail to include μ.)

TABLE E.1 1000 95% Confidence Intervals

(24.21, 25.94)	(24.77, 26.22)	(24.71, 26.01)	(25.43, 27.01)*	(24.22, 26.19)	(25.12, 26.56)	(24.70, 26.54)	(24.50, 25.95)
(23.99, 25.39)*	(25.33, 26.45)	(24.62, 25.89)	(24.79, 26.16)	(24.85, 26.11)	(23.92, 25.67)	(25.30, 26.84)	(24.71, 26.39)
(24.58, 26.10)	(24.47, 25.50)	(25.14, 26.56)	(24.79, 25.72)	(24.22, 25.79)	(25.03, 26.27)	(24.63, 25.83)	(25.09, 26.84)
(24.80, 26.18)	(24.95, 26.07)	(24.61, 26.27)	(25.06, 26.29)	(24.69, 26.25)	(24.64, 25.88)	(24.96, 26.15)	(25.04, 25.92)
(24.59, 26.09)	(24.99, 26.15)	(24.65, 26.29)	(25.08, 26.28)	(24.74, 26.02)	(25.19, 26.43)	(24.54, 25.73)	(24.17, 25.94)
(24.86, 25.81)	(24.40, 25.92)	(24.86, 25.80)	(24.58, 25.91)	(23.69, 25.25)*	(24.64, 25.77)	(24.88, 26.12)	(24.63, 25.92)
(24.84, 25.92)	(24.97, 26.47)	(25.13, 26.38)	(24.39, 26.32)	(24.15, 26.16)	(24.80, 26.01)	(24.87, 26.54)	(24.66, 25.95)
(24.64, 25.93)	(24.79, 26.18)	(24.68, 25.93)	(24.17, 25.83)	(24.29, 26.14)	(24.43, 25.62)	(24.58, 25.99)	(25.11, 26.33)
(24.85, 26.26)	(25.11, 26.44)	(24.58, 25.99)	(24.89, 26.25)	(24.79, 26.36)	(25.23, 26.42)	(25.30, 26.45)	(25.09, 26.53)
(25.23, 26.34)	(24.58, 26.35)	(24.81, 25.88)	(24.37, 25.62)	(24.79, 26.07)	(24.82, 26.19)	(24.66, 26.56)	(24.56, 26.12)
(25.11, 26.18)	(24.73, 25.83)	(24.46, 25.51)	(24.92, 25.87)	(24.46, 26.45)	(24.40, 25.61)	(24.49, 26.45)	(24.61, 26.02)
(25.20, 26.34)	(24.94, 25.99)	(24.53, 25.90)	(24.73, 26.07)	(24.56, 25.75)	(24.49, 25.85)	(24.49, 26.24)	(25.28, 26.26)
(24.26, 25.48)	(24.50, 26.26)	(25.11, 26.56)	(23.96, 25.61)	(24.94, 26.46)	(24.63, 26.31)	(24.37, 25.95)	(24.67, 26.18)
(24.56, 26.17)	(24.48, 25.76)	(24.72, 26.02)	(25.63, 26.40)*	(24.82, 26.09)	(24.97, 26.02)	(24.51, 25.78)	(24.88, 26.18)
(24.72, 25.91)	(24.94, 26.24)	(24.91, 26.39)	(25.46, 26.82)*	(24.84, 26.37)	(25.05, 26.07)	(24.72, 25.78)	(25.20, 26.60)
(24.83, 26.09)	(24.62, 25.96)	(25.10, 26.29)	(24.78, 26.24)	(24.50, 26.30)	(24.60, 26.20)	(25.38, 26.62)	(24.86, 26.20)
(24.31, 25.89)	(24.35, 25.73)	(24.67, 26.07)	(24.86, 26.58)	(24.52, 25.55)	(24.81, 26.47)	(24.45, 26.32)	(23.94, 25.77)
(24.76, 25.56)	(25.04, 26.20)	(24.25, 26.06)	(24.33, 25.69)	(24.99, 26.16)	(25.13, 26.43)	(24.68, 26.26)	(24.69, 25.68)
(25.11, 26.45)	(24.69, 26.00)	(24.45, 25.42)	(24.72, 26.50)	(24.54, 25.80)	(24.57, 25.76)	(25.10, 26.16)	(24.60, 25.90)
(24.21, 25.86)	(25.00, 26.52)	(25.05, 26.55)	(24.47, 25.76)	(24.76, 26.07)	(24.80, 25.95)	(24.45, 25.89)	(24.82, 26.86)
(25.16, 26.21)	(24.62, 26.01)	(24.45, 25.88)	(25.60, 26.70)*	(24.32, 25.83)	(25.11, 26.12)	(24.59, 25.81)	(24.11, 25.86)
(24.53, 25.86)	(24.89, 26.21)	(24.52, 25.83)	(25.08, 26.22)	(25.22, 26.71)	(24.25, 25.39)*	(24.73, 26.34)	(24.67, 26.24)
(24.84, 26.31)	(24.68, 25.55)	(24.69, 26.28)	(24.82, 26.02)	(25.29, 26.44)	(24.84, 26.21)	(24.83, 26.03)	(24.90, 26.26)
(24.84, 26.14)	(25.13, 26.52)	(25.07, 26.44)	(24.52, 25.88)	(24.62, 26.04)	(24.84, 25.88)	(24.34, 25.71)	(24.25, 25.51)
(24.50, 25.88)	(24.49, 25.73)	(25.20, 26.23)	(24.64, 26.30)	(24.73, 26.16)	(25.16, 26.30)	(24.66, 25.87)	(25.20, 26.59)
(24.44, 25.85)	(24.29, 25.29)*	(24.46, 25.94)	(24.46, 25.45)	(24.66, 25.78)	(24.07, 25.62)	(24.48, 25.88)	(24.99, 26.61)
(24.41, 25.97)	(25.04, 26.23)	(24.32, 25.57)	(24.76, 25.85)	(24.60, 26.03)	(25.04, 26.50)	(24.39, 26.27)	(24.97, 26.13)
(25.41, 26.81)*	(24.17, 25.61)	(25.10, 26.48)	(24.78, 26.18)	(24.85, 26.01)	(24.87, 25.88)	(24.62, 26.00)	(25.21, 26.39)

(24.98, 26.33)	(24.11, 26.09)	(24.74, 26.05)	(24.11, 25.65)	(24.47, 25.83)	(24.85, 26.32)	(24.37, 25.63)	(24.66, 25.95)
(24.83, 26.27)	(25.07, 26.29)	(25.24, 26.56)	(24.13, 25.65)	(25.41, 26.47)*	(24.64, 26.04)	(24.42, 25.94)	(24.61, 25.61)
(25.07, 26.38)	(24.90, 26.44)	(24.44, 25.72)	(24.63, 25.95)	(25.28, 26.37)	(24.63, 25.96)	(24.63, 25.71)	(24.98, 26.38)
(24.61, 26.21)	(24.93, 26.12)	(24.41, 25.79)	(24.95, 26.06)	(24.92, 25.88)	(24.69, 26.10)	(24.84, 26.22)	(25.40, 26.67)
(24.41, 25.51)	(24.93, 26.11)	(24.90, 25.91)	(25.18, 26.44)	(24.90, 26.48)	(25.06, 26.12)	(25.23, 26.18)	(24.87, 26.06)
(24.36, 25.85)	(24.45, 25.68)	(24.84, 25.91)	(24.83, 26.40)	(24.69, 26.04)	(24.68, 26.11)	(24.95, 26.17)	(24.94, 26.00)
(24.56, 25.98)	(25.19, 26.24)	(24.27, 25.82)	(24.98, 26.00)	(24.85, 25.98)	(25.00, 25.82)	(24.72, 25.99)	(24.98, 26.48)
(25.19, 26.85)	(24.38, 25.92)	(24.59, 26.19)	(24.76, 26.42)	(24.56, 26.00)	(24.57, 26.29)	(24.60, 26.12)	(24.99, 26.18)
(24.23, 26.01)	(24.28, 25.56)	(24.14, 25.37)*	(24.64, 26.29)	(24.96, 26.40)	(24.89, 26.33)	(24.43, 25.11)*	(24.88, 26.53)
(25.17, 26.78)	(24.52, 26.20)	(24.29, 25.86)	(24.26, 25.97)	(24.74, 26.17)	(24.81, 26.20)	(24.97, 26.15)	(24.85, 26.43)
(24.42, 25.86)	(24.61, 26.04)	(25.13, 26.18)	(24.09, 25.73)	(24.83, 26.19)	(25.18, 26.53)	(25.01, 26.66)	(24.52, 25.97)
(25.43, 26.53)*	(25.00, 26.32)	(24.86, 25.70)	(24.52, 25.65)	(24.67, 26.00)	(25.17, 26.69)	(24.73, 25.98)	(24.58, 25.93)
(24.85, 26.42)	(25.49, 26.78)*	(25.52, 26.38)*	(24.14, 25.61)	(25.03, 26.43)	(24.38, 26.29)	(25.16, 26.73)	(24.92, 26.35)
(24.24, 26.00)	(24.00, 25.11)*	(24.85, 26.03)	(24.73, 26.02)	(25.19, 26.92)	(24.70, 26.18)	(24.85, 26.40)	(24.84, 26.31)
(24.75, 26.11)	(24.82, 26.07)	(24.62, 25.84)	(25.42, 26.48)*	(24.75, 26.03)	(24.32, 25.70)	(25.19, 26.45)	(24.71, 26.31)
(24.42, 25.94)	(24.71, 26.39)	(24.15, 25.84)	(24.72, 25.98)	(25.22, 26.51)	(24.16, 25.76)	(25.18, 25.94)	(25.04, 26.87)
(24.50, 25.73)	(24.96, 26.19)	(25.35, 26.41)	(24.58, 25.92)	(24.23, 25.64)	(25.18, 26.28)	(24.36, 26.29)	(24.18, 25.74)
(25.11, 26.24)	(24.14, 25.70)	(24.94, 25.85)	(24.59, 25.96)	(25.02, 26.60)	(24.67, 26.26)	(24.83, 26.38)	(24.43, 25.38)*
(25.06, 26.28)	(24.26, 25.69)	(24.30, 25.42)	(25.03, 26.56)	(24.47, 26.10)	(24.52, 25.77)	(25.12, 26.44)	(25.55, 26.84)*
(24.53, 26.19)	(24.77, 26.25)	(25.15, 26.49)	(24.96, 26.12)	(25.13, 26.32)	(24.69, 25.99)	(24.63, 26.15)	(25.37, 26.48)
(24.56, 26.21)	(24.44, 25.76)	(24.27, 25.50)	(25.09, 26.03)	(24.76, 26.01)	(24.63, 26.14)	(24.36, 25.59)	(24.36, 26.04)
(24.82, 25.75)	(24.68, 25.98)	(25.01, 26.78)	(24.83, 26.36)	(25.39, 26.46)	(24.39, 25.52)	(25.07, 26.11)	(24.97, 26.26)
(25.05, 26.31)	(24.89, 26.20)	(24.12, 25.66)	(24.79, 26.43)	(24.98, 26.15)	(25.05, 26.57)	(24.81, 26.08)	(24.32, 25.69)
(24.76, 26.37)	(25.30, 26.54)	(25.05, 26.20)	(24.38, 25.72)	(24.89, 26.37)	(24.34, 25.68)	(24.64, 25.75)	(24.54, 26.08)
(24.52, 26.21)	(24.85, 25.86)	(25.16, 25.95)	(24.71, 25.92)	(24.65, 25.36)*	(25.00, 26.48)	(24.58, 25.61)	(24.66, 26.26)
(24.74, 25.92)	(24.45, 25.68)	(24.89, 26.23)	(24.80, 26.04)	(24.84, 26.24)	(24.15, 25.62)	(25.48, 27.00)*	(25.00, 26.34)
(24.71, 26.55)	(25.01, 26.44)	(24.67, 26.20)	(24.94, 26.35)	(24.79, 26.36)	(24.98, 26.31)	(24.61, 26.21)	(23.94, 25.88)
(24.55, 25.94)	(24.79, 26.08)	(24.94, 25.90)	(24.73, 25.87)	(24.40, 25.68)	(25.24, 26.39)	(25.18, 26.61)	(24.42, 25.61)

(*continues*)

TABLE E.1 (*Continued*)

(24.36, 25.77)	(24.35, 25.70)	(24.85, 26.06)	(24.56, 25.51)	(24.23, 25.69)	(24.41, 25.71)	(24.52, 25.89)	(24.53, 25.55)
(25.57, 26.72)*	(24.85, 26.11)	(24.98, 26.41)	(24.71, 26.00)	(25.26, 26.38)	(24.71, 26.36)	(24.66, 25.99)	(24.89, 25.86)
(25.46, 26.24)*	(25.25, 26.39)	(24.70, 26.22)	(24.59, 25.88)	(24.43, 25.54)	(25.09, 26.50)	(24.83, 26.22)	(24.87, 26.11)
(24.33, 26.14)	(25.00, 26.19)	(23.88, 25.22)*	(24.42, 26.05)	(24.68, 26.02)	(24.93, 26.18)	(24.85, 26.20)	(25.52, 26.77)*
(25.00, 25.95)	(24.51, 26.03)	(24.24, 25.97)	(24.92, 26.35)	(24.45, 26.33)	(24.94, 26.08)	(24.07, 26.38)	(24.84, 26.27)
(24.95, 26.22)	(24.94, 26.28)	(24.75, 26.00)	(24.78, 26.37)	(25.45, 27.04)*	(24.20, 25.91)	(24.71, 26.08)	(24.14, 25.83)
(25.14, 26.48)	(24.63, 26.28)	(24.91, 25.91)	(25.08, 26.23)	(24.47, 25.85)	(24.54, 26.04)	(24.95, 26.74)	(25.00, 26.40)
(24.60, 25.65)	(24.75, 26.31)	(24.23, 26.01)	(24.94, 26.36)	(24.66, 26.01)	(23.84, 25.91)	(24.78, 26.42)	(25.13, 26.34)
(24.97, 26.37)	(24.42, 26.73)	(24.63, 25.53)	(24.48, 25.73)	(24.44, 25.66)	(24.14, 25.11)*	(24.84, 26.33)	(25.05, 26.14)
(24.18, 25.96)	(24.59, 25.61)	(24.73, 25.86)	(24.63, 25.97)	(24.09, 25.90)	(25.47, 26.45)*	(24.69, 25.92)	(24.58, 25.77)
(24.67, 25.80)	(24.59, 26.10)	(23.87, 25.38)*	(24.93, 26.46)	(24.69, 26.16)	(24.66, 25.94)	(24.48, 25.95)	(24.33, 26.04)
(24.30, 25.75)	(25.33, 26.18)	(24.18, 26.00)	(24.58, 26.18)	(25.04, 26.25)	(24.33, 26.15)	(24.58, 26.13)	(24.32, 25.70)
(24.39, 25.48)	(24.89, 26.44)	(24.79, 26.14)	(24.62, 26.04)	(25.29, 26.58)	(24.81, 26.29)	(24.35, 25.28)*	(25.32, 26.60)
(24.04, 25.44)	(24.83, 26.22)	(24.43, 25.72)	(24.38, 26.20)	(24.63, 25.63)	(24.40, 26.17)	(25.09, 26.50)	(24.93, 26.45)
(24.89, 26.28)	(24.05, 25.43)	(24.78, 26.06)	(24.50, 25.94)	(24.54, 26.01)	(24.97, 25.89)	(24.58, 25.87)	(24.86, 26.11)
(24.76, 26.25)	(24.37, 25.53)	(24.85, 25.86)	(24.26, 25.48)	(25.14, 26.41)	(24.17, 25.84)	(24.77, 26.03)	(25.26, 26.53)
(25.08, 26.29)	(24.60, 26.08)	(25.24, 26.50)	(24.79, 26.10)	(25.17, 26.53)	(24.85, 25.88)	(24.79, 26.18)	(24.64, 26.11)
(24.52, 25.99)	(24.89, 26.14)	(24.29, 25.84)	(24.68, 25.99)	(24.41, 26.17)	(25.18, 26.57)	(24.10, 25.54)	(25.23, 26.47)
(24.97, 26.44)	(24.59, 25.63)	(24.80, 26.00)	(24.91, 26.29)	(24.56, 25.92)	(24.65, 26.02)	(24.34, 25.42)	(24.86, 26.32)
(24.54, 25.79)	(25.13, 26.68)	(24.60, 25.78)	(24.49, 25.87)	(24.94, 26.19)	(25.13, 26.39)	(24.47, 25.60)	(24.61, 25.77)
(24.74, 26.67)	(25.03, 26.54)	(24.69, 26.59)	(24.84, 26.37)	(24.81, 26.35)	(24.64, 25.95)	(24.89, 26.07)	(24.50, 25.91)
(24.34, 25.83)	(25.10, 26.12)	(24.87, 26.48)	(25.00, 26.54)	(24.64, 26.00)	(24.71, 26.09)	(24.43, 26.05)	(24.54, 25.69)
(24.37, 25.61)	(24.49, 25.97)	(24.48, 25.60)	(24.65, 26.73)	(25.18, 26.41)	(25.09, 26.44)	(24.95, 26.40)	(24.64, 26.01)
(24.58, 25.81)	(24.47, 25.66)	(24.39, 25.72)	(24.12, 25.71)	(24.58, 25.85)	(25.77, 27.28)*	(24.67, 25.86)	(24.78, 26.43)
(24.44, 26.16)	(24.70, 26.25)	(24.48, 25.68)	(24.32, 25.64)	(24.65, 25.50)	(24.75, 26.21)	(24.25, 25.69)	(24.10, 25.83)
(24.72, 26.08)	(24.73, 26.17)	(25.44, 26.34)*	(24.43, 25.75)	(25.27, 26.74)	(24.13, 25.49)	(25.04, 26.28)	(24.63, 25.80)

(24.27, 25.94)	(24.93, 26.73)	(24.43, 25.84)	(24.29, 25.52)	(24.55, 26.35)	(25.13, 26.54)	(24.27, 25.63)	(25.17, 26.45)
(25.09, 26.67)	(24.44, 25.88)	(24.59, 26.63)	(24.92, 26.37)	(25.10, 26.29)	(24.57, 26.27)	(24.49, 25.90)	(24.76, 26.08)
(24.64, 26.13)	(24.46, 25.84)	(24.31, 25.89)	(25.34, 26.66)	(24.87, 26.31)	(24.17, 25.47)	(25.28, 26.57)	(24.75, 26.14)
(24.44, 26.00)	(24.46, 25.63)	(24.75, 26.33)	(25.03, 26.03)	(24.87, 26.36)	(24.45, 25.83)	(24.85, 26.15)	(24.12, 25.75)
(25.53, 26.94)*	(23.89, 25.24)*	(24.78, 26.12)	(24.95, 26.19)	(24.81, 26.05)	(24.76, 26.13)	(24.38, 25.77)	(24.45, 26.24)
(25.72, 26.82)*	(24.36, 25.75)	(24.63, 26.25)	(24.65, 26.19)	(25.04, 25.80)	(25.01, 26.66)	(24.48, 25.96)	(24.88, 26.48)
(24.45, 26.06)	(25.19, 26.42)	(24.64, 26.18)	(24.95, 26.25)	(24.43, 26.35)	(24.89, 26.11)	(24.19, 25.61)	(24.52, 26.26)
(24.95, 26.13)	(25.17, 26.46)	(24.70, 26.19)	(24.59, 25.94)	(24.71, 26.06)	(24.84, 26.45)	(24.41, 25.60)	(24.76, 25.96)
(24.83, 26.42)	(24.62, 26.12)	(24.39, 25.41)	(25.18, 26.22)	(24.75, 26.33)	(24.47, 26.37)	(24.45, 25.74)	(25.19, 26.37)
(24.75, 26.15)	(25.33, 26.63)	(24.69, 25.90)	(24.67, 26.29)	(24.89, 26.24)	(24.35, 26.27)	(24.95, 26.01)	(24.77, 25.94)
(24.69, 25.95)	(24.74, 26.22)	(24.61, 26.15)	(25.10, 26.15)	(24.95, 26.40)	(24.46, 26.08)	(25.07, 26.68)	(25.45, 26.28)*
(24.88, 26.52)	(25.04, 26.48)	(24.71, 25.90)	(24.86, 26.18)	(24.45, 25.96)	(24.82, 25.94)	(24.61, 26.09)	(24.87, 26.67)
(24.32, 25.93)	(24.74, 26.00)	(24.42, 25.72)	(25.14, 26.53)	(24.74, 25.97)	(24.35, 26.08)	(24.70, 25.75)	(25.11, 26.29)
(24.55, 25.95)	(24.33, 25.56)	(24.22, 25.61)	(24.59, 26.25)	(25.21, 26.31)	(25.03, 26.40)	(25.09, 26.54)	(24.58, 25.72)
(25.18, 26.27)	(24.10, 25.13)*	(24.64, 25.69)	(25.16, 26.15)	(24.95, 26.03)	(24.91, 26.08)	(24.83, 25.95)	(24.49, 25.69)
(24.60, 26.16)	(24.83, 26.07)	(25.63, 26.64)*	(24.77, 26.01)	(24.41, 25.84)	(25.38, 26.21)	(24.53, 25.86)	(24.17, 25.80)
(24.69, 25.83)	(25.00, 26.41)	(25.06, 26.19)	(24.32, 26.34)	(25.11, 26.42)	(23.99, 25.83)	(24.21, 25.43)	(24.84, 25.99)
(24.89, 26.16)	(24.99, 26.73)	(24.85, 26.24)	(24.91, 26.12)	(24.79, 26.03)	(24.52, 25.83)	(24.74, 26.21)	(24.37, 26.01)
(24.95, 26.36)	(24.37, 25.92)	(24.81, 26.10)	(24.89, 26.40)	(24.80, 25.90)	(24.60, 26.12)	(24.84, 26.32)	(25.02, 26.19)
(24.84, 25.95)	(25.06, 26.51)	(25.09, 25.96)	(25.44, 26.34)*	(25.09, 26.27)	(25.19, 26.86)	(24.97, 26.15)	(25.31, 26.21)
(24.65, 26.09)	(24.96, 26.36)	(24.88, 26.18)	(24.94, 26.05)	(24.70, 25.78)	(25.10, 26.22)	(24.63, 26.06)	(24.77, 26.36)
(24.86, 26.81)	(24.22, 25.71)	(24.98, 26.46)	(24.29, 25.43)	(24.53, 26.00)	(24.58, 26.31)	(24.27, 25.58)	(25.25, 26.68)
(24.65, 26.09)	(25.23, 26.49)	(25.25, 26.83)	(24.97, 25.94)	(24.88, 26.13)	(24.37, 25.41)	(25.37, 26.31)	(24.80, 26.43)
(24.73, 26.32)	(24.42, 25.81)	(24.63, 25.70)	(24.66, 25.43)	(25.58, 26.41)*	(24.22, 26.04)	(24.43, 26.08)	(24.81, 26.14)
(25.14, 26.47)	(24.92, 26.19)	(24.48, 25.91)	(25.13, 26.42)	(24.15, 25.45)	(24.84, 26.14)	(24.77, 26.08)	(25.22, 26.36)
(24.27, 25.36)*	(24.37, 25.56)	(25.35, 26.84)	(24.73, 26.10)	(24.80, 26.20)	(24.35, 25.81)	(24.30, 25.74)	(24.15, 25.64)

(*continues*)

TABLE E.1 (*Continued*)

(24.70, 26.04)	(24.88, 26.14)	(24.45, 25.86)	(24.30, 26.08)	(24.66, 25.43)	(24.46, 26.12)	(24.38, 25.55)	(25.13, 26.01)
(24.60, 26.25)	(25.52, 26.51)*	(24.68, 26.20)	(24.87, 26.18)	(24.57, 25.65)	(25.60, 26.77)*	(24.91, 26.44)	(24.88, 26.14)
(24.20, 25.79)	(24.52, 25.70)	(25.58, 26.83)*	(25.15, 25.94)	(24.87, 26.34)	(24.70, 26.12)	(24.40, 25.56)	(24.93, 26.21)
(24.62, 25.87)	(24.54, 26.05)	(24.07, 25.80)	(25.01, 26.40)	(25.08, 26.52)	(24.15, 25.80)	(24.59, 26.22)	(24.65, 25.90)
(25.21, 26.21)	(24.85, 26.46)	(25.22, 26.74)	(25.07, 26.45)	(24.05, 25.59)	(24.08, 25.41)	(24.24, 25.36)*	(24.67, 26.08)
(24.40, 25.75)	(25.29, 26.31)	(24.51, 26.07)	(24.80, 26.23)	(24.82, 26.58)	(24.81, 26.06)	(24.74, 26.04)	(24.67, 25.76)
(24.79, 25.71)	(23.90, 25.01)*	(24.61, 26.39)	(24.89, 26.53)	(24.86, 26.49)	(24.54, 25.86)	(24.54, 26.25)	(24.67, 25.82)
(24.22, 25.61)	(24.39, 26.25)	(24.81, 26.08)	(25.41, 26.35)*	(24.62, 25.97)	(25.02, 26.13)	(24.39, 25.76)	(24.53, 25.72)
(24.44, 25.73)	(24.75, 26.44)	(25.03, 25.84)	(24.53, 26.10)	(25.05, 26.12)	(25.21, 26.39)	(24.37, 25.81)	(25.16, 26.25)
(24.46, 25.86)	(24.92, 26.30)	(24.65, 26.25)	(24.85, 26.22)	(24.22, 25.46)	(25.24, 26.42)	(24.63, 25.97)	(24.43, 25.65)
(24.92, 25.76)	(24.65, 26.04)	(24.96, 26.33)	(24.97, 26.55)	(25.30, 26.55)	(25.15, 26.62)	(24.82, 25.93)	(25.05, 26.51)
(25.02, 26.02)	(25.01, 26.38)	(24.71, 26.19)	(24.45, 25.26)*	(24.55, 26.16)	(24.90, 26.09)	(24.71, 26.29)	(25.14, 26.43)
(24.98, 26.21)	(24.78, 26.11)	(24.82, 26.44)	(24.75, 26.34)	(25.21, 26.27)	(24.55, 25.71)	(24.94, 26.42)	(24.34, 25.64)
(25.13, 26.29)	(24.36, 25.74)	(24.38, 25.42)	(24.83, 26.40)	(24.23, 25.42)	(24.34, 25.89)	(24.29, 26.05)	(24.01, 25.61)
(24.80, 26.15)	(24.77, 26.08)	(24.52, 26.03)	(24.42, 25.97)	(25.47, 26.93)*	(24.56, 25.93)	(24.53, 25.74)	(24.72, 26.14)
(25.34, 26.62)	(24.29, 25.38)*	(24.89, 26.23)	(24.89, 26.19)	(24.85, 26.08)	(24.84, 26.24)	(24.89, 26.15)	(24.60, 25.76)
(25.38, 26.61)	(25.28, 26.67)	(24.76, 26.06)	(25.10, 26.19)	(24.65, 25.90)	(24.55, 26.19)	(24.96, 26.05)	(24.48, 25.98)

APPENDIX F

COMPANION WEBSITE

F.1 INTRODUCTION

The book's companion website has several programs and instructional worksheets to aid the student in the learning process. The companion website contains the software ADJUST, MATRIX, and STATS that were available with the previous edition of the book. Additionally, Mathcad® worksheets demonstrate the programming of many of the problems presented in this book. For those who do not own MATHCAD, html files of these worksheets have been created. These files can be viewed using your computer's web browser. Microsoft Excel® spreadsheets on the companion website demonstrate the solutions to many of the example problems presented in this book.

All software contained on the companion website is Windows-based and will run on most Windows-based computers. Each program has its own installation program on the companion website. Each program has different system memory requirements, although all will run in less than 2 megabytes of memory. The MATHCAD worksheets and Excel spreadsheets can be individually copied to your machine. The MATHCAD e-book will be available in the MATHCAD help system if the entire *e-book* subdirectory is copied to the MATHCAD *handbook* subdirectory.

This software is "freeware" and as such can be freely distributed with this book. However, it is not intended for commercial use, not guaranteed to be computationally correct, nor supported in any manner. It is simply provided to aid your understanding of the topics contained in this book.

F.2 FILE FORMATS AND MEMORY MATTERS

Most options in the software packages ADJUST, MATRIX, and STATS use formatted text files for data entry. Each package is equipped with an editor that allows the user to create these files. The help file for each software program describes the format of the data files. Many of these files have been described throughout the book. It is also possible to create data files using other text file editors such as Notepad. The file-reading routines contained in ADJUST, MATRIX, and STATS can use either commas, spaces, or tabs as text delimiters. It is therefore important to avoid commas in large numbers such as coordinates, since the software will read each part of the number individually. For example, a coordinate of 675,301.213 will be read as the numbers 675 and 301.213, since the comma is a text delimiter. The use of the TAB delimiter is especially useful with MATRIX, since this allows the user to "cut and paste" values from spreadsheets. Furthermore, station and point identifiers should not contain any delimiter characters. Thus, a station called BM RED must not be entered with a space or it will be interpreted as BM and RED. For example, it could be entered as BM_RED or simply RED.

F.3 SOFTWARE

F.3.1 ADJUST

ADJUST is the main computational program. It contains programs that either support computation of problems in the book or perform the computations. There are several least-squares programs contained in ADJUST and several supporting options. For example, the traverse option in ADJUST can compute initial approximations for unknown stations in a horizontal plane survey. The "Horizontal Data" under Least-Squares Adjustments can perform the least-squares adjustment of the data, as discussed in Chapter 15 through 17. New to ADJUST is the ability to read the National Geodetic Survey (NGS) binary little endian (le) data files for geoid height and deflection of the vertical. This ability can be used in software such as the three-dimensional geodetic adjustment, as discussed in Chapter 24. If the user lives in a region not supported by the NGS, the geoid height and deflection of the vertical components can be entered into the data files where it is appropriate. Additionally, ADJUST now has state plane coordinate system editors for both SPCS 83 and SPCS 2022. These two United States map projections have their parameters stored

in binary files in the Windows Common Files subdirectory. The SPCS 2022 editor can be used by users in other countries to enter their regional plane coordinate systems. It supports the single parallel Lambert conformal conic, transverse Mercator, and oblique Mercator map projections. Additionally, in several options in ADJUST, the software will accept angles in the sexagesimal or grad system of units.

File formats required by this software are discussed in the accompanying Help file and the file *Documentation to Accompany ADJUST*, which is loaded in the "Adjust data files" at the time of installation. Along with the Help files, these sample data files can be viewed and compared with the accompanying example problems in the book to assist you in creating your own files.

F.3.2 STATS

STATS is a statistical package that computes the basic statistical properties of simple data sets as well as derives the critical values for the normal, t, χ^2, F, and τ distributions. It can also develop confidence intervals for the three distributions and performs hypothesis testing for the t, χ^2, and F distributions.

F.3.3 MATRIX

MATRIX performs simple matrix operations. This software can be used to solve many of the least-squares problems presented in this book. It can read and write files of matrices, perform the operations of scaling, transposition, addition, subtraction, multiplication, and inversing of matrices, and allow the user to view the results of these operations. New additions to matrix include the ability to select a submatrix from an existing matrix and augment and stack two matrices. These operations allow the user to add constraint equations using Helmert's method (see Chapter 20) and perform the adjustment internally in MATRIX. Additionally, this adjustment can be performed using the new addition of "Constraints using Helmert Method" under the numerical operations menu.

When solving least-squares problems with this package, a spreadsheet can be used to compute the matrices for the problem. These matrices can then be cut and pasted into the MATRIX editor. Once saved to disk, the data files can be read by the software and manipulated.

F.3.4 MATHCAD Worksheets

The companion website contains an electronic MATHCAD book. This book should be copied into a subdirectory in the HANDBOOK directory under MATHCAD so that the electronic book can be opened from the MATHCAD Help Menu.

The worksheets demonstrate most of the numerical examples contained in this book. Most can be modified to compute other problems in the book. Some of the worksheets read data files. These files can be created by any text editor such as

those contained in ADJUST, MATRIX, and STATS, or by another package such as Notepad. With many spreadsheets, it is possible to save a comma-separated values (.csv) text file. The format of the file is demonstrated by the numerical example problem being solved in the original worksheet. Thus, when the original worksheet is modified, it should be saved with a different name. You should compare the file formats in these worksheets with the example problem listed in the book to determine the proper format for your data.

These worksheets also provide a guide as to how various problems are solved using programming. The MATHCAD language is very similar to traditional higher-level programming languages. However, MATHCAD does not provide global variables. Thus, all variables, especially those that will be modified, must be passed from function to function in a worksheet and between worksheets. MATHCAD reads the executable commands and variables from the top to the bottom of the worksheet and from left to right on a line. This means that the location of a command or variable on the worksheet can be critical to the worksheet performing properly. Another difference between a traditional programming language and MATHCAD is that MATHCAD has both subscripted variables and array elements. Unfortunately, while the subscript and array element are entered differently, they look visually the same on the worksheet. Thus, it is easy to confuse a subscripted variable with a matrix element. That is, a_0 may look like a reference to a matrix element when in fact it is a subscripted variable. The user should refer to the MATHCAD help system to learn how to distinguish between, and use, subscripted variables and matrix elements. The MATHCAD code in Figure F.1 depicts references to matrix elements.

Figure F.1 contains a function to compute the azimuth of a line based on the coordinates of the endpoints. Except for the slight language-specific differences, the code in the MATHCAD and C functions is very similar. Thus, with some modifications, the code in the MATHCAD worksheets can serve as a model when developing similar code in a traditional programming language. One advantage in MATHCAD is that the equations look like the equations which appear in this book.

Mathcad Azimuth Computation Function:

$$Az(crds,i,j) := \left| \begin{array}{l} dx \leftarrow crds_{j,1} - crds_{i,1} \\ dx \leftarrow crds_{j,1} - crds_{i,1} \\ \alpha \leftarrow atan2(dy, dx) \\ \alpha \leftarrow \alpha + 2 \cdot \pi \text{ if } \alpha < 0 \\ \text{return } \alpha \end{array} \right.$$

C Language Azimuth Computation Function:

```
double az(crdstruct crds[10], int i, int j)
{
   double dx, dy, alpha;
    dx = crds[j][1] - crds[i][1];
    dy = crds[j][2] - crds[i][2];
    alpha = atan2(dy,dx);
    if (alpha<0) alpha = alpha + 2*PI();
    return alpha;
}
```

FIGURE F.1 Comparison of a Mathcad function and a C function.

F.3.5 Spreadsheets

Several times in the book, spreadsheets are discussed as an aid in analyzing data. Samples of EXCEL® spreadsheets are available on the companion website. These spreadsheets can be viewed and analyzed as appropriate. These spreadsheets demonstrate much of the computational power that is available in a spreadsheet. Additionally, the statistical tables from Appendix D are also available for use from a spreadsheet called *tables.xls*.

F.3.6 MATLAB Script Files

On the companion website instructors and students will find MATLAB scripts that solve the examples of Chapters 26 and 27. The scripts can be used to solve the numerical problems by changing the problem's variable input information. The scripts are annotated with comments to help identify the variables. For some problems in Chapter 27, the input file .mat is given. The file can be entered into MATLAB by simply dragging it into the MATLAB Workspace, where they script can be run. The scripts can be viewed in any text editor for those who do not have MATLAB; however, MATLAB is needed to run the scripts.

F.4 USING THE SOFTWARE AS AN INSTRUCTIONAL AID

Many of the problems presented in this book can be solved using the software on the companion website. However, it would not be wise to solve all problems with this software since true understanding can only be gained by solving the problem yourself. Still some problems are so repetitive or long, it is extremely difficult to solve correctly without the aid of software. In these cases, the reader is often referred to the software on the companion website. For example, in Appendixes A through C, the matrix operations of addition, subtraction, multiplication, and inversing are presented. Although the student is expected to solve the accompanying problems in these appendixes by hand, the MATRIX program should be used to solve the remainder of the problems presented in this book. This frees the reader of the matrix operations so that emphasis in learning can be placed on the topics presented in the chapters. However, the software can be used to check solutions. For example, many of the least-squares adjustment routines in ADJUST have the option to create a file of matrices. When selected, this option creates the matrices used in the solution and provides the user with a method of checking matrices they have created in a spreadsheet. The MATHCAD worksheets are also available to check the intermediate steps in a solution.

APPENDIX G

ANSWERS TO SELECTED PROBLEMS

Note: Only partial answers are given to many of the problems to provide the reader with a check of their work.

Chapter 1

6(c)	Systematic
1.11(d)	Systematic
1.13(a)	Instrumental
1.14	-0.004 m

Chapter 2

2.1	$S = \pm 0.003$
2.2(d)	200.534 m or 200.533 m. Both appear twice.
2.4(a)	10.4
2.6(c)	$S^2 = 0.7077$, $S = \pm 0.84''$

Chapter 3

3.1 1.43951
3.7 0.389; 1.478, 3.528
3.13(a) 449.99 ± 0.015
3.14 (33.13, 36.04)

Chapter 4

4.2(a) $\frac{\alpha}{2} = 0.05, \quad \chi^2_{0.95,4} = 0.711 \quad \chi^2_{0.05,4} = 9.488$
4.3(a) $\frac{\alpha}{2} = 0.05, \quad t_{0.05,4} = 2.132$
4.4(a) $F_{0.95,20,5} = \frac{1}{2.711} = 0.369 \;\; F_{0.10,20,5} = 4.558$
4.5(a) $\chi^2 = \frac{4(1.56)}{1} = 6.24$
4.8(a) $132.183 < \mu < 132.189$
4.14 $4.410 < \sigma^2 < 59.965$
4.18(a) $0.355 < \frac{\sigma_1^2}{\sigma_2^2} < 3.589$

Chapter 5

5.8(a) Two-tailed test
5.13(a) Fail to reject H_0.
5.14(a) Reject H_0.
5.16(a) Fail to reject H_0.

Chapter 6

6.1 ± 9.7 mm
6.5 ± 9.5 mm
6.9 $V = 130{,}900 \pm 300$ ft^3
6.13 $70{,}200 \pm 100$ ft^3

Chapter 7

7.4 ± 0.012 ft
7.9 $\pm 11.5''$
7.13 Yes, estimated misclosure $\pm 45.1''$

Chapter 8

8.1(a) Direct
8.3 Latitude $= 597.35 \pm 0.005$ ft
8.8 $\sigma = \pm 21.3''$
8.12(b) ± 0.038 ft
8.14(b) ± 0.038 ft
8.20(d) ± 0.05 ft

Chapter 9

9.1 8.08 ft
9.3 −157.873 ft
9.5 0.44 ft
9.8 ±2.7 mm
9.13 12.006 ft

Chapter 10

10.1 $56°14'47.5'' \pm 0.4''$
10.2 $\pm 1.7''$
10.6 153.776

Chapter 11

11.1 $A = 2.321$
11.6 $v_1 = -0.038$
11.7 $x = 2.030$
11.12 $1 = 114°23'00.2''$

Chapter 12

12.1 $C = 108.05$
12.2 $S_0 = \pm 0.0082$
12.8 $16 + 00 = 766.08$; $v_{16+00-17+00} = -0.010$

Chapter 13

13.1(a) ±0.71
13.2(a) Adjustment passes χ^2 test
13.3 $S_X = \pm 0.005$
13.9 $\Delta\text{Elev}_{YX} = 2.31$; $v_{YX} = 0.000$; $S_{YX} = \pm 0.006$

Chapter 14

14.1 $x = 4231.59$
14.2(b) ±1.60
14.3(a) $x_c = 12{,}196.92 \pm 0.005$, $y_c = 10{,}969.94 \pm 0.014$
14.7(a) $x_F = 2884.70\mathbf{8} \pm 0.003$

Chapter 15

15.1 $X = 9920.87$, $Y = 11{,}295.26$
15.4 $X = 9925.83$
15.5(b) $S_0 = \pm 0.10$
15.8(a) $X_B = 9155.53 \pm 0.019$

Chapter 16

16.1(b) $X = 13{,}769.95 \pm 0.016$
16.2(a) ± 1.37
16.4(b) $X = 10{,}109.155 \pm 0.0045$

Chapter 17

17.1(a) $X = 933{,}597.220$ m, $Y = -4{,}847{,}253.121$ m, $Z = 4{,}025{,}830.514$ m
17.2(a) $\varphi = 48°06'12.04369''$ N, $\lambda = 122°08'08.28458''$ W, $h = 13.407$ m
17.3(a) B: (1,646,621.505, −4,215,186.645, 4,479,591.200)
17.4(a) $\Delta X = 4.11$ ppm; $\sigma_{99.7\%} = \pm 0.009$
17.5(a) B: (48,807.312, −4,666,418.132, 4,333,541.203)
17.6(a) $\Delta X = 0.37$ ppm ; $\sigma_{99.7\%} = \pm 0.012$ m

Chapter 18

18.1(a) $a = 2.9734324$; $b = 1.3865338$
18.1(c) $\theta = 25°00'00''$; Scale = 3.28082
18.3(a) $a_1 = 2.9734277$; $b_1 = -1.3865081$
18.5(a) $a_1 = 0.2111053$; $c_2 = -50.658$
18.8 Projective transformation

Chapter 19

19.1 $t = -30.3934°$; $S_u = \pm 0.036$
19.2 $S_u = \pm 1.13$
19.6 $S_u = \pm 0.052$
19.22 $S_u = \pm 0.089$
19.25(a) 37%

Chapter 20

20.1 $x_B = 30{,}433.560$; $y_B = 39{,}803.631$
20.7 $\text{Elev}_D = 1302.514$ m
20.12 $S_x = S_y = \pm 0.267$

Chapter 21

21.1 See Section 21.2.
21.10 No blunders detected
21.16 ± 0.17 ft

Chapter 22

22.1 $b = 4499.38$ ft and $m = 1.00$
22.4(a) $a = 2.9734321$; $b = 1.3865337$; $T_X = 680{,}000.031$; $Ty = 320{,}000.025$
22.4(c) Scale = 3.28081908; $\theta = 25°00'00''$
22.6(a) $a = 2.9734272$; $b = -1.3865074$
22.8 Conformal transformation

Chapter 23

23.1 458.794 m
23.3 13°56′27.2″
23.5 98°10′13.1″
23.7 −2°36′09.6″
23.9 $0.49570de_A - 0.86768dn_A + 0.03754du_A - 0.49573de_C + 0.86767dn_C - 0.03750du_C = -0.00825$
23.11 $108.43156dn_B - 436.78988de_B + 0.00003du_B + 609.99716dn_A + 847.21603de_A + 0.00000du_A - 718.40986dn_C - 410.40641de_C + 0.00005du_C = -7.97''$

Chapter 24

24.2 See Section 24.1, paragraph 2.
24.5 See Section 24.1, paragraph 5.
24.6 *XYZ* in meters = (1,160,598.896, −4,655,901.446, 4,188,487.204)
24.13 (512.148, 595.384, 28.420)

Chapter 25

25.3 No
25.12 Too large, 5.66 mm
25.14 Outside the 99% CI

Chapter 26

26.1(a) $x_{ls} = 39.305$, $y_{ls} = 7.060$, $z_{ls} = 4.026$
26.2(a) $r = 250.105$ m, $\theta = 302°30'25''$, $\alpha = 6°03'06''$
26.5(a) $\sigma_r = 0.003$ m
26.8(a) $\sigma_{BWangle} = \pm 20.6''$

Chapter 27

27.1(a) $n = (a, b, c) = (0.0847, -0.0869, 0.9926)$
27.8(a) $h = 4.9998$, $k = 5.0008$, $p = 5.0022$, $r = 0.2524$
27.12(b) $h = 10.0019$, $k = 20.0017$, $p = 1.5612$, $r = 0.0994$, $c_1 = 0.0074$, $c_2 = -0.0102$, $c_3 = 0.9999$
27.14(a) $a_{00} = 5.0267$, $a_{10} = -0.0102$, $a_{01} = -0.0033$, $a_{11} = 0.2172$

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